Dynamics around Libration Points in the Binary Asteroid System

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Abstract: *Using two homogeneous ellipsoids to simulate the two asteroids in a binary asteroid system, equation of motion (EOM) for this binary system truncated at the 2nd order of non-spherical terms is given first. Also truncated at the 2nd order of non-spherical terms, EOM for a massless body in this system is given. Dynamical equilibrium points usually disappear in this system. If the dynamics of this binary system are regular, special quasi-periodic orbits called dynamical substitutes exist. They are around the geometrical libration points of the circular restricted three-body problem (CRTBP) of the binary system. Truncated at the 2nd order of non-spherical terms, these special solutions are given. Their stability properties are briefly studied. Some numerical simulations are made. Several special cases are discussed.*

Keywords: *Binary Asteroid, Libration Point, Dynamical Substitute, Stability.*

1. Introduction

The binary system is an interesting phenomenon in the asteroid population [1,2]. They may come from a single asteroid by breakup events caused by thermal effects [3], gravitational tides [4], or by colliding with each other[5]. Generally, there is a strong coupling between the mutual orbit and the rotations of the two asteroids[6-8]. Besides the dynamical evolution of the asteroid system itself, it would be interesting to study the dynamics of a massless particle in such a system [9]. A good point to start such studies is the libration points[10,11]. With the aid of some dissipative mechanisms [12-14], the binary system may collide with each other again, break apart, or evolve into a synchronous or double synchronous state [15,16]. In this work, we address this problem in a binary asteroid system in a synchronous state. That is, we'll assume a primary with a rotation period different from (usually shorter than) the orbital period and a secondary with a rotation period same as the orbital period. Many binary asteroid systems observed are locked in such a state [17,18]. Our Earth-Moon system is also in such a synchronous state.

The tri-axial homogeneous ellipsoid model is often used to simulate the asteroid. Different from a single ellipsoid[19], the mutual potential between the two ellipsoids can't be expressed in close forms but usually in literal expansions[20-24]. In this study, the mutual potential is truncated at the 2nd order. To further simply the studies, the work is carried out for the planar case. Generally, the dynamical equilibrium points no longer exist in this system. However, special quasi-periodic orbits with several basic frequencies exist around the geometrical libration points of the CRTBP model of this system. These orbits are usually called dynamical substitutes in literature [25-28]. If the mutual distance between the two asteroids is not very large compared with the size of the asteroids, these dynamical substitutes may reside inside the asteroids. This causes their non-existence in practice. This problem is most serious when the dynamical substitutes are very large due to resonances.

As a first step of this work, the EOM of the binary asteroid system is given. Assuming primary axial rotations of the two asteroids, analytical solution of the mutual orbit along with the rotations truncated at the 2nd order of non-spherical terms is obtained. Next, we deduce the EOM of a massless body in such a binary system. Truncated at the 2nd order of non-spherical terms, the EOM can be taken as the EOM of the CRTBP plus some perturbative term. Further expanding the EOM around the geometrical libration points of the CRTBP, we can get analytical solutions for the dynamical substitutes. Dynamical substitutes around both collinear libration points and triangular libration points are studied. Some numerical simulations are done. It is found that the stability property of the motions around the triangular libration points may change due to resonances between the external perturbation frequency and the intrinsic frequencies. At the end of the work, several special cases are briefly discussed.

2. EOM of the Binary System

Dynamics of the full two body problem composed of two ellipsoids are very complicate[29-32]. Except some special cases, usually no analytical solutions exist. Numerical approaches are often taken to study the dynamics of this system.

To simplify our studies, we restrict the mutual potential of the two ellipsoids be truncated at the 2nd order and we restrict the studies in the planar case. Fig. 1 shows the geometry of the binary system and the coordinates used in our study. O is the barycenter of the system. A is the secondary asteroid and B is the primary asteroid. $O-XY$ indicates the inertial frame and $O-xy$ indicates the synodic frame. The vector $\mathbf{S}$ is the relative position vector between the barycenters of the two ellipsoids. The two angles θ_A and θ_B are rotation angles of the body fixed frames of asteroids A and B with respect to the inertial frame. ϕ is the rotation angle of the body fixed frame of asteroid B with respect to the body fixed frame of asteroid A. The two angles θ and Θ are respectively the quadrant angles of the vector $\mathbf{S}$ in the body fixed frame of asteroid A and the inertial frame. From Fig. 1, we know

$$\Theta = \theta + \theta_A, \qquad \delta = \theta - \phi$$

Not all the five angles $\theta_A, \theta_B, \phi, \theta, \Theta$ are necessary to describe the orbital motion and rotations of the two asteroids. Only three of them are necessary. In the following, we use θ_A, θ, ϕ and the norm of the vector $s = \|\mathbf{S}\|$ as independent variables

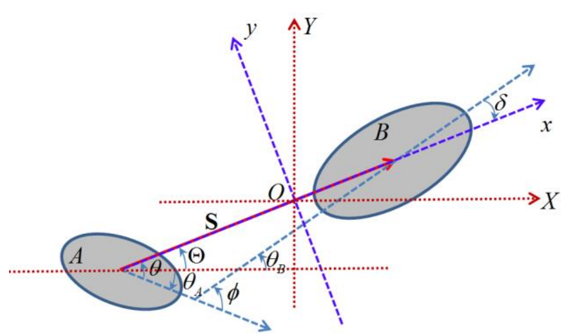

Figure 1. A picture illustrating the coordinates and angles in the binary asteroid system

The semi-axes and the mass of the two ellipsoids are denoted as

$$a_A, b_A, c_A, m_A; \qquad a_B, b_B, c_B, m_B$$

Truncated at the 2nd order of the gravity, the non-spherical coefficients J_2 and J_{22} are given by[33]

$$J_2^* = \frac{a_*^2 + b_*^2 - 2c_*^2}{10\bar{a}_*^2}, \qquad J_{22}^* = \frac{a_*^2 - b_*^2}{20\bar{a}_*^2}$$

where $*$ indicates A or B. The reference semi-major axis $\bar{a}_*$ of each ellipsoid is defined as

$$\bar{a}_* = \left(a_* b_* c_*\right)^{1/3}$$

In this work, the following units are used: the mass unit is the sum of the masses of A and B, the length unit is the distance a between the barycenters of the two asteroids, and the time unit is chosen such that the gravitational constant $G=1$. Introducing

$$\alpha_A = \frac{a_A}{a}, \quad \alpha_B = \frac{a_B}{a}$$

The mutual potential truncated at the 2nd order is[32]

$$U = -m\left[\frac{1}{S} + \frac{1}{S^3}\left(A_1 + A_2 \cos 2\theta + A_3 \cos 2\delta\right)\right] \tag{1}$$

where

$$m = \mu(1-\mu), \qquad \mu = \frac{m_A}{m_A + m_B}$$

$$A_1 = \frac{1}{2}\left(J_2^B \alpha_B^2 + J_2^A \alpha_A^2\right), \qquad A_2 = 3J_{22}^A \alpha_A^2, \qquad A_3 = 3J_2^B \alpha_B^2$$

