THE FLIGHT MAINTENANCE IN THE VICINITY OF A LIBRATION CENTRE AND THE ONE-IMPULSE TRANSFER TRAJECTORY TO THE LIMITED ORBIT IN THIS REGION

P E Eliasberg, T A Timokhava & M N Boyarski

Space Research Institute Moscow USSR

ABSTRACT

The problem of spacecraft (SC) motion control aimed at long-term keeping of the SC in the vicinity of one of collinear centers L_1 or L_2 in the restricted elliptic three-body problem is considered in the paper. The osculating parameters method is used for the purpose. The equitions of SC motion are obtained in these parameters. These equitions were employed to find the necessary integral condition for orbit to be limited. The method of search for limited orbit under this condition was constructed as well as method of calculation of correction impulses necessary for a long-term maintenance of the limited orbit. The method of search for a one-impulse transfer trajectory between Earth satellite orbit and about libration center is also described. The numeric calculations were made on the basis of these methods.

Keywords: Three-body Problem, Libration Center, Limited Orbit, Spacecraft Motion Control

1. INTRODUCTION

The solution of some space research problems demands long-term keeping of a spacecraft (SC) in the vicinity of one of libration centers L_1 or L_2 of three-body problem, which are situated near smaller attracting body. The selection of orbits for such SC a and its motion control were discussed in (Refs. 1-6). Different modifications of a small-parameter method were used in these works. As a result, the existence of limited orbits was shown. Moving on this orbits, the SC do not moves far away from a libration center. These orbits are unstable and demand maintenance corrections. The closer is the real orbit to a limited one, the smaller are the correction implulses needed for the orbit maintenance.

The accuracy of limited orbit determination by small parameter method is mainly defined by the value of ratio $\mathcal{E}$ of SC distance from the libration center to the distance between the libration center and the center of smaller attracting body (Ref.7). In the works mentioned above, satisfactory accuracy is provided when $\mathcal{E}$ is not greater than $\mathcal{E}_{max}=0.1$. With greater computational errors lead to the significant fuel overspare for the orbit maintenance.

From the other side, as it was shown in (Ref.6) transition from the transfer trajectory between the smaller attracting body and the limited orbit about libration center with $\mathcal{E}=0.1$ demands large injection impulse (in the Sun-Earth system $V \cong 300$ m/s). In this paper it is shown, that when $\mathcal{E}_{max}=0.5$ this impulse is not needed. So the necessety arises to solve the mentioned problems for larger $\mathcal{E}_{max}$.

The new method of solving above mentioned problems is suggested in the paper, which is based on osculating parameters method. It provides the possibility to obtain solutions in the wide range of $\mathcal{E}_{max}$ (in the Sun-Earth system up to $\mathcal{E}_{max}=0.8$). At the same time, high accuracy can be achieved, which is sufficient for application purposes and is limited only by our knowledge of forces acting on the SC and by the performance of computer available. The problem is solved for the case of SC moving in the attraction field of two spherical bodies, which in their turn move relative to each other on Kepler elliptic orbits (i.e., under assumptions of classic restricted elliptic three-bodies problem). The method used can be immediately extended to the case of motion taking into account perturbations from other celestial bodies.

Also analised are the ways of selection of the transfer orbit between circular Earth satellite orbit and orbit about libration center and it is shown that one-impulse trajectory of this transition exists.

2. THE OSCULATING PARAMETERS METHOD.

Let us consider motion of a SC in the field of attraction of two bodies S_1 and S_2 which have masses M_1 and M_2 respectively and let $M_1 > M_2$. In all practical cases (Sun-planet, planet - its satellite) $M_1 >> M_2$. We shall adopt a rectangular coordinate system Oxyz, centered at the center of masses O of bodies S_1 and S_2. Ox axis of the system is directed from the center of S_1 to the center of S_2 and Oy axis lies in the orbital plane of S_1 and S_2 at the right angle to Ox in the direction of motion of S_1 and S_2. We shall analise the SC motion in the neighborhood of one of the libration centers L_1 or L_2 situated on Ox near S_2. The distance OL between any of these centers and the center O is (Refs.3,8,9):

Proceedings of the Second International Symposium on Spacecraft Flight Dynamics, Darmstadt, FR Germany, 20–23 October 1986 (ESA SP-255, Dec. 1986)

$$OL = (1-\mu+\gamma)r$$

where $\mu = M_1/(M_1+M_2)$, is the distance between S_1 and S_2 and is the root of quintic equation :

$$1-\mu+\gamma - \frac{1-\mu}{(1+\gamma)^2} \pm \frac{\mu}{\gamma^2} = 0$$

in which sign '+' corresponds to L_1, and '-' - to L_2.

This equation has one real root for each libration center, and $\gamma < 0$ for L_1, $\gamma > 0$ for L_2.

In Appendix the values of μ and γ are given for the system Sun-Earth&Moon barycenter, which will be further called the Sun-Earth system.

