The Applicability of Semi-Analytical Method for Different Orbits in Long Term Prediction

Dawei Wang[1], **Jingshi Tang**[2]
[(1)(2)] *School of Astronomy and Space Science, Nanjing University, Nanjing, China,*
sj_cbl@163.com

Abstract: *We hope to give some useful suggestions when dealing with the long term (100-years) prediction for a large number of targets, especially retired satellites and space debris. Obviously, we should take care of the speed and accuracy, therefore we consider the question from two aspects, the integrating method and the dynamic models. Firstly, for integrating method, we choose semi-analytical method which uses numerical integrator to calculate averaged system. It will get a larger step size and faster speed compared with traditional numerical method. Although semi-analytical method is not a new method but the details, especially for a long term orbit prediction, has not been explained clearly. Our work will discuss some details when realize the method. Secondly, for dynamic models, there are quite a lot perturbation, we should choose some main factors to satisfy both accuracy and speed. Since we have not finished the work totally, there only show partial results..*

Keywords: *semi-analytical method, long term prediction.*

1. Introduction

Semi-analytical method is a common method for long term orbit prediction, especially in the evolution of the space debris and solar system dynamics. It uses numerical integrator to calculate averaged system. Because of averaged right hand function, it will have a larger integrating step size and faster speed compared with traditional numerical method. The key point is how to get the averaged acceleration. There exists two ways, numerical and analytical method. We have tried both and will show their performance lately. Of course, integrating speed and accuracy is what we care about.

2. The theory of semi-analytical method

Orbit prediction can be written in the following ordinary differential equation as shown in Eq. (1)

$$\begin{cases} \dot{\vec{X}} = \vec{F}(t_0, t, \vec{X}) \\ \vec{X}(t_0) = \vec{X_0} \end{cases}, \tag{1}$$

where represents the six dimensional orbit state which can be position-velocity or orbital elements. In this paper, we will adopt orbital elements whose acceleration is perturbation, which may lead to high integrating speed.

The key idea of semi-analytical method is to eliminate the short period terms of perturbation functions. Then, integrating the remaining averaged system by numerical integrator. The averaged system can be shown as Eq. (2)

$$\frac{d\bar{\sigma}}{dt} = \bar{f}_{\varepsilon}(t_0, t, \sigma) \quad , \tag{2}$$

where represent averaged orbital elements, represent osculating elements and are averaged perturbation functions. After averaged, the system only has secular and long period variation. Usually, the averaged system can get several orbital period as an integrating step, which leads to high integrating speed. Since we are going to integrate averaged system, we should use mean or quasi-mean elements.

We have two ways to get averaged perturbation functions, numerical and analytical. We will explain the two methods below.

2.1. Numerical method to get averaged system

Numerical method to get averaged system is not to give an explicit averaged perturbation functions, but try to give the perturbation functions value at the time when the value is to be used.

For example, at time *t*, the six quasi-mean elements are ($a,i,\Omega,\xi,\eta,\lambda$), we need the averaged perturbation functions. Then we integrate the perturbation functions from *t* to *t*+*T*, where *T* is one orbit period. When we dealing with the integration, at each *t* between [*t*,*t*+*T*], we should add short period to quasi-mean elements and get the osculating elements before we put the elements in perturbation functions. The perturbation functions are ordinary formulas in traditional numerical integrating method. At last, we get the integrating result divided by *T* and get the averaged perturbation value at time *t*. The whole process likes Eq. (3)

$$\bar{f} = \frac{1}{T}\int_{t}^{t+T} f(\bar{\sigma}+\sigma_s)dt \, , \tag{3}$$

where $\bar{\sigma}$ are six quasi-mean elements ($a,i,\Omega,\xi,\eta,\lambda$), the former five elements remain unchanged during integrating time, but the last λ should change the value at different time *t* following the two-body and J_2 influence. Meanwhile the six σ_s also have different value at different time. And $\bar{\sigma}+\sigma_s$ will transform the quasi-mean elements into osculating elements at each time between *t* and *t+T*.

Now the question is how to calculate the integration Eq. (3). Here we have tried different numerical methods, like Guass-Legendre, Newton-Cotes and RKF7(8). The common principle of them are using polynomial sum calculation replaces integration, just like Eq. (4)

$$\frac{1}{T}\int_{t}^{t+T} f(\bar{\sigma}+\sigma_s)dt = \frac{1}{T}\sum_{k=1}^{n} A_k f_k(\bar{\sigma}+\sigma_s) \, , \tag{4}$$

Where A_k is weighting coefficient and different numerical method has different way to choose *n*, A_k and $f_k(\bar{\sigma}+\sigma_s)$. As our experience, these three methods have the similar performance in accuracy but RKF7(8) will cost much more time than the others, so we choose Guass-Legendre to calculate the integration.