Using the variables described in Fig. 1, EOM of the orbital motion and rotations of this bianry system is[32]

$$\begin{cases} \ddot{S} = S\left(\dot{\theta} + \dot{\theta}_A\right)^2 - \dfrac{1}{m}\dfrac{\partial U}{\partial S} \\ \ddot{\theta} = -2\dfrac{\dot{S}}{S}\left(\dot{\theta} + \dot{\theta}_A\right) - \left(\dfrac{1}{I_z^A} + \dfrac{1}{mS^2}\right)\dfrac{\partial U}{\partial \theta} - \dfrac{1}{I_z^A}\dfrac{\partial U}{\partial \phi} \\ \ddot{\phi} = -\dfrac{1}{I_z^A}\dfrac{\partial U}{\partial \theta} - \left(\dfrac{1}{I_z^A} + \dfrac{1}{I_z^B}\right)\dfrac{\partial U}{\partial \phi} \\ \ddot{\theta}_A = \dfrac{1}{I_z^A}\dfrac{\partial U}{\partial \theta} + \dfrac{1}{I_z^A}\dfrac{\partial U}{\partial \phi} \end{cases} \tag{2}$$

From Eq. (1), we have

$$\frac{\partial U}{\partial S} = m\left[\frac{1}{S^2} + \frac{3}{S^4}\left(A_1 + A_2 \cos 2\theta + A_3 \cos 2\delta\right)\right]$$
$$\frac{\partial U}{\partial \theta} = \frac{2m}{S^3}\left(A_2 \sin 2\theta + A_3 \sin 2\delta\right)$$
$$\frac{\partial U}{\partial \phi} = -\frac{2m}{S^3} A_3 \sin 2\delta$$

Substituting these equations in Eq. (2), we have

$$\begin{cases} \ddot{S} = S\left(\dot{\theta} + \dot{\theta}_A\right)^2 - \left[\frac{1}{S^2} + \frac{3}{S^4}\left(A_1 + A_2 \cos 2\theta + A_3 \cos 2\delta\right)\right] \\ \ddot{\theta} = -2\frac{\dot{S}}{S}\left(\dot{\theta} + \dot{\theta}_A\right) - \frac{2}{S^5}\left(A_2 \sin 2\theta + A_3 \sin 2\delta\right) - \frac{2mA_2}{I_z^A}\frac{\sin 2\theta}{S^3} \\ \ddot{\phi} = \frac{2mA_3}{I_z^B}\frac{\sin 2\delta}{S^3} - \frac{2mA_2}{I_z^A}\frac{\sin 2\theta}{S^3} \\ \ddot{\theta}_A = \frac{2mA_2}{I_z^A}\frac{\sin 2\theta}{S^3} \end{cases} \tag{3}$$

If the non-spherical terms A_1, A_2, A_3 all equal zero, from Eq. (3) we have

$$\ddot{S} = S\left(\dot{\theta} + \dot{\theta}_A\right)^2 - 1/S^2, \quad \ddot{\theta} = -2\dot{S}\left(\dot{\theta} + \dot{\theta}_A\right)/S, \quad \dot{\phi} = \text{constant}, \quad \dot{\theta}_A = \text{constant}$$

This means the rotations and the orbital motion are decoupled. This can only happen when the two asteroids are homogeneous spheroids. In this case, the binary system is reduced to the two body problem. But even if one of the two asteroids is not a homogeneous spheroid, the rotations and the orbital motion are coupled. That's what we called spin-orbit coupling in literature[34,35]. There are some equilibrium states of the spin-orbit coupling problem. Two important ones are the synchronous state and the double synchronous state, as illustrated in Fig. 2.

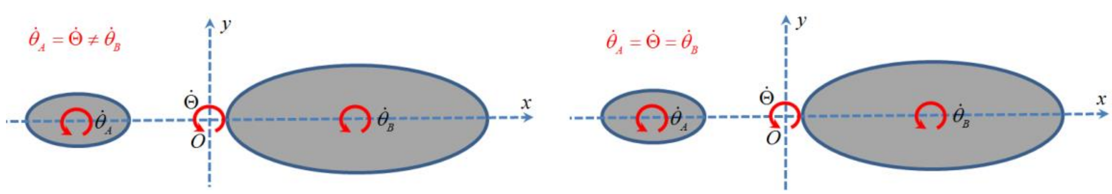

Figure 2. The synchronous state (left) and the double synchronous state (right) of the binary system

3. EOM of A Massless Body

In this work, we'll describe the dynamics around the libration points in a binary asteroid system which is trapped in a synchronous state. To this end, we have to describe the motion of the two ellipsoids first. To further simplify the studies, we assume a near-circular orbit of the two ellipsoids. Expanding Eq. (3) around the following state (ω_B is the mean rotation frequency of the asteroid B)

$$S_0 = 1,\ \dot{S}_0 = 0,\ \theta_0 = 0,\ \dot{\theta}_0 = 0,\ \dot{\phi}_0 = \omega_B - 1,\ \left(\dot{\theta}_A\right)_0 = 1 \tag{4}$$

and only retaining the linear terms, we have

$$\begin{cases} \Delta\ddot{S} - 3\Delta S - 2\left(\Delta\dot{\theta} + \Delta\dot{\theta}_A\right) = -3\left(A_1 + A_2 + A_3 \cos 2\delta_0\right) \\ \Delta\ddot{\theta} + 2\Delta\dot{S} = -2A_3 \sin 2\delta_0 \\ \Delta\ddot{\phi} = \left(2mA_3 / I_z^B\right) \sin 2\delta_0 \\ \Delta\ddot{\theta}_A = 0 \end{cases} \tag{5}$$

We require the system is locked in a synchronous state, so no secular terms are allowed in Eq. (5). Truncated at the 2nd order of non-spherical terms, it's easy to obtain

$$\begin{cases} \Delta S = \Delta S_0 + \alpha \cos 2\delta_0 \\ \Delta\theta = \beta \sin 2\delta_0 \\ \Delta\phi = \gamma \sin 2\delta_0 \\ \Delta\theta_A = 0 \end{cases} \tag{6}$$

where ($\omega_\delta = \dot{\Theta} - \dot{\theta}_B$)

$$\Delta S_0 = \left(A_1 + A_2\right),\ \alpha = \frac{\left(3\omega_\delta - 2\right)A_3}{\left(4\omega_\delta^2 - 1\right)\omega_\delta},\ \beta = \frac{\left(4\omega_\delta^2 - 6\omega_\delta + 3\right)A_3}{2\left(4\omega_\delta^2 - 1\right)\omega_\delta^2},\ \gamma = -\frac{mA_3}{2\omega_\delta^2 I_z^B}$$