We shall solve our problem using dimensionless variables ξ, η and ζ defined by the following transformation:

$$x = (1-\mu+\gamma+\xi)r, \quad y = \eta r, \quad z = \zeta r \tag{1}$$

which describes the SC motion in the right rectangular coordinate system $L\xi\eta\zeta$ centered at the libration center L, axes of which are parallel to these of Oxyz system and, generally speaking, variable length scale proportional to the distance r between centers of S_1 and S_2. The true anomaly ϑ of S_2 in its motion about S_1 is used as independent variable. In these variables the equations of SC motion can be written in form (Refs.3,7,8):

$$\begin{aligned}
&\xi'' - 2\eta' - (2B_0+1)\xi = a_\xi, \\
&\eta'' + 2\xi' + (B_0-1)\eta = a_\eta, \\
&\zeta'' + B_0\zeta = a_\zeta \\
&B_0 = \frac{1-\mu}{(1+\gamma)^3} + \frac{\mu}{|\gamma|^3}
\end{aligned} \tag{2}$$

where a_ξ, a_η and a_ζ are perturbation accelerations which include non-linear terms relative to values of ξ, η and ζ and eccentricity e of S_1 orbit about S_2.

Let us adopt well-known solution of (2) with $a_\xi = a_\eta = a_\zeta = 0$ (Refs.3,7-10) as an intermediate orbit.It can be writen as :

$$\begin{aligned}
&\xi = \frac{a}{k}\cos\varphi + \alpha + \beta, \\
&\eta = a\sin\varphi - \ell\alpha + \ell\beta, \\
&\zeta = d\sin\psi, \\
&a = a_0, \quad \varphi = \varphi_0 - \omega(\vartheta - \vartheta_0), \\
&\alpha = \alpha_0 e^{\lambda(\vartheta-\vartheta_0)}, \\
&\beta = \beta_0 e^{-\lambda(\vartheta-\vartheta_0)}, \\
&d = d_0, \quad \psi = \psi_0 + \Omega(\vartheta-\vartheta_0), \\
&\omega = \sqrt{\tfrac{1}{2}\left(2 - B_0 + \sqrt{9B_0^2 - 8B_0}\right)}, \\
&\lambda = \sqrt{\tfrac{1}{2}\left(B_0 - 2 + \sqrt{9B_0^2 - 8B_0}\right)}, \\
&k = \frac{\omega^2 + 2B_0 + 1}{2\omega}, \quad \ell = \frac{-\lambda^2 + 2B_0 + 1}{2\lambda}, \quad \Omega = \sqrt{B_0}
\end{aligned} \tag{3}$$

where $a, \varphi, \alpha, \beta, d, \psi$ are the orbit parameters and index '0' marks values of corresponding parameters in the initial moment $\vartheta = \vartheta_0$ The values of $\omega, \lambda, k, \ell, \Omega$ for the Sun-Earth system are given in Appendix.

So introduced intermediate orbit parameters have simple kinematic sence (Ref.7). Quantities a and φ are the semimajor axis and the phase of elliptic motion in $L\xi\eta$ plane, while d and ψ are the amplitude and the phase of harmonic oscillation along $L\zeta$ axis. Both motions are performed relative to center O', which moves along one of the hyperbolas centered at L and bounded by asymptotas M_1N_1 and M_2N_2 defined by equations $\eta = -\ell\xi$ and $\eta = \ell\xi$ - respectively. Parameters α and β are proportional to distances from asymptotas M_2N_2 and M_1N_1 respectively. Directions of all motions are shown by arrows on Fig.2. The big arrow shows the adopted direction of S_1 and S_2 about each other. As it changes, so do all other motion directions. The solution (3) shows, that with $\alpha_0 = 0$ and without perturbations α remains 0 for all ϑ. In this case O' center moves along straight line M_2N_2, asymptotically approaching L. The SC motion is than limited and asymptotically approaches the quasiperiodic motion about the libration center L with different angular velocities ω and Ω. But this motion is unstable: with the smallest perturbation the center O' passes to a hyperbolic trajectory and moves away from the libration center asymptotically approaching M_1N_1 line. With given values of phase coordinates ξ, η, ζ, ξ' and η', ζ' orbital parameters are defined by formulae derived from (3):

$$\begin{aligned}
&a = k\sqrt{A_1^2 + A_2^2}, \quad \sin\varphi = \frac{A_1}{\sqrt{A_1^2+A_2^2}}, \quad \cos\varphi = \frac{A_2}{\sqrt{A_1^2+A_2^2}} \\
&d = \sqrt{A_5^2 + A_6^2}, \quad \sin\psi = \frac{A_6}{\sqrt{A_5^2+A_6^2}}, \quad \cos\psi = \frac{A_5}{\sqrt{A_5^2+A_6^2}} \\
&\alpha = \frac{A_3 + A_4}{2}, \quad \beta = \frac{A_4 - A_3}{2}, \\
&A_1 = \frac{\ell\xi' + \lambda\eta}{D_1}, \quad A_2 = -\frac{\ell\lambda\xi + \eta'}{D_2}, \quad A_3 = \frac{k\xi' - \omega\eta}{D_1}, \\
&A_4 = \frac{\omega k\xi + \eta'}{D_2}, \quad A_5 = \frac{\zeta'}{\Omega}, \quad A_6 = \zeta, \\
&D_1 = \omega\ell + \lambda k, \quad D_2 = -\lambda\ell + \omega k.
\end{aligned} \tag{4}$$

The values of D_1 and D_2 for the Sun-Earth system are given in Appendix.