After many attempts, we choose 3-order Guass-Legendre formula as our solution. 3-order means *n*=3, in other words, to calculate one integration, we need to calculate 3 times $A_k f_k(\bar{\sigma}+\sigma_s)$. What's more, for the accuracy, we separate one orbit period time *T* into 8 parts, and use 3-order Guass-Legendre method in each part. That's to say, we have to calculate 24 times $A_k f_k(\bar{\sigma}+\sigma_s)$ to get one averaged perturbation value at time *t*. Compared with traditional numerical method, the semi-analytical method's integrating step size must be 24 times larger, or it will cost more integrating time than traditional one.

Unfortunately, for a low orbit satellite, only consider J2 and 10^{-13} integrator truncation error, the semi-analytical method's step size will be 160-320 minutes, but it only lasts for 5 years, then the step size can't keep this level and decreases rapidly by error accumulation. If we use traditional numerical method, the step size can keep around 5 minutes level for 100 years. The result is semi-analytical method will cost more time than traditional numerical method.
What's worse, to deal with tesseral harmonic terms, the earth rotation terms (m-daily) can't be eliminated clearly in one orbit period. We have to use numerical method to integrate 24 hours, because its period is 24 hours,12 hours and so on, much larger than orbit period which is about 90 minutes for low earth orbit. Under this situation, semi-analytical will cost much more integrating time and get even smaller step size.
So the numerical method to get averaged system is not efficient. But it can be a good accuracy standard for semi-analytical method, because it don't leave out high order perturbation functions, the error mainly comes from numerical method.

2.2. Analytical method to get averaged system

Analytical method to get averaged system is to give an explicit averaged perturbation functions which can be explained as Eq. (5)

$$\overline{f} = f_{1c} + f_{2c} + f_{2l} + \left(\frac{\partial f_{1s}}{\partial \sigma} \sigma_s \right)_{c,l} \tag{5}$$

where f_{1c}, f_{2c} are one and two order secular perturbation, f_{2l} is two order long period perturbation, $\left(\frac{\partial f_{1s}}{\partial \sigma} \sigma_s \right)_{c,l}$ is cross terms which have secular and long period terms respectively. Now we only take all the orders of zonal harmonic terms and coordinate additional perturbations into account. Lately, we will add tesseral harmonic terms, third body and solar pressure.
Although the right hand functions are not complete, but the performance of this method has been quite exciting on integrating speed compared with traditional numerical method, and the accuracy is as good as numerical method to get averaged perturbation which can be treated as a standard level for semi-analytical accuracy. Lately, we will give some examples to show the results.

3. Examples

3.1. Numerical average and analytical average examples

To compare the two methods clearly, we only choose J_2 as perturbation function. Then we take traditional numerical method as standard whose integrator is RKF7(8). Then we use numerical and analytical method to get averaged perturbations respectively, and compare the two results with the standard. The result can be seen in Fig. 1 and Fig. 2.
As shown in Fig. 1, the difference between numerical average method and traditional numerical integration. The original osculating elements (a,e,i,Ω,ω,M) are (7000,0.001, 10, 290, 250, 320) , the unit of semi-major axis and angle is km and degree respectively. The prediction time is 100 years. The horizontal axis unit is year.
So, whether the numerical average results are reasonable? We will give our opinions. Fig. 1 shows some details conform to the orbit theory.

Firstly, let's consider *a, e, i*. As we known, under J2 influence, *a, e, i* only have short period terms. So compared with traditional numerical method the difference of *a, e, i* mainly comes from short period terms which are functions of $\sin\lambda$ or $\cos\lambda$. We can see from Fig. 1, in 100 years time scale, λ has a difference around 18 degrees. So, in a word, the difference of *a, e, i* mainly comes from the error of λ.

Secondly, let's see λ. Although the difference of λ is not linear, but 18 degrees in 100 years equals to a slope of 8×10^{-8} under normalized unit. Numerical average method expects no difference compared with traditional numerical method, but the sum of numerical error and the error of semi-major axis *a* lead to the result. $\lambda=\omega+M$, and the main error comes from *M* whose secular term is closely linked with *a*. This is obviously expressed in Eq. (6)

$$M = M_0 + n(t-t_0) + M_1(t-t_0) + \cdots , \tag{6}$$

where $n= a^{-3/2}$. The error in semi-major axis *a* will directly affect *M* and the error of *M* will keep increasing with integrating time. And the imprecise *M* will affect *a, e, i* in turn.