Next, we give EOM of a massless small body in this binary system. In the barycenter-centered inertial frame. Denote the position vector of the small body in this frame as $\mathbf{R}$, we have

$$\ddot{\mathbf{R}} = R_z\left(-\theta_A\right)\mathbf{F}_A + R_z\left(-\theta_B\right)\mathbf{F}_B \tag{7}$$

where $R_z(*)$ means a rotation of the angle $*$ around the z axis. The two forces $\mathbf{F}_A$ and $\mathbf{F}_B$ are expressed in the body-fixed frame of A and B. They have the following form

$$\mathbf{F}_A = \begin{pmatrix} -\mu X_A \left\{ \frac{1}{R_A^3} + \frac{3J_2^A \alpha_A^2}{2R_A^5} + \frac{3J_{22}^A \alpha_A^2}{R_A^5} \left[\frac{5\left(X_A^2 - Y_A^2\right)}{R_A^2} - 2 \right] \right\} \\ -\mu Y_A \left\{ \frac{1}{R_A^3} + \frac{3J_2^A \alpha_A^2}{2R_A^5} + \frac{3J_{22}^A \alpha_A^2}{R_A^5} \left[\frac{5\left(X_A^2 - Y_A^2\right)}{R_A^2} + 2 \right] \right\} \end{pmatrix}$$

$$\mathbf{F}_B = \begin{pmatrix} -(1-\mu) X_B \left\{ \frac{1}{R_B^3} + \frac{3J_2^B \alpha_B^2}{2R_B^5} + \frac{3J_{22}^B \alpha_B^2}{R_B^5} \left[\frac{5\left(X_B^2 - Y_B^2\right)}{R_B^2} - 2 \right] \right\} \\ -(1-\mu) Y_B \left\{ \frac{1}{R_B^3} + \frac{3J_2^B \alpha_B^2}{2R_B^5} + \frac{3J_{22}^B \alpha_B^2}{R_B^5} \left[\frac{5\left(X_B^2 - Y_B^2\right)}{R_B^2} + 2 \right] \right\} \end{pmatrix}$$

where the subscripts A and B respectively indicate that the position vector of the small body is expressed in the body-fixed frame of A (centered at the barycenter of A) and in the body-fixed frame of B (centered at the barycenter of B). Denote the position vector in the barycenter-centered synodic frame as $\mathbf{r}$, we have

$$\mathbf{R} = R_z(-\Theta)\mathbf{r}, \qquad \dot{\mathbf{R}} = R_z(-\Theta)\dot{\mathbf{r}} + \dot{\Theta}\frac{\partial R_z(-\Theta)}{\partial \Theta}\mathbf{r}$$

$$\ddot{\mathbf{R}} = R_z(-\Theta)\ddot{\mathbf{r}} + 2\dot{\Theta}\frac{\partial R_z(-\Theta)}{\partial \Theta}\dot{\mathbf{r}} + \ddot{\Theta}\frac{\partial R_z(-\Theta)}{\partial \Theta}\mathbf{r} + \dot{\Theta}^2\frac{\partial R_z^2(-\Theta)}{\partial \Theta^2}\mathbf{r}$$

Substituting the above relations to Eq. (7) and multiplying both sides with $R_z(\Theta)$, we have

$$\ddot{\mathbf{r}} + 2\dot{\Theta}\begin{pmatrix}-\dot{y}\\ \dot{x}\end{pmatrix} - \dot{\Theta}^2\begin{pmatrix}x\\ y\end{pmatrix} + \ddot{\Theta}\begin{pmatrix}-y\\ x\end{pmatrix} = R_z(\Theta-\theta_A)\mathbf{F}_A + R_z(\Theta-\theta_B)\mathbf{F}_B = \tilde{\mathbf{F}}_A + \tilde{\mathbf{F}}_B \tag{8}$$

From Fig. 1, we know

$$\Theta - \theta_A = \theta, \qquad \Theta - \theta_B = \delta$$

Substitute the relation to the right side of Eq. (8), we have

$$\tilde{\mathbf{F}}_A = \begin{pmatrix} -\mu\left(x+(1-\mu)S\right)\left\{\frac{1}{\bar{r}_A^3} + \frac{3J_2^A\alpha_A^2}{2\bar{r}_A^5} + \frac{15J_{22}^A\alpha_A^2}{\bar{r}_A^7}\left[\cos 2\theta\left(\left(x+(1-\mu)S\right)^2 - y^2\right) - 2\sin 2\theta\left(x+(1-\mu)S\right)y\right]\right\} - \mu\frac{6J_{22}^A\alpha_A^2}{\bar{r}_A^5}\left[y\sin 2\theta - \left(x+(1-\mu)S\right)\cos 2\theta\right] \\ -\mu y\left\{\frac{1}{\bar{r}_A^3} + \frac{3J_2^A\alpha_A^2}{2\bar{r}_A^5} + \frac{15J_{22}^A\alpha_A^2}{\bar{r}_A^7}\left[\left(\cos 2\theta\left(\left(x+(1-\mu)S\right)^2 - y^2\right) - 2\sin 2\theta\left(x+(1-\mu)S\right)y\right)\right]\right\} - \mu\frac{6J_{22}^A\alpha_A^2}{\bar{r}_A^5}\left[\left(x+(1-\mu)S\right)\sin 2\theta + y\cos 2\theta\right] \end{pmatrix}$$

$$\tilde{\mathbf{F}}_B = \begin{pmatrix} -(1-\mu)(x-\mu S)\left\{\frac{1}{\bar{r}_B^3} + \frac{3J_2^B\alpha_B^2}{2\bar{r}_B^5} + \frac{15J_{22}^B\alpha_B^2}{\bar{r}_B^7}\left[\cos 2\delta\left((x-\mu S)^2 - y^2\right) - 2\sin 2\delta(x-\mu S)y\right]\right\} - (1-\mu)\frac{6J_{22}^B\alpha_B^2}{\bar{r}_B^5}\left[y\sin 2\delta - (x-\mu S)\cos 2\delta\right] \\ -(1-\mu)y\left\{\frac{1}{\bar{r}_B^3} + \frac{3J_2^B\alpha_B^2}{2\bar{r}_B^5} + \frac{15J_{22}^B\alpha_B^2}{\bar{r}_B^7}\left[\cos 2\delta\left((x-\mu S)^2 - y^2\right) - 2\sin 2\delta(x-\mu S)y\right] - (1-\mu)\frac{6J_{22}^B\alpha_B^2}{\bar{r}_B^5}\left[(x-\mu S)\sin 2\delta + y\cos 2\delta\right]\right\} \end{pmatrix}$$

where

$$\bar{r}_A = \sqrt{\left(x+(1-\mu)S\right)^2 + y^2 + z^2}, \quad \bar{r}_B = \sqrt{(x-\mu S)^2 + y^2 + z^2} \tag{9}$$

Truncated at the 2nd order, from Eq. (6), we have

$$\dot{\Theta} = 1 + 2\omega_\delta\beta\cos 2\delta, \quad \dot{\Theta}^2 = 1 + 4\omega_\delta\beta\cos 2\delta, \quad \ddot{\Theta} = -4\omega_\delta^2\beta\sin 2\delta$$

Substituting the above equation into Eq. (8), we have

$$\ddot{\mathbf{r}} + 2\begin{pmatrix}-\dot{y}\\ \dot{x}\end{pmatrix} - \begin{pmatrix}x\\ y\end{pmatrix} = -4\omega_\delta\beta\cos 2\delta\begin{pmatrix}-\dot{y}\\ \dot{x}\end{pmatrix} + 4\omega_\delta\beta\cos 2\delta\begin{pmatrix}x\\ y\end{pmatrix} + 4\omega_\delta^2\beta\sin 2\delta\begin{pmatrix}-y\\ x\end{pmatrix} + \tilde{\mathbf{F}}_A + \tilde{\mathbf{F}}_B \tag{10}$$

Further expanding Eq. (9) around the equilibrium state Eq. (4), truncated at the 2^{nd} order, we have

$$\tilde{\mathbf{F}}_A = \tilde{\mathbf{F}}_A^0 + \tilde{\mathbf{F}}_A^\varepsilon, \quad \tilde{\mathbf{F}}_B = \tilde{\mathbf{F}}_B^0 + \tilde{\mathbf{F}}_B^\varepsilon \tag{11}$$