Let us now consider the orbit defined by the solution of complete system (2) taking into account perturbations a_ξ, a_η and a_ζ. We shall call this orbit a perturbed orbit. It is characterized by the law of change of phase vector :

$$\Phi = \{\xi, \eta, \zeta, \xi', \eta', \zeta'\}, \quad \Phi = \Phi(\vartheta). \tag{5}$$

With the aid of (4) and (5) one can find for each value of parameters $a, \varphi, \alpha, \beta, d$ and ψ, which we shall call osculating parameters of perturbed orbit in given moment (the moment of osculation). Using these parameters, it is possible to construct corresponding osculating intermediate orbit. The law of motion along this orbit $\Phi = \Phi_c(\vartheta)$ is defined by (3). Evidently, in the moment of osculation

$$\Phi_c(\vartheta) = \Phi(\vartheta) \tag{6}$$

The vector $\Phi(\vartheta)$ will later change according to the complete system (2) and vector $\Phi_0(\vartheta)$ according to the same system with $a_\xi = a_\eta = a_\zeta = 0$.

Let $p(\vartheta) = p[\Phi(\vartheta)]$ be one of the osculating parameters of perturbed orbit. It can be obtained from (2), (5) and (6) that

$$\frac{dp}{d\vartheta} = \frac{\partial p}{\partial \xi}\xi' + \frac{\partial p}{\partial \eta}\eta' + \frac{\partial p}{\partial \zeta}\zeta' + \frac{\partial p}{\partial \xi'}\xi'' + \frac{\partial p}{\partial \eta'}\eta'' + \frac{\partial p}{\partial \zeta'}\zeta'' = \left(\frac{dp}{d\vartheta}\right)_0 + \frac{\partial p}{\partial \xi'}a_\xi + \frac{\partial p}{\partial \eta'}a_\eta + \frac{\partial p}{\partial \zeta'}a_\zeta, \tag{7}$$

where $\left(\frac{dp}{d\vartheta}\right)_0$ is derivative of p with respect to ϑ when the motion is performed along osculating intermediate orbit. We obtain from (3) that

$$\left(\frac{da}{d\vartheta}\right)_0 = \left(\frac{dd}{d\vartheta}\right)_0 = 0, \quad \left(\frac{d\varphi}{d\vartheta}\right)_0 = -\omega, \quad \left(\frac{d\alpha}{d\vartheta}\right)_0 = \lambda\alpha, \tag{8}$$
$$\left(\frac{d\beta}{d\vartheta}\right)_0 = -\lambda\beta, \quad \left(\frac{d\psi}{d\vartheta}\right)_0 = \Omega.$$

We shall use (4) in order to obtain the values of $\frac{\partial p}{\partial \xi'}$, $\frac{\partial p}{\partial \eta'}$ and $\frac{\partial p}{\partial \zeta'}$. Substituting these values together with (8) into (7) we find:

$$\begin{aligned}
&a' = \frac{k\ell}{D_1}\sin\varphi \cdot a_\xi - \frac{k}{D_2}\cos\varphi \cdot a_\eta, \\
&\varphi' = -\omega + \frac{k}{a}\left(\frac{\ell}{D_1}\cos\varphi \cdot a_\xi + \frac{1}{D_2}\sin\varphi \cdot a_\eta\right), \\
&\alpha' - \lambda\alpha = \frac{1}{2}\left(\frac{k}{D_1}a_\xi + \frac{1}{D_2}a_\eta\right), \\
&\beta' + \lambda\beta = -\frac{1}{2}\left(\frac{k}{D_1}a_\xi - \frac{1}{D_2}a_\eta\right), \\
&d' = \frac{\cos\psi}{\Omega}a_\zeta, \quad \psi' - \Omega = -\frac{\sin\psi}{\Omega d}\cdot a_\zeta.
\end{aligned} \tag{9}$$

So we obtained the system of equations of motion in osculating parameters. It is completely equivalent to the initial system (2) for every a_ξ, a_η and a_ζ can be succesfully used for qualitative analisis of motion along perturbed orbit (Ref.7).

3. THE NECESSARY CONDITION OF THE LIMITEDNESS OF ORBIT.

We shall call a 'limited orbit' such sulution of system (2) or (3) on which a SC remains in the region D defined by

$$\max\left[\xi(\vartheta), \eta(\vartheta), \zeta(\vartheta)\right] \le M \tag{10}$$

for arbitrary $\vartheta > \vartheta_0$. M is a constant. It is assumed, that the selection of M provides for the absence of bodies of noticable masses in the region D. In this case the perturbation accelerations a_ξ, a_η and a_ζ in the region D are limited, i.e.

$$\max[|a_\xi|, |a_\eta|, |a_\zeta|] < N \tag{11}$$

Now we shall define the necessary condition of the limitedness of orbit (Ref.11). Let suggest, that such an orbit exists. The equations of motion along this orbit may be written as:

$$a = a(\vartheta), \varphi = \varphi(\vartheta), \alpha = \alpha(\vartheta), \beta = \beta(\vartheta), d = d(\vartheta), \psi = \psi(\vartheta).$$