Lastly, the Ω has a slope of 1.34×10^{-8} under normalized unit. This is due to Ω_1 which is the secular term of Ω under J_2. The Ω_1 can be expressed as Eq. (7)

$$\Omega_1 = \frac{3}{2}\frac{J_2}{p^2} n\left(2-\frac{5}{2}\sin^2 i\right) , \tag{7}$$

where $p=a(1-e^2)$, $n= a^{-3/2}$ and a has an error of 10^{-5}, J_2 is 10^{-3},

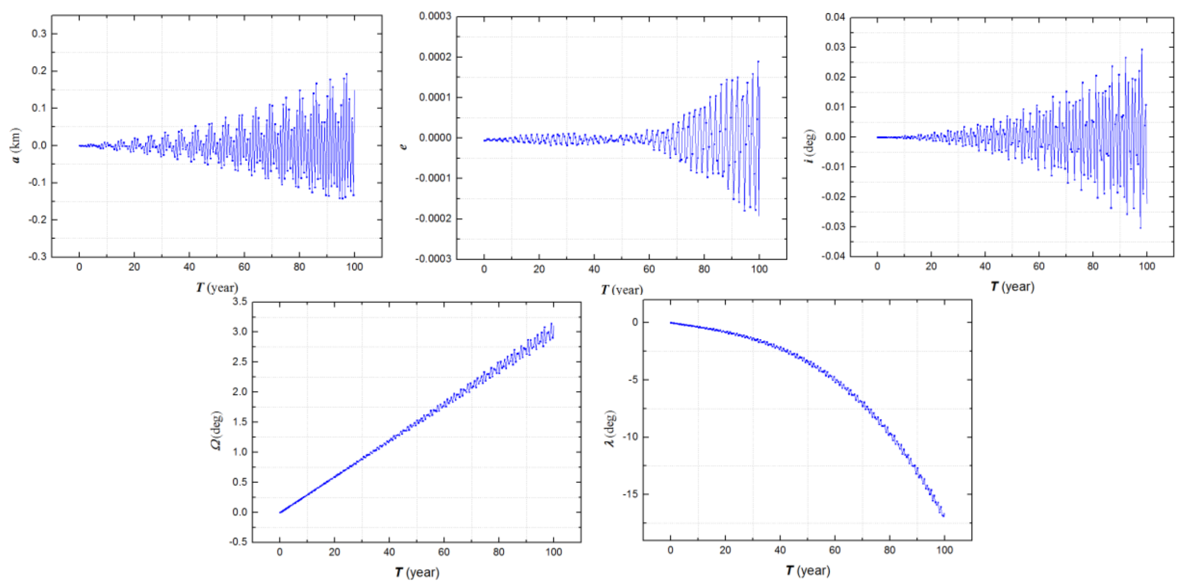

Fig. 1. The difference of orbit elements between numerical average method and traditional numerical method for 100 years. Only consider J_2 and the original elements (a,e,i,Ω,ω,M) are (7000,0.001, 10, 290, 250, 320) , the unit of semi-major axis and angle is km and degree respectively.

so the Ω_1 has 10^{-8} is not hard to understand.

The accuracy of semi-analytical by numerical average is hard to increase based on our current experience.

As shown in Fig. 2, the difference between analytical average method and traditional numerical integration. The original parameters are the same as former in Fig. 1. Although the shape has a little difference, but the error magnitude is nearly the same as Fig. 1. Analytical average method

has nearly the same accuracy as numerical average in 100 years prediction where the later one can be a standard principle as we explained before. That's a good news for us, because the two kinds of semi-analytical methods have a huge difference on integrating time, and the analytical average is much faster than numerical average. We find the reason is step size changing.
The situation of step size changing, we can see from Fig. 3. when integrating, numerical average will only keep a large step size for 5 years, and then its step size will keep on decreasing. As our analysis before, if numerical average can't afford 24 times larger step size than traditional numerical method, it will spend even more time finish the integration.

Table 1. Time spending of three kinds of methods

	Numerical average	Analytical average	Traditional numerical
Model	J_2	J_2	J_2
Time cost	1260.64 min	10.18 s	412.01 min

But for analytical average method, it will keep a large step size during the whole integration, what's more, its step size is so large and even reach 25 times orbit period.