Where

$$\mathbf{F}_A^0 = \begin{pmatrix} -\dfrac{\mu(x+1-\mu)}{r_A^3} \\ -\dfrac{\mu y}{r_A^3} \end{pmatrix}, \quad \mathbf{F}_B^0 = \begin{pmatrix} -\dfrac{(1-\mu)(x-\mu)}{r_B^3} \\ -\dfrac{(1-\mu)y}{r_B^3} \end{pmatrix}$$

$$\tilde{\mathbf{F}}_A^\varepsilon = \begin{pmatrix} -\mu(1-\mu)\left[\dfrac{1}{r_A^3} - \dfrac{3(x+1-\mu)^2}{r_A^5}\right]\Delta S - \mu(x+1-\mu)\left[\dfrac{3J_2^A\alpha_A^2}{2r_A^5} + \dfrac{15J_{22}^A\alpha_A^2}{r_A^7}\left[(x+1-\mu)^2 - y^2\right]\right] + \mu\dfrac{6J_{22}^A\alpha_A^2}{r_A^5}(x+1-\mu) \\ \mu(1-\mu)\left[\dfrac{3(x+1-\mu)y}{\bar{r}_A^5}\right]\Delta S - \mu y\left[\dfrac{3J_2^A\alpha_A^2}{2r_A^5} + \dfrac{15J_{22}^A\alpha_A^2}{r_A^7}\left[(x+1-\mu)^2 - y^2\right]\right] - \mu\dfrac{6J_{22}^A\alpha_A^2}{r_A^5}y \end{pmatrix}$$

$$\tilde{\mathbf{F}}_B^\varepsilon = \begin{pmatrix} \mu(1-\mu)\left[\dfrac{1}{r_B^3} - \dfrac{3(x-\mu)^2}{r_B^5}\right]\Delta S - (1-\mu)(x-\mu)\left[\dfrac{3J_2^B\alpha_B^2}{2r_B^5} + \dfrac{15J_{22}^B\alpha_B^2}{r_B^7}\left[\cos 2\delta_0\left((x-\mu)^2 - y^2\right) - 2\sin 2\delta_0 (x-\mu)y\right]\right] - (1-\mu)\dfrac{6J_{22}^B\alpha_B^2}{r_B^5}\left[y\sin 2\delta_0 - (x-\mu)\cos 2\delta_0\right] \\ -\mu(1-\mu)\left[\dfrac{3(x-\mu)y}{r_B^5}\right]\Delta S - (1-\mu)y\left[\dfrac{3J_2^B\alpha_B^2}{2r_B^5} + \dfrac{15J_{22}^B\alpha_B^2}{r_B^7}\left[\cos 2\delta_0\left((x-\mu)^2 - y^2\right) - 2\sin 2\delta_0 (x-\mu)y\right]\right] - (1-\mu)\dfrac{6J_{22}^B\alpha_B^2}{r_B^5}\left[(x-\mu)\sin 2\delta_0 + y\cos 2\delta_0\right] \end{pmatrix}$$

where

$$r_A = \sqrt{(x+1-\mu)^2 + y^2 + z^2}, \quad r_B = \sqrt{(x-\mu)^2 + y^2 + z^2}$$

Substitute Eq. (11) into Eq. (8), we have

$$\begin{aligned} &\ddot{\mathbf{r}} + 2\begin{pmatrix} -\dot{y} \\ \dot{x} \end{pmatrix} - \begin{pmatrix} x \\ y \end{pmatrix} = \tilde{\mathbf{F}}_A^0 + \tilde{\mathbf{F}}_B^0 \\ &-4\omega_\delta \beta \cos 2\delta_0 \begin{pmatrix} -\dot{y} \\ \dot{x} \end{pmatrix} + 4\omega_\delta \beta \cos 2\delta_0 \begin{pmatrix} x \\ y \end{pmatrix} + 4\omega_\delta^2 \beta \sin 2\delta_0 \begin{pmatrix} -y \\ x \end{pmatrix} + \tilde{\mathbf{F}}_A^\varepsilon + \tilde{\mathbf{F}}_B^\varepsilon \end{aligned} \tag{12}$$

Eq. (12) is the final form of the EOM for a massless small body in the binary system truncated at the 2^{nd} order of non-spherical terms. Obviously the first line of Eq. (12) is the EOM for the CRTBP, and the second line is the perturbing term.

4. Libration Points

Denote the positions of five libration points (three collinear ones and two triangular ones) of the CRTBP as (x_0, y_0). Expanding Eq. (12) around these libration points and only retaining the 2^{nd} order non-spherical terms, we have

$$\begin{cases} \ddot{\xi} - 2\dot{\eta} - \Omega_{xx}\xi - \Omega_{xy}\eta = D_0 + D_1\cos 2\delta_0 + D_2 \sin 2\delta_0 \\ \ddot{\eta} + 2\dot{\xi} - \Omega_{xy}\xi - \Omega_{yy}\eta = E_0 + E_1\cos 2\delta_0 + E_2 \sin 2\delta_0 \end{cases} \tag{13}$$

where

$$\Omega_{xx} = 1 + \mu\left[\frac{3\left(x_0 + 1 - \mu\right)^2}{\left(r_A^0\right)^5} - \frac{1}{\left(r_A^0\right)^3}\right] + \left(1-\mu\right)\left[\frac{3\left(x_0 - \mu\right)^2}{\left(r_B^0\right)^5} - \frac{1}{\left(r_B^0\right)^3}\right]$$
$$\Omega_{xy} = \mu\frac{3\left(x_0 + 1 - \mu\right)y_0}{\left(r_A^0\right)^5} + \left(1-\mu\right)\frac{3\left(x_0 - \mu\right)y_0}{\left(r_B^0\right)^5}$$
$$\Omega_{yy} = 1 + \mu\left[\frac{3y_0^2}{\left(r_A^0\right)^5} - \frac{1}{\left(r_A^0\right)^3}\right] + \left(1-\mu\right)\left[\frac{3y_0^2}{\left(r_B^0\right)^5} - \frac{1}{\left(r_B^0\right)^3}\right]$$

The coefficients $D_0, D_1, D_2, E_0, E_1, E_2$ are given in appendix A. Due to the existence of the terms on the right side of Eq. (13), the geometrical libration points (x_0, y_0) are no longer equilibrium points. However, a special solution with the same basic frequency as $2\omega_\delta$ exists. We call this solution as dynamical substitute. Denoting it as $\left(\bar{\xi}, \bar{\eta}\right)$, we have

$$\begin{cases} \bar{\xi} = \alpha_0 + \alpha_1 \cos 2\delta_0 + \alpha_2 \sin 2\delta_0 \\ \bar{\eta} = \beta_0 + \beta_1 \cos 2\delta_0 + \beta_2 \sin 2\delta_0 \end{cases} \tag{14}$$