With the aid of these relations we can write righthand sides of equations (9) as functions of the argument ϑ:

$$\begin{aligned}
&f_a(\vartheta) = \frac{k\ell}{D_1}\sin\varphi \cdot a_\xi - \frac{k}{D_2}\cos\varphi \cdot a_\eta, \\
&f_\alpha(\vartheta) = \frac{1}{2}\left(\frac{k}{D_1}a_\xi + \frac{1}{D_2}a_\eta\right), \\
&f_\beta(\vartheta) = -\frac{1}{2}\left(\frac{k}{D_1}a_\xi - \frac{1}{D_2}a_\eta\right), \\
&f_d(\vartheta) = \frac{1}{\Omega}\cos\psi \cdot a_\zeta.
\end{aligned} \tag{12}$$

Integrating (9) we obtain, that on orbit under consideration

$$\begin{aligned}
&a(\vartheta) = a_0 + \int_{\vartheta_0}^{\vartheta} f_a(\tau)\,d\tau, \\
&\alpha(\vartheta) = \left\{\alpha_0 + \int_{\vartheta_0}^{\vartheta} f_\alpha(\tau)e^{-\lambda(\tau-\vartheta_0)}d\tau\right\}e^{\lambda(\vartheta-\vartheta_0)}, \\
&d(\vartheta) = d_0 + \int_{\vartheta_0}^{\vartheta} f_d(\tau)\,d\tau, \\
&\beta(\vartheta) = \left\{\beta_0 + \int_{\vartheta_0}^{\vartheta} f_\beta(\tau)e^{\lambda(\tau-\vartheta_0)}d\tau\right\}e^{-\lambda(\vartheta-\vartheta_0)}.
\end{aligned} \tag{13}$$

It follows from (11) and (12) that on the limited orbit :

$$|f_a(\vartheta)| \le N_a, \quad |f_\alpha(\vartheta)| \le N_\alpha,$$
$$|f_\beta(\vartheta)| \le N_\alpha, \quad |f_d(\vartheta)| \le N_d, \tag{14}$$

where $N_a = Nk\left(\frac{\ell}{D_1} + \frac{1}{D_2}\right)$, $N_\alpha = \frac{N}{2}\left(\frac{k}{D_1} + \frac{1}{D_2}\right)$, $N_d = \frac{N}{\Omega}$.

Substituting these unequalities into (13) and taking into account that $a > 0$, $d > 0$ we obtain that for $\vartheta > \vartheta_0$

$$\begin{aligned}
&0 \le a(\vartheta) \le a_0 + N_a(\vartheta - \vartheta_0), \\
&|\beta(\vartheta)| \le |\beta_0| + \frac{N_\alpha}{\lambda}, \\
&0 \le d(\vartheta) \le d_0 + N_d(\vartheta - \vartheta_0).
\end{aligned} \tag{15}$$

Let us denote

$$\delta\alpha_0 = \left|\alpha_0 + \int_{\vartheta_0}^{\infty} f_\alpha(\tau)e^{-\lambda(\tau-\vartheta_0)}d\tau\right|. \tag{16}$$

It immediately follows from (14) that infinit integral in the righthand side of (16) converges on the limited orbit. With the aid of (13), (14) and (16) we find, that:

$$\delta\alpha_0 e^{\lambda(\vartheta-\vartheta_0)} + \frac{N_\alpha}{\lambda} \ge |\alpha(\vartheta)| \ge \delta\alpha_0 e^{\lambda(\vartheta-\vartheta_0)} - \frac{N_\alpha}{\lambda} \tag{17}$$

Substituting (15) and (17) into (3) we obtain:

$$\begin{aligned}
&\delta\alpha_0 e^{\lambda(\vartheta-\vartheta_0)} + R_1 \ge |\xi| \ge \delta\alpha_0 e^{\lambda(\vartheta-\vartheta_0)} - R_1, \\
&\ell\delta\alpha_0 e^{\lambda(\vartheta-\vartheta_0)} + R_2 \ge |\eta| \ge \ell\delta\alpha_0 e^{\lambda(\vartheta-\vartheta_0)} - R_2, \\
&R_1 = \frac{a_0 + N_a(\vartheta-\vartheta_0)}{k} + |\beta_0| + \frac{2N_\alpha}{\lambda}, \\
&R_2 = a_0 + N_a(\vartheta-\vartheta_0) + \ell\left(|\beta_0| + \frac{2N_\alpha}{\lambda}\right).
\end{aligned} \tag{18}$$

Than the necessary condition of the limitedness of the orbit is $\delta\alpha_0 = 0$. Or, according to (16):

$$\alpha_0 = -\int_{\vartheta_0}^{\infty} f_\alpha(\tau)e^{-\lambda(\tau-\vartheta_0)}d\tau. \tag{19}$$