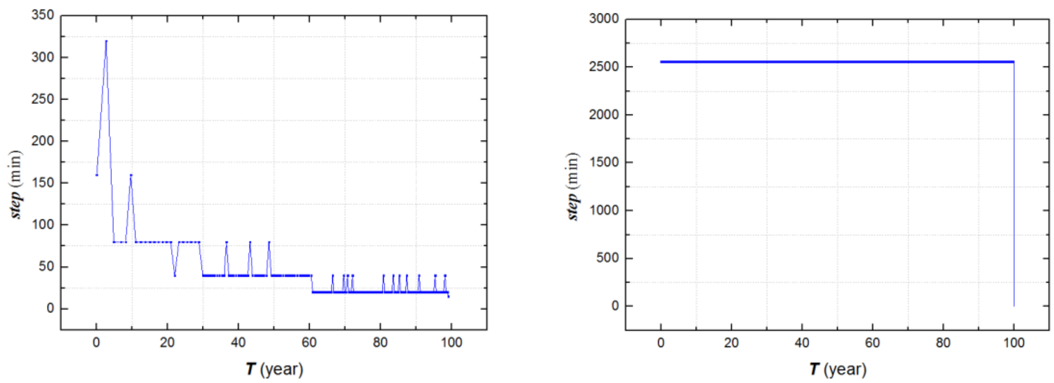

Fig. 3. Numerical average (left) and analytical average (right) integrating step size

The cost of time for numerical average, analytical average and traditional numerical method have been expressed in Table 1, and the analytical average has an obvious advantage on speed which only costs 10 seconds while others cost several hours.

3.2. Analytical average with full zonal harmonic terms

This part will display one examples of analytical average method with full zonal harmonic terms (50 orders). We use JGM-3 Gravity Model and the original elements are (7000,0.001,30,60,80,100). The results also compare with the traditional numerical method, we can see the difference between the two methods in Fig. 4. The situation is similar with Fig. 2, and we can see even consider the whole zonal harmonic terms, the result compared with standard traditional numerical method is not get much worse. It's a good condition for us. And this result also meets with the analysis before. The slope of Ω and λ are three order magnitude, smaller than Eq. (5) where we only consider two order perturbation functions. The whole integrating time is about 15 minutes for full order zonal harmonic terms, and the step size can reach 2500 minutes.

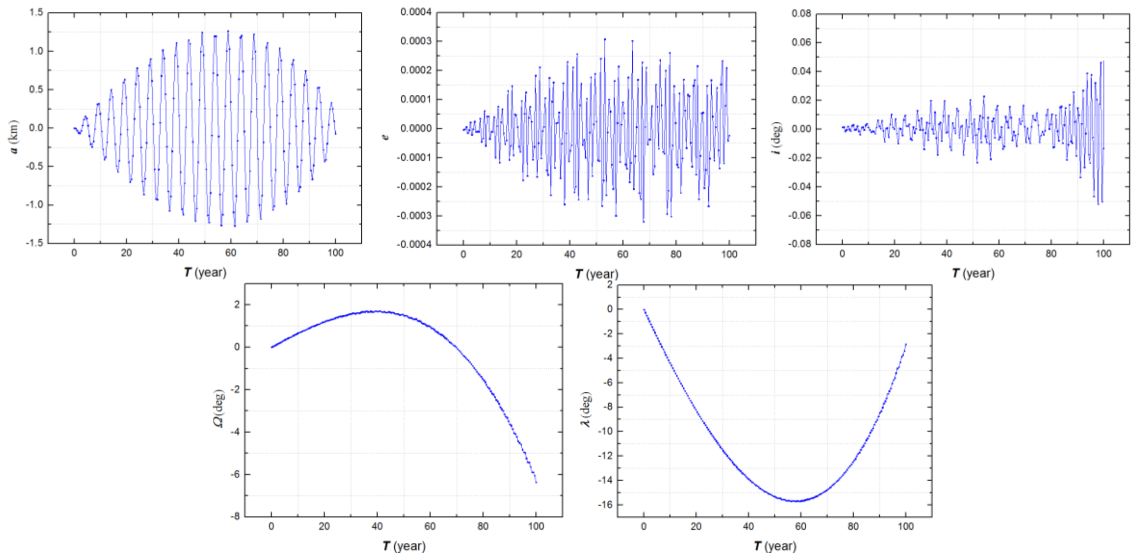

Fig. 4. The difference of orbit elements between analytical average method and traditional numerical method for 100 years. Consider 50 orders of zonal harmonic terms.The original elements (a,e,i,Ω,ω,M) are (7000,0.001, 30, 60, 80, 100) , the unit of semi-major axis and angle is km and degree respectively.

4. Conclusions

We have tried two kinds of semi-analytical methods, one is using numerical method to get averaged perturbation value, the other is using analytical method to give explicit averaged perturbation functions. After many attempts, we find the numerical average method can be semi-analytical method standard on accuracy, and analytical average method can be an ideal tool for long term prediction, not only on accuracy but also on integrating speed.

We have used analytical average method to try full orders zonal harmonic terms, and its accuracy after 100 years is at the same magnitude as only consider J_2 compared with traditional numerical method. Lately, we will add earth rotation and resonance of tesseral harmonic terms, third body and solar pressure to improve the method. Then we will test its performance in different kinds of orbits.

5. References

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