where

$$\alpha_0 = \frac{\Omega_{xy}E_0 - \Omega_{yy}D_0}{\Omega_{xx}\Omega_{yy} - \Omega_{xy}^2}, \quad \beta_0 = \frac{\Omega_{xy}D_0 - \Omega_{xx}E_0}{\Omega_{xx}\Omega_{yy} - \Omega_{xy}^2}$$
$$\alpha_1 = \frac{\Omega_{xy}E_1 + 4\omega_\delta E_2 - \left(4\omega_{\delta_0}^2 + \Omega_{yy}\right)D_1}{\left(4\omega_{\delta_0}^2 + \Omega_{xx}\right)\left(4\omega_{\delta_0}^2 + \Omega_{yy}\right) - \left(16\omega_{\delta_0}^2 + \Omega_{xy}^2\right)}, \quad \alpha_2 = \frac{\Omega_{xy}E_2 - 4\omega_{\delta_0} E_1 - \left(4\omega_{\delta_0}^2 + \Omega_{yy}\right)D_2}{\left(4\omega_{\delta_0}^2 + \Omega_{xx}\right)\left(4\omega_{\delta_0}^2 + \Omega_{yy}\right) - \left(16\omega_{\delta_0}^2 + \Omega_{xy}^2\right)}$$
$$\beta_1 = \frac{\Omega_{xy}D_1 - 4\omega_{\delta_0} D_2 - \left(4\omega_{\delta_0}^2 + \Omega_{xx}\right)E_1}{\left(4\omega_{\delta_0}^2 + \Omega_{xx}\right)\left(4\omega_{\delta_0}^2 + \Omega_{yy}\right) - \left(16\omega_{\delta_0}^2 + \Omega_{xy}^2\right)}, \quad \beta_2 = \frac{\Omega_{xy}D_2 + 4\omega_{\delta_0} D_1 - \left(4\omega_{\delta_0}^2 + \Omega_{xx}\right)E_2}{\left(4\omega_{\delta_0}^2 + \Omega_{xx}\right)\left(4\omega_{\delta_0}^2 + \Omega_{yy}\right) - \left(16\omega_{\delta_0}^2 + \Omega_{xy}^2\right)}$$

For collinear libration points, we have $y_0 = 0$ and $\Omega_{xy} = 0$. Eigenvalues of Eq. (13) satisfy

$$\lambda^4 + \left(4 - \Omega_{xx} - \Omega_{yy}\right)\lambda^2 + \Omega_{xx}\Omega_{yy} = 0 \tag{15}$$

Studies show that motions around the collinear libration points are unstable [36], so generally the dynamical substitute given by Eq. (14) is also believed to be unstable.

For the triangular libration points, the eigenvalues of Eq. (13) satisfy

$$\lambda^4 + \left(4 - \Omega_{xx} - \Omega_{yy}\right)\lambda^2 + \Omega_{xx}\Omega_{yy} - \Omega_{xy}^2 = 0 \tag{16}$$

A well-known fact is that motions around the triangular libration points are stable if the mass parameter μ is smaller than $\mu_c = 0.03852\cdots$ [36]. In this case, the eigenvalues of Eq. (16) are of the following form

$$\lambda_{1,2} = \pm i\omega_l < \sqrt{2}/2, \qquad \lambda_{34} = \pm i\omega_s > \sqrt{2}/2$$

where the subscripts l and s indicate the long period and the short period component[36].

For triangular libration points, generally the dynamical substitute given by Eq. (14) is believed to be stable if $\mu < \mu_c$. However, exceptions appear when the external perturbation frequency $2\omega_\delta$ is close to one of the eigenvalues. In this case, nonlinear terms unconsidered in this study may cause the instability of the motions due to this commensurability. See following simulations for more details.

One remark is made here. Eq. (6) is only a special solution to Eq. (3), truncated at the 2nd order of non-spherical terms. This solution corresponds to the exact synchronous state of the binary system. If the exact synchronous state is stable, it allows free oscillations around this state, which will introduce more basic frequencies to the motion of the binary system (see appendix B). As a result, the dynamical substitute in Eq. (14) will change from a periodic orbit (with the basic frequency $2\omega_\delta$) to a quasi-periodic orbit. Besides, one extra frequency will appear if we consider an elliptic orbit instead of a near-circular orbit of the two asteroids. The problem becomes even more complicate if the rotational planes of the two asteroids don't coincide with the orbit plane. Nevertheless, in our study, we only consider the simplest exact synchronous state.

5. Numerical Simulations

In this section, we do some numerical simulations. The parameters for the binary system are

$$a_A = 200m,\ b_A = 180m,\ c_A = 160m,\ \rho_A = 2\times10^3 kg\cdot m^{-3}$$
$$a_B = 1000m,\ b_B = 950m,\ c_B = 900m,\ \rho_B = 2\times10^3 kg\cdot m^{-3}$$

where ρ_* is the density of the asteroid A or B. The unit distance between the two asteroids is

$$a = 5000m$$

The mean rotation rate of the asteroid ω_B is different in different simulations. We simultaneously integrate the EOM of the binary system Eq. (3) and the EOM of the massless body Eq. (8).The integrator used in our work is the RKF7(8) order integrator, with a truncation order of 10^{-15}.

5.1 Collinear Libration Points

Fig. 3 and Fig. 4 show the numerical results (solid line) and the analytical results (dashed line). The initial condition of the numerical orbit is same as the analytical orbit. The rotational frequency ω_B is 3 in Fig. 3 and is 2 in Fig. 4. In both cases, the binary system is regular (i.e., no chaotic motion appears in the orbital motion or the rotations). The arrows in the figures indicate the motion direction. When the rotation frequency ω_B changes, the size and the motion direction of the dynamical substitute also change. Obviously, due to the inherent strong instability of the motions around the collinear libration points, all the numerical orbits diverge quickly.

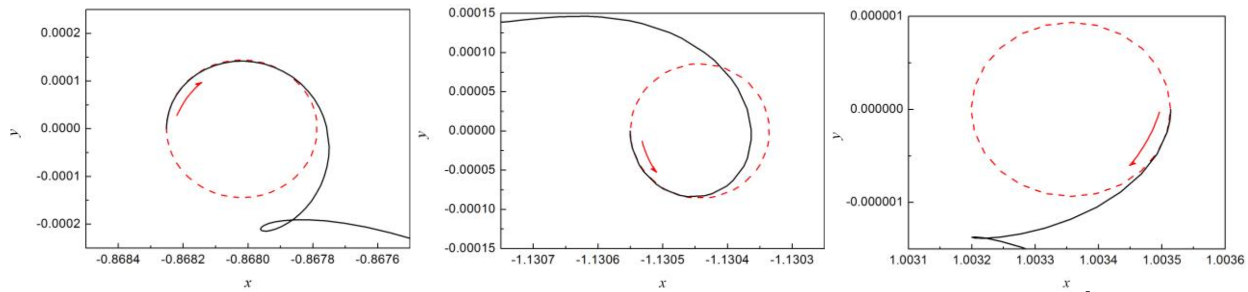

Figure 3. Analytical expression of the dynamical substitute truncated at the 2nd order (red dashed line) and the numerical integrated orbit with the same initial condition (black solid line). ω_B=3