Using (13), it is easy to show, that under condition (19) the unequality holds:

$$|\alpha(\vartheta)| \le \frac{N_\alpha}{\lambda}. \tag{20}$$

The condition obtained is not always sufficient, because parameters $a(\vartheta)$ and $d(\vartheta)$ may in some cases infinitely grow when $\vartheta \to \infty$. In particular, this may happen when resonance occurs between components of vector $\{a_\xi, a_\eta, a_\zeta\}$ and elliptic or oscillating motion of the SC. For instance, it is sufficient to put a_ξ in the righthand side of the first relation in (12) proportional to $\sin\varphi$ and $a_\eta = 0$ in order to ensure the secular increase of $a(\vartheta)$ which leads to unlimitedness of the orbit in accordance with (3), (15) and (20) under condition (19). So, if for some initial conditions

$$a(\vartheta_0)=a_0,\ \varphi(\vartheta_0)=\varphi_0,\ \beta(\vartheta_0)=\beta_0,$$
$$d(\vartheta_0)=d_0,\quad \psi(\vartheta_0)=\psi_0. \tag{21}$$

limited orbit exists, than it can be found by solving systems (2) or (9) under restrictions (19) and (21). The limitedness of so obtained orbit should be verified (for instance, by numeric integration of (2) or (9) on sufficiently long period).

4. THE CONSTRUCTION OF A LIMITED ORBIT.

In order to find a limited orbit by numerical integration of equations of motion under conditions (19) and (21) we shall use the method of successive iterations (Raf.12). Let adopt a limited intermediate orbit defined under (21) and $\alpha_0=0$ as a zero approximation. For this orbit we compute the values of perturbation accelerations and restricting ourselves by terms of second order :

$$a_\xi^{(0)}=-3B_1\left(\xi^2-\frac{\eta^2+\zeta^2}{2}\right)-(2B_0+1)\xi e\cos\vartheta,$$
$$a_\eta^{(0)}=3B_1\xi\eta-(B_0-1)\eta e\cos\vartheta, \tag{22}$$
$$B_0=\frac{1-\mu}{(1+\gamma)^3}+\frac{\mu}{|\gamma|^3},\quad B_1=\frac{1-\mu}{(1+\gamma)^4}+\frac{\mu}{|\gamma|\gamma^3}$$

Substituting (22) into righthand side of (12) and making use of (3) we obtain corresponding approximate expression for function $f_\alpha(\vartheta)$:

$$f_\alpha^{(0)}=f_\alpha^{(0)}(a_0,\varphi_0,\beta_0,d_0,\psi_0,\vartheta)$$

and with the aid of (19) the first approximation for :

$$\alpha_0^{(1)}=-\int_{\vartheta_0}^{\infty} f_\alpha^{(0)}(a_0,\varphi_0,\beta_0,d_0,\psi_0,\tau)\,e^{-\lambda(\tau-\vartheta_0)}\,d\tau.$$

After integration we obtain :

$$\alpha_0^{(1)}=-\frac{3B_1}{2}\Big[a_0^2T_2(U_1-U_2\cos\varphi_0+U_3\sin^2\varphi_0+U_4\sin\varphi_0\cos\varphi_0)+d_0^2U_5\left(\frac{2\Omega^2}{\lambda}+2\Omega\sin\psi_0\cos\psi_0+\lambda\sin^2\psi_0\right)-\beta_0^2U_6+a_0\beta_0T_1(U_7\sin\varphi_0-U_8\cos\varphi_0)\Big]+$$
$$+\frac{e}{2}\Big\{\beta_0F\left[\frac{k}{D_1}(1+2B_0)+\frac{\ell}{D_2}(1-B_0)\right]+\frac{a_0}{2}\left[\frac{1+2B_0}{D_1}\times(\lambda P_1\cos\varphi_0+P_2\sin\varphi_0)+\frac{1-B_0}{D_2}(\lambda P_1\sin\varphi_0-P_2\cos\varphi_0)\right]\Big\},$$

$$T_1=\frac{1}{4\lambda^2+\omega^2},\quad T_2=\frac{1}{\lambda^2+4\omega^2},\quad P_1=E_1+E_2,$$
$$U_1=\omega\left[\frac{2\omega k}{\lambda D_1}\left(\frac{1}{k^2}-\frac{1}{2}\right)+\frac{1}{kD_2}\right], \tag{23}$$
$$U_2=\frac{1}{kD_1},\ U_3=\frac{\lambda k}{2D_1}+\frac{2\omega}{kD_2},\quad P_2=-E_1(1-\omega)+E_2(1+\omega),$$
$$U_4=-\frac{2\omega k}{D_1}\left(\frac{1}{k^2}+\frac{1}{2}\right)+\frac{\lambda}{kD_2},\quad U_5=\frac{k}{2D_1(\lambda+4\Omega^2)},$$
$$U_6=\frac{1}{3\lambda}\left[\frac{k}{D_1}\left(1-\frac{\ell^2}{2}\right)-\frac{\ell}{D_2}\right],\quad E_1=\frac{1}{\lambda^2+(1-\omega)^2},$$
$$U_7=\frac{2k}{D_1}\left(\lambda\ell-\frac{\omega}{k}\right)+\frac{1}{D_2}\left(\frac{\ell\omega}{k}+2\lambda\right),\quad E_2=\frac{1}{\lambda^2+(1+\omega)^2},$$
$$U_8=\frac{k}{D_1}\left(\frac{4\lambda}{k}+\ell\omega\right)+\frac{1}{D_2}\left(\omega-\frac{2\lambda\ell}{k}\right),\quad F=\frac{2\lambda}{4\lambda^2+1}$$

where a_0, φ_0, β_0, d_0 and ψ_0 are the values of orbit parameters for $\vartheta=\vartheta_0$. The construction of next approximation is difficult because the orbit which corresponds to the initial condition $\alpha(\vartheta_0)=\alpha_0^{(1)}$ is, in general terms, unlimited, and the problem of convergence of the integral in (19) remains open. In order to bypass the difficulty we present (19) in form:

$$\alpha_0=-\int_{\vartheta_0}^{\vartheta_1} f_\alpha(\tau)e^{-\lambda(\tau-\vartheta_0)}d\tau-\int_{\vartheta_1}^{\infty} f_\alpha(\tau)e^{-\lambda(\tau-\vartheta_0)}d\tau, \tag{24}$$

where ϑ_1 is some value of ϑ . With the aid of (14) it is easy to show, that for the limited orbit:

$$\left|\int_{\vartheta_1}^{\infty} f_\alpha(\tau)e^{-\lambda(\tau-\vartheta_0)}d\tau\right|\le\frac{N_\alpha}{\lambda}e^{-\lambda(\vartheta_1-\vartheta_0)} \tag{25}$$

It follows that, with proper choice of ϑ_1 , it is possible to make second term in righthand side of (24) as small as needed in absolute value and substitute the necessary condition (19) of orbit limitedness by approximate condition:

$$\alpha_0=-\int_{\vartheta_0}^{\vartheta_1} f_\alpha(\tau)e^{-\lambda(\tau-\vartheta_0)}d\tau. \tag{26}$$

We can use the method of successive iterations to solve system (2) under conditions (21) and (26):

$$\alpha^{(i+1)}=-\int_{\vartheta_0}^{\vartheta_1} f_\alpha^{(i)}(\tau)e^{-\lambda(\tau-\vartheta_0)}d\tau,\quad i=1,2,\ldots \tag{27}$$

where $\alpha_0^{(i)}$ and $f_\alpha^{(i)}(\tau)$ (i=1,2,...) are the i^{th} approximation of function α_0 (ϑ_0) and corresponding value of f_α (τ), obtained by numerical integration of equations (2) or (9) with initial conditions (21) and α $(\vartheta_0)=\alpha_0^{(i)}$. For the sake of simplification it is possible to omit computation of integral in (27). To this end we can use expression (13):

$$\alpha^{(i)}(\vartheta_1)=\left[\alpha_0^{(i)}+\int_{\vartheta_0}^{\vartheta_1} f_\alpha^{(i)}(\tau)e^{-\lambda(\tau-\vartheta_0)}d\tau\right]e^{\lambda(\tau-\vartheta_0)},$$

where $\alpha^{(i)}(\vartheta_1)$ is the value of α on orbit of i^{th} approximation for $\vartheta=\vartheta_1$. By substituting (27) into this expression, we obtain:

$$\alpha_0^{(i+1)}=\alpha_0^{(i)}-\alpha_0^{(i)}(\vartheta_1)e^{-\lambda(\vartheta_1-\vartheta_0)}. \tag{28}$$

The computation is performed until i=n, for which with required accuracy:

$$\alpha_0^{(n)}=\alpha_0^{(n+1)}. \tag{29}$$

The final accuracy of solution obtained can be characterized by the value of $\delta\alpha_0$, determined by formula (16). The convergence and accuracy of sugessted process depends on the selection of value ϑ_1. It is necessary to take into account that with unequality (25) the error decreases as difference ($\vartheta_1-\vartheta_0$) increases. From the other side, with too large value of this difference the orbit of the first approximation can leave the region where the convergence is ensured. In this case it may be expidient to solve the problem in few stages. On the first stage we use smaller value of $\vartheta_1-\vartheta_0$. So obtained value of α_0 is later used as the first apprpximation for the next solution with larger ϑ_1 and so on. By this way arbitrary accuracy can be achieved, which is limited only by the number of digits of processor used and accuracy of knowlege of forces acting on the SC.

The method presented in the paper was numerically tested for the case of SC motion in the gravity field of the Sun-Earth system under assumption of classical restricted elliptic three body problem. Various values of a_o and β_o were used (see tab.1). $\vartheta_\iota = 0$, $\varphi_o = \pi$, $d_o = 1E-3$, $\psi_o = 0$.

Table 1.