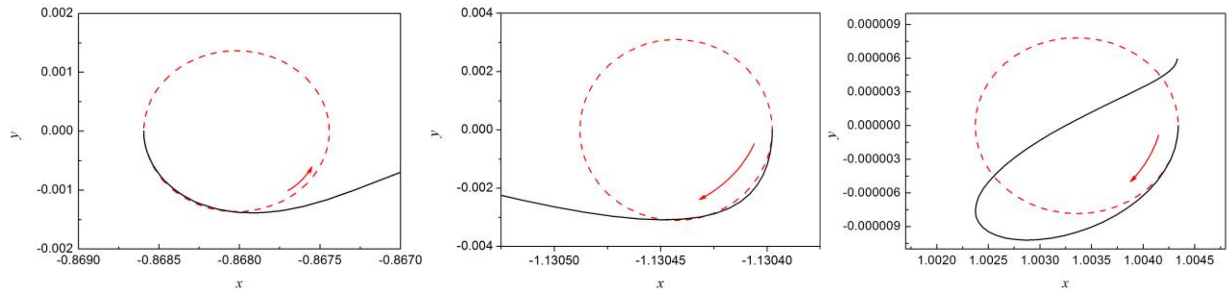

Figure 4. Same as Fig. 3, but for ω_B=2

For collinear libration points, the eigenvalues of Eq. (15) are usually of the following form

$$\lambda_{1,2} = \pm d_1, \qquad \lambda_{1,2} = \pm i d_2$$

Obviously, if the perturbing frequency $|2\omega_\delta|$ equals d_2, then the dynamical substitute given by Eq. (14) has an infinite size. Of course, infinity is impossible due to nonlinear effects, but the size of the dynamical substitute should be larger when $|2\omega_\delta|$ is closer to d_2. Taking the L_1 point as an example ($d_2 = 2.28358$). Fig. 5 shows the analytical results given by Eq. (14). From inside to outside, $\omega_B = 2.12,\ 2.13,\ 2.14$, generating a $|2\omega_\delta|$ value closer and closer to d_2. The size of the dynamical substitute is larger when $|2\omega_\delta|$ is closer to d_2.

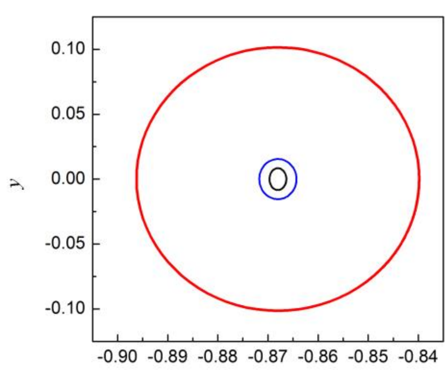

Figure 5. Dynamical substitutes around the point L_1. From inside out, ω_B=2.12, 2.13, 2.14

One remark is that the stability of the motion usually doesn't change. It's always unstable. One problem is that with the increase of the size of the dynamical substitute and considering the finite size of the asteroids, the dynamical substitute may hit the surface of the asteroid. This leads to their non-existence in practice.

5.2 Triangular Libration Points

Taking the trailing L_5 point as an example, Fig. 6 shows the dynamical substitute for $\omega_B = 0.8$ (left), $\omega_B = 1.6$ (middle) and $\omega_B = 2.4$. The red dashed line is the analytic orbit and the black solid line is the numerical integrated orbit with the same initial condition as the analytic orbit.

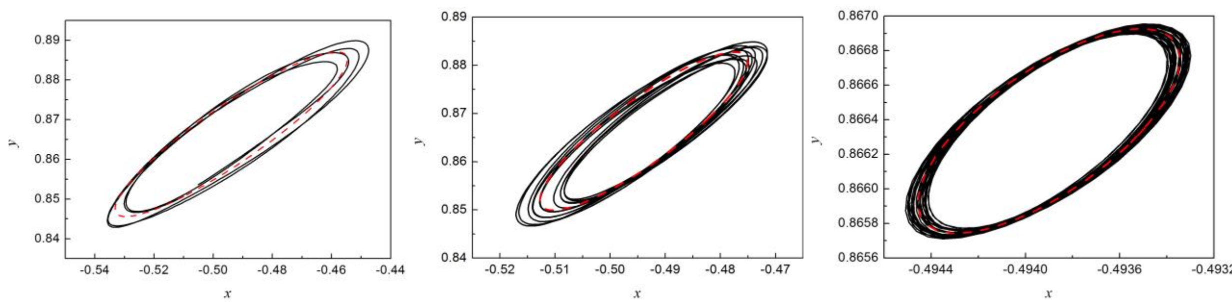

Figure 6. Analytical expression of the dynamical substitute truncated at the 2nd order (red dashed line) and the numerical integrated orbit with the same initial condition (black solid line). From left to right, $\omega_B = 0.8,\ 1.6,\ 2.4$

Generally, motions around the triangular libration points should still be stable. However, when the external perturbation frequency $|2\omega_\delta|$ is close to one of the basic frequencies ω_l or ω_s, motions around the triangular libration points will become unstable due to the resonances. For the binary system in this simulation, we have $\omega_l = 0.21699$ and $\omega_s = 0.97617$. Fig. 7 shows the results when $\omega_B = 0.87500$ (which generates a $|2\omega_\delta|$ value close to ω_l) and $\omega_B = 0.51100$ (which generates a $|2\omega_\delta|$ value close to ω_s). The initial conditions are given by Eq. (14). Obviously, motions around these points become unstable. The orbital motion and rotations of the two asteroids are regular, so the instability should be caused by the resonance between the external perturbation frequency and the intrinsic basic frequencies of the motion around the triangular libration points.

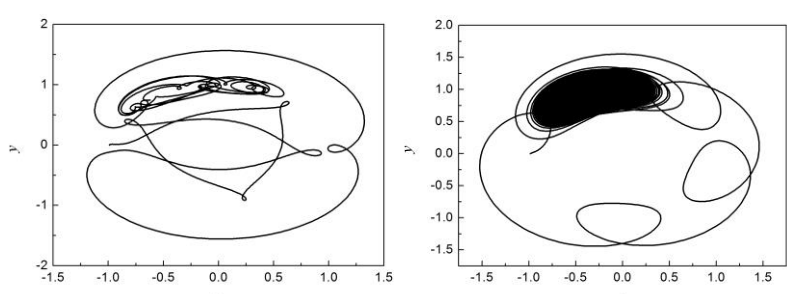

Figure 7. Instability of the motions caused by the resonance between the external perturbation frequency $|2\omega_\delta|$ and the intrinsic frequency ω_l (left) and ω_s (right).

6. Special Cases

Several special cases of the binary system will be discussed in this section:

(1) The most special case is that the two asteroids are spheroids. In this case, the problem is actually reduced to the conventional two-body problem. The rotations of the two asteroids are decoupled from their orbital motion. The positions and the stability of the libration points are same as the CRTBP model of this binary system.

(2) When the two ellipsoids are trapped in a double-synchronous state, i.e. $\delta_0 = 0,\ \omega_\delta = 0$, Eq. (5) becomes

$$\begin{cases} \Delta\ddot{S} = 3\Delta S + 2\left(\Delta\dot{\theta} + \Delta\dot{\theta}_A\right) - 3\left(A_1 + A_2 + A_3\right) \\ \Delta\ddot{\theta} = -2\Delta\dot{S} \\ \Delta\ddot{\phi} = 0 \\ \Delta\ddot{\theta}_A = 0 \end{cases} \tag{17}$$

The special solution of Eq. (17) is

$$\Delta S = A_1 + A_2 + A_3$$

which is a constant. This means the distance between the two asteroids is a constant. As a result, the dynamical equilibrium points still exist in this binary system. However, if components of free motions are considered (see the remark in section 4 and the appendix), the equilibrium points disappear. Instead, the dynamical substitute exists.

(3) If the asteroid A is a spheroid, the discussions above still apply. We only have to set $A_2 = 0$ in the above solutions.

(4) If the asteroid B is a spheroid, the rotation of the asteroid B is decoupled from the orbital motion. In this case, the distance between the two asteroids is constant. We can simply set $A_3 = 0$ in Eq. (17). This case is same as the double-synchronous case.