No	Initial values a_o	Initial values β_o	Parameter α_o: 1st approximation	Parameter α_o: Final value	$\mathcal{E}_{max}$
1	1E-3	0	-1.9401372E-5	-1.9976433E-5	~0.1
2	2E-3	0	-3.7939901E-5	-3.9803861E-5	~0.2
3	3E-3	0	-6.6923685E-5	-7.1545174E-5	~0.3
4	4E-3	0	-1.0635272E-4	-1.1567592E-4	~0.4
5	5E-3	0	-1.5622701E-4	-1.7250149E-4	~0.5
6	6E-3	0	-2.1654656E-4	-2.4206307E-4	~0.6
7	7E-3	0	-2.8731137E-4	-3.2399551E-4	~0.7
8	8E-3	0	-3.6852143E-4	-4.1732749E-4	~0.8
9	5E-3	2E-3	-1.4708004E-4	-5.3007091E-4	~0.5

The calculations were performed in two stages: for ϑ_1 =3 and ϑ_1 =6. They were completed when equality (4.8) kept with accuracy 1E-10 (or 15 m). Each calculation took no more than a few minutes of BESM-6 computer CPU time. The results are presented in table 1.

The limitedness of orbits so obtained was tested by numerical integration on interval $0<\vartheta<6.6$ (i.e., somewhat more than a year). All orbits remained limited on this interval, but by the end of the interval the SC drift from the calculated orbit became significant. It is caused by the error $\delta\alpha_o$ in accordance with unequality (18). The ratios $\mathcal{E}_{max}$ of the maximum SC distance from the libration center to the distance between the libration center and Earth for every orbit are given in Table 1. For orbits under consideration $\mathcal{E}_{max} \cong a_o/\gamma$.

The projections of orbits 5 and 9 on the planes $L\xi\eta$ and $L\xi\zeta$ are shown at Fig.3. It can be seen, that on the major part of the analysed interval, the SC motion is close to periodic one in both planes (with different revolution periods in two planes). The maximum deviations of the SC from the libration center L along axes ξ, η and ζ are close to a_o, a_o/K and d_o respectively, exept for the beginning and ending parts of the interval. In the beginning, the periodic motion is disturbed by fading with the time influence of the accepted β_o, and on the final part - by the growing influence of error $\delta\alpha_o$.

5. THE ORBIT MAINTENANCE CORRECTIONS.

From the unequalities (17) and (18) it follows, that inevitable errors $\delta\alpha_o$ of the SC injection to the limited orbit lead to the growth of the SC distance from the libration center. In this connection periodic maintenance corrections, bringing SC back to limited orbit, become necessary. In this section we describe the method for the calculation of these correction parameters. We restrict ourselves by the case of impulsive correction which results in a stepwise change of derivatives ξ', η' and ζ' without change of coordinates themselves. This method may be easily extended to the case of non-impulsive correction.

Let $\rho = \{\xi, \eta, \zeta, \xi', \eta', \zeta'\}$ is SC phase vector and $\tilde{\rho}$ is the value of this vector before correction, $\Delta\rho = \{0,0,0,\Delta\xi', \Delta\eta', \Delta\zeta'\}$ is the correction impulse phase vector and $\mathcal{E} = \{0,0,0,\mathcal{E}_\xi, \mathcal{E}_\eta, \mathcal{E}_\zeta\}$ is the unit vector of direction, along which the impulse is applied.

Let assume the moment $\vartheta = \vartheta_K$ and the unit vector $\mathcal{E}$ to be given (the problem of their selection is discussed in (Ref.2)). The SC phase vector ρ_K after correction may be written as:

$$\rho_K = \rho(\vartheta_K) = \tilde{\rho} + \chi\mathcal{E}, \tag{30}$$

where $\chi = \pm|\Delta\rho|$.

The problem consists in the selection of χ, for which solution of system (2) with initial conditions (30) satisfies the approximate integral limitedness condition (26).

We shall apply the successive iterations method to the solution of this problem. Let denote by $\chi^{(i)}$, $\rho_K^{(i)}$ the values of χ and ρ_K obtained on each step of the solution and by $\alpha^{(i)}(\vartheta)$ the function $\alpha(\vartheta)$ obtained by the numerical integration of (2) with initial conditions $\rho(\vartheta_K) = \rho_K^{(i)}$. As the zero approximation we shall adopt

$$\chi^{(0)} = 0\ , \quad \rho_K^{(0)} = \tilde{\rho}. \tag{31}$$

In the same way as (28) we can write:

$$\alpha_K^{(i+1)} = \alpha_K^{(i)} - \alpha^{(i)}(\vartheta_1)\, e^{-\lambda(\vartheta_1 - \vartheta_K)}. \tag{32}$$

It follows from (4) that:

$$\alpha = C_1\xi - C_2\eta + C_3\xi' + C_4\eta' \tag{33}$$

where

$$C_1 = \frac{\omega K}{2D_2}\ , \quad C_2 = \frac{\omega}{2D_1}\ , \quad C_3 = \frac{K}{2D_1}\ , \quad C_4 = \frac{1}{2D_2}\ .$$

The numerical values of constants C_1, C_2, C_3 and C_4 for the Sun-Earth system are given in the Appendix. It follows from (30) and (33) that:

$$\begin{aligned} \alpha_K^{(i)} &= \alpha_K^{(0)} + \chi^{(i)}(C_3\mathcal{E}_\xi + C_4\mathcal{E}_\eta), \\ \alpha_K^{(i+1)} &= \alpha_K^{(0)} + \chi^{(i+1)}(C_3\mathcal{E}_\xi + C_4\mathcal{E}_\eta). \end{aligned} \tag{34}$$

With the aid of (32) and (33) we obtain, that:

$$\chi^{(i+1)} = \chi^{(i)} - \frac{\alpha^{(i)}(\vartheta)\, e^{-\lambda(\vartheta - \vartheta_K)}}{C_3\mathcal{E}_\xi + C_4\mathcal{E}_\eta} \tag{35}$$

Using (31) and (35) it is possible to obtain every approximation $\chi^{(i)}$ of χ. Calculations should be perfirmed until the equality

$$\chi^{(n)} = \chi^{(n+1)} \tag{36}$$

holds with the required accuracy.