7. Conclusions

Dynamics around the libration points in the binary asteroid system are studied. In our study, the binary system is restricted to be captured in the synchronous state. Truncated at the 2^{nd} order of the mutual potential, the orbital motion and rotations of the binary system are given first. Then EOM of a massless small body in this system is given. Usually dynamical equilibrium points of the small body no longer exist. Assuming that the motion of the binary system is regular, special quasi-periodic orbits called dynamical substitutes exist around the geometrical libration points of the CRTBP model of the binary system. For the collinear libration points, motions around them are generally unstable. For the triangular libration points, even the mass parameter μ is smaller than 0.03852…, motions around them may be unstable due to resonances between the external perturbation frequency and the intrinsic frequencies.

One remark is made here. The method in this study can be easily generated to the general case of the binary system, as long as the dynamics (rotations and orbital motion) of the binary system are regular. In this case, the angle θ_0 in Eq. (4) is not a constant any more, and $\dot{\theta}_0 = \omega_A - 1$ where ω_A is the mean rotational frequency of the asteroid A. The dynamical substitute expressed by Eq. (14) changes from a periodic orbit of the basic frequency $2\omega_\delta$ to a quasi-periodic orbit of two basic frequencies $2\omega_\delta$ and $2\omega_\theta = 2(\omega_A - 1)$.

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Appendix A

$$r_A^0=\sqrt{(x_0+1-\mu)^2+y_0^2},\qquad r_B^0=\sqrt{(x_0-\mu)^2+y_0^2}$$

$$D_0=\mu(1-\mu)\left[\frac{3(x_0+1-\mu)^2}{(r_A^0)^5}-\frac{1}{(r_A^0)^3}+\frac{1}{(r_B^0)^3}-\frac{3(x_0-\mu)^2}{(r_B^0)^5}\right]\Delta S_0$$
$$-\mu(x_0+1-\mu)\left[\frac{3J_2^A\alpha_A^2}{2(r_A^0)^5}+\frac{15J_{22}^A\alpha_A^2}{(r_A^0)^7}\left[(x_0+1-\mu)^2-y_0^2\right]\right]$$
$$+\mu(x_0+1-\mu)\frac{6J_{22}^A\alpha_A^2}{(r_A^0)^5}-(1-\mu)(x_0-\mu)\frac{3J_2^B\alpha_B^2}{2(r_B^0)^5}$$

$$D_1=4\omega_{\delta_0}(\dot{y}_0+x_0)\beta+\mu(1-\mu)\left[\frac{3(x_0+1-\mu)^2}{(r_A^0)^5}-\frac{1}{(r_A^0)^3}+\frac{1}{(r_B^0)^3}-\frac{3(x_0-\mu)^2}{(r_B^0)^5}\right]\alpha$$
$$-\frac{15J_{22}^B\alpha_B^2}{(r_B^0)^7}(1-\mu)(x_0-\mu)\left[(x_0-\mu)^2-y_0^2\right]+\frac{6J_{22}^B\alpha_B^2}{(r_B^0)^5}(1-\mu)(x_0-\mu)$$

$$D_2=-4\omega_{\delta_0}^2y_0\beta+\frac{30J_{22}^B\alpha_B^2}{(r_B^0)^7}(1-\mu)(x_0-\mu)^2y_0-\frac{6J_{22}^B\alpha_B^2}{(r_B^0)^5}(1-\mu)y_0$$

$$E_0=\mu(1-\mu)\left[\frac{3(x_0+1-\mu)y_0}{(r_A^0)^5}-\frac{3(x_0-\mu)y_0}{(r_B^0)^5}\right]\Delta S_0$$
$$-\mu y_0\left[\frac{3J_2^A\alpha_A^2}{2(r_A^0)^5}+\frac{15J_{22}^A\alpha_A^2}{(r_A^0)^7}\left[(x_0+1-\mu)^2-y_0^2\right]\right]$$
$$-\mu y_0\frac{6J_{22}^A\alpha_A^2}{(r_A^0)^5}-(1-\mu)y_0\frac{3J_2^B\alpha_B^2}{2(r_B^0)^5}$$

$$E_1=4\omega_{\delta_0}(y_0-\dot{x}_0)\beta+\mu(1-\mu)\left[\frac{3(x_0+1-\mu)y_0}{(r_A^0)^5}-\frac{3(x_0-\mu)y_0}{(r_B^0)^5}\right]\alpha$$
$$-\frac{15J_{22}^B\alpha_B^2}{(r_B^0)^7}(1-\mu)y_0\left[(x_0-\mu)^2-y_0^2\right]-\frac{6J_{22}^B\alpha_B^2}{(r_B^0)^5}(1-\mu)y_0$$

$$E_2=4\omega_{\delta_0}^2x_0\beta+\frac{30J_{22}^B\alpha_B^2}{(r_B^0)^7}(1-\mu)(x_0-\mu)y_0^2-\frac{6J_{22}^B\alpha_B^2}{(r_B^0)^5}(1-\mu)(x_0-\mu)$$

Appendix B

From Eq. (3), we know that θ_A can be arbitrarily chosen. It means that Eq. (3) is a 7-dimensional dynamical system with variables $S,\theta,\phi,\dot{S},\dot{\theta},\dot{\phi},\dot{\theta}_A$. Re-denote Eq. (6) as $\Delta\bar{S},\Delta\bar{\theta},\Delta\bar{\phi},\Delta\bar{\theta}_A$, and denote the exact synchronous solution as (truncated at the 2nd order of the non-spherical terms)

$$\begin{cases} \bar{S} = 1 + \Delta\bar{S} \\ \bar{\theta} = 0 + \Delta\bar{\theta} \\ \bar{\phi} = \phi_0 + \Delta\bar{\phi} \\ \bar{\theta}_A = t + \Delta\bar{\theta}_A \end{cases} \tag{B1}$$

Expand first three equations of Eq. (3) around this exact synchronous state, we have (truncated at the 2nd order of the non-spherical terms)

$$\begin{cases} \Delta\ddot{S} = \left[\left(\dot{\bar{\theta}} + \dot{\bar{\theta}}_A\right)^2 + \dfrac{2}{\bar{S}^3} + 12\left(A_1 + A_2 + A_3\cos 2\delta_0\right)\right]\Delta S + 2\bar{S}\left(\dot{\bar{\theta}} + \dot{\bar{\theta}}_A\right)\Delta\dot{\theta} \\ \quad + 2\bar{S}\left(\dot{\bar{\theta}} + \dot{\bar{\theta}}_A\right)\Delta\dot{\theta}_A + 6A_3\sin 2\delta_0\Delta\theta - 6A_3\sin 2\delta_0\Delta\phi \\ \Delta\ddot{\theta} = \left[\dfrac{2\dot{\bar{S}}}{\bar{S}^2}\left(\dot{\bar{\theta}} + \dot{\bar{\theta}}_A\right) + 10A_3\sin 2\delta_0\right]\Delta S - \dfrac{2}{\bar{S}}\left(\dot{\bar{\theta}} + \dot{\bar{\theta}}_A\right)\Delta\dot{S} - \dfrac{2\dot{\bar{S}}}{\bar{S}}\Delta\dot{\theta} - \dfrac{2\dot{\bar{S}}}{\bar{S}}\Delta\dot{\theta}_A \\ \quad - 4\left(A_2 + A_3\cos 2\delta_0 + \dfrac{mA_2}{I_z^A}\right)\Delta\theta + 4A_3\cos 2\delta_0 \cdot \Delta\phi \\ \Delta\ddot{\phi} = -\dfrac{6mA_3}{I_z^B}\sin 2\delta_0 \cdot \Delta S + \left(\dfrac{4mA_3}{I_z^B}\cos 2\delta_0 - \dfrac{4mA_2}{I_z^A}\right)\Delta\theta - \dfrac{4mA_3}{I_z^B}\cos 2\delta_0 \cdot \Delta\phi \end{cases} \tag{B2}$$

If we require the momentum of the binary system is conserved, we have

$$\Delta\dot{\theta}_A = -\frac{2m\bar{S}}{I_z}\left(\dot{\bar{\theta}} + \dot{\bar{\theta}}_A\right)\Delta S - \frac{m\bar{S}^2}{I_z}\Delta\dot{\theta} - \frac{I_z^B}{I_z}\Delta\dot{\phi}, \qquad I_z = I_z^A + I_z^B + m\bar{S}^2$$

Substitute this relation to Eq. (B2), we have

$$\begin{cases} \Delta\ddot{S} = \left[\left(\dot{\bar{\theta}} + \dot{\bar{\theta}}_A\right)^2 + \dfrac{2}{\bar{S}^3} + 12\left(A_1 + A_2 + A_3\cos 2\delta_0\right) - \dfrac{4m\bar{S}^2}{I_z}\left(\dot{\bar{\theta}} + \dot{\bar{\theta}}_A\right)^2\right]\Delta S + 2\bar{S}\left(\dot{\bar{\theta}} + \dot{\bar{\theta}}_A\right)\left(1 - \dfrac{m\bar{S}^2}{I_z}\right)\Delta\dot{\theta} \\ \quad - \dfrac{2I_z^B}{I_z}\bar{S}\left(\dot{\bar{\theta}} + \dot{\bar{\theta}}_A\right)\Delta\dot{\phi} + 6A_3\sin 2\delta_0\Delta\theta - 6A_3\sin 2\delta_0\Delta\phi \\ \Delta\ddot{\theta} = \left[\dfrac{2\dot{\bar{S}}}{\bar{S}^2}\left(\dot{\bar{\theta}} + \dot{\bar{\theta}}_A\right) + 10A_3\sin 2\delta_0 + \dfrac{4m\dot{\bar{S}}}{I_z}\left(\dot{\bar{\theta}} + \dot{\bar{\theta}}_A\right)\right]\Delta S - \dfrac{2}{\bar{S}}\left(\dot{\bar{\theta}} + \dot{\bar{\theta}}_A\right)\Delta\dot{S} - \dfrac{2\dot{\bar{S}}}{\bar{S}}\left(1 - \dfrac{m\bar{S}^2}{I_z}\right)\Delta\dot{\theta} \\ \quad + \dfrac{2I_z^B\dot{\bar{S}}}{I_z\bar{S}}\Delta\dot{\phi} - 4\left(A_2 + A_3\cos 2\delta_0 + \dfrac{mA_2}{I_z^A}\right)\Delta\theta + 4A_3\cos 2\delta_0 \cdot \Delta\phi \\ \Delta\ddot{\phi} = -\dfrac{6mA_3}{I_z^B}\sin 2\delta_0 \cdot \Delta S + \left(\dfrac{4mA_3}{I_z^B}\cos 2\delta_0 - \dfrac{4mA_2}{I_z^A}\right)\Delta\theta - \dfrac{4mA_3}{I_z^B}\cos 2\delta_0 \cdot \Delta\phi \end{cases}$$

Introduce

$$\Delta X = \left(\Delta S, \Delta\theta, \Delta\phi, \Delta\dot{S}, \Delta\dot{\theta}, \Delta\dot{\phi}\right)$$

We can rewrite Eq. (B2) as

$$\Delta X = A \cdot \Delta X \tag{B3}$$

where

$$A = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ a_{41} & a_{42} & a_{43} & 0 & a_{45} & a_{46} \\ a_{51} & a_{52} & a_{53} & a_{54} & a_{55} & a_{56} \\ a_{61} & a_{62} & a_{63} & 0 & 0 & 0 \end{bmatrix}$$

with

$$a_{41} = \left(\dot{\bar{\theta}} + \dot{\bar{\theta}}_A\right)^2 + \frac{2}{\bar{S}^3} + 12\left(A_1 + A_2 + A_3 \cos 2\delta_0\right) - \frac{4m\bar{S}^2}{I_z}\left(\dot{\bar{\theta}} + \dot{\bar{\theta}}_A\right)^2$$

$$a_{42} = 6A_3 \sin 2\delta_0$$

$$a_{43} = -6A_3 \sin 2\delta_0$$

$$a_{45} = 2\bar{S}\left(\dot{\bar{\theta}} + \dot{\bar{\theta}}_A\right)\left(1 - \frac{m\bar{S}^2}{I_z}\right)$$

$$a_{46} = -\frac{2I_z^B}{I_z}\bar{S}\left(\dot{\bar{\theta}} + \dot{\bar{\theta}}_A\right)$$

$$a_{51} = \frac{2\dot{\bar{S}}}{\bar{S}^2}\left(\dot{\bar{\theta}} + \dot{\bar{\theta}}_A\right) + 10A_3 \sin 2\delta_0 + \frac{4m\dot{\bar{S}}}{I_z}\left(\dot{\bar{\theta}} + \dot{\bar{\theta}}_A\right)$$

$$a_{52} = -4\left(A_2 + A_3 \cos 2\delta_0 + \frac{mA_2}{I_z^A}\right)$$

$$a_{53} = 4A_3 \cos 2\delta_0$$

$$a_{54} = -\frac{2}{\bar{S}}\left(\dot{\bar{\theta}} + \dot{\bar{\theta}}_A\right)$$

$$a_{55} = -\frac{2\dot{\bar{S}}}{\bar{S}}\left(1 - \frac{m\bar{S}^2}{I_z}\right)$$

$$a_{56} = \frac{2I_z^B\dot{\bar{S}}}{I_z\bar{S}}$$

$$a_{61} = -6mA_3 / I_z^B$$

$$a_{62} = \frac{4mA_3}{I_z^B}\cos 2\delta_0 - \frac{4mA_2}{I_z^A}$$

$$a_{63} = -\frac{4mA_3}{I_z^B}\cos 2\delta_0$$

Obviously, the matrix $A(t)$ is a periodic matrix with the basic frequency $2\omega_\delta$ (i.e., a period of $T = \pi/\omega_\delta$). Using the Floquet theory, we can know the stability of the synchronous state. Denote the state transition matrix as $M(t)$. Using the parameters in section 4 and $\omega_B = 0.8$, eigenvalues of the Monodromy matrix $M(T)$ are of the following form

$$\lambda_{1,2} = 1.01307 \pm 0.08963i$$
$$\lambda_{3,4} = -0.79495 \pm 0.60654i$$
$$\lambda_{5,6} = -0.98333 \pm 0.00444i$$

This means that the synchronous state is stable. There are three extra frequencies besides the frequency $2\omega_\delta$ will appear in the solution to Eq. (3) if free motions are considered.

One remark is that the synchronous state is not always stable. Its stability depends on the mutual distance between the two asteroids, similar to the case of double synchronous state studied in literature [30,32].