In order to pass from the dimensionless value χ to the corresponding dimensional value $\Delta V = \pm\sqrt{\Delta\dot{x}^2+\Delta\dot{y}^2+\Delta\dot{z}^2}$ we differentiate the transformation (1) with respect to time and make use of the fact, that ihe impulsive correction does not lead to the change of SC coordinates. Than

$$\Delta V = V_u(\vartheta_\kappa)\chi , \tag{37}$$

where $V_u(\vartheta_\kappa)$ is the transversal component $V_u = r\dot{\vartheta}$ of S_2 velocity relative to S_1 in the moment $\vartheta = \vartheta_\kappa$.

In order to make the estimation of errors of the method, we made the calculation of SC motion in the vicinity of libration center L_2 of the Sun-Earth system on interval $0<\vartheta<20$ (approx. 3 years and 2 month) under elliptic problem assumptions and following initial conditions: $\vartheta_0=0$, $a_0 = 5E-3$, $\varphi_0=\pi$, $\beta_0=0$, $d_0 = 1E-3$, $\psi_0=0$, assuming the absence of measurement errors and correction execution errors. The correction was made in the direction of Ox axis with constant step $\varkappa$ along angle ϑ in moments

$$\vartheta_\kappa = \vartheta_0 + j\varkappa , \quad j = 1,2.$$

$\varkappa$ was taken equal to 0.5 (somewhat less than a month). The value of obtained by the numerical integration of (2) with calculated initial conditions after previous correction was used as a value $\hat{\rho}$ of the vector ρ for some moment of correction $\vartheta = \vartheta_\kappa$. The value $\vartheta-\vartheta_\kappa$ in (35) was equal to 6. The calculation was performed until equality (36) had been achieved with error less than 1E-10.

As a result, it was found that total correction impulse necesssary for orbit maintenance during one year is ~0.0015 m/s, i.e. it is negligebly small. In real flight the correction cost would be determined by the errors of measurement and correction execution. In order to estimate this expenditure one can use results of (Refs.2,11).

Fig. 4-6 shows graphs of change of the osculating parameters $a, \alpha, \beta, d, \delta\varphi$ and $\delta\psi$ on all analised interval as well as projections of the orbit on planes $L\xi\eta$ and $L\xi\zeta$. It can be seen, that the orbits remain limited on all interval (unlike orbits without correction). In the same time there are slow, but significant perturbations of a and d, which may be explained by the resonance effects connected with the proximity of frequences ω and Ω to each other and to the doubled frequency of the Earth revolution around the Sun.

In order to verify the character of the slow perturbations of perameters a and d we made calculations of these parameters values on interval $0< \vartheta <180$, corresponding to about 28.7 years. The results are presented at Fig 7. It is seen, that these perturbations have long-periodic nature with period which depends on initial values of φ_0 and ψ_0. Besides, there are some secular perturbations of phases of elliptic and osculating motions φ and ψ. They do not affect the limitedness of the orbit ahd are connected to the perturbations of average frequences of the analised oscillations (Fig. 8).

6. ONE-IMPULSE TRAJECTORY OF TRANSFER TO THE LIMITED ORBIT

The analysis of SC motion on the transfer trajectory to the limited orbit in the vicinity of libration center is rather complicated problem since we cannot employ conventional methods of orbit selection and correction.

The following assumptions were used for the search of transfer trajectory:

- the spacecraft is launched from the low circular Earth orbit (LEO) with given altitude and inclination;
- the SC is speeded up by the thrust parallel to the SC velocity;
- the SC is launched from the intermediate LEO at the moment which ensures the SC crossing of the Moon orbit at sufficiently large distance from the Moon;
- the apsides axis of the SC osculating orbit after injection into the transfer trajectory should lie in the ecliptic plane;
- the transfer from LEO to the transfer trajectory is impulsive.

These assumptions resulted in the fixed values of perigee altitude, perigee passage moment and inclination of the osculating in the injection moment transfer orbit and in a simple relation between the perigee longitude and the longitude of ascending node Ω. So, only the values of semimajor axis a and Ω were left for the selection of transfer trajectory.

The process of search for the nominal transfer trajectory was reduced to the search for the maximum of function $T_1(a,\Omega)$, which is the period of time for which the SC does not move to a distance more than R from the libration center. We adopted R equal to 1 million km. The initial values of a and Ω were selected in the interactive mode. Such value of was searched for, with which the trajectory could return to the Earth or leave it depending on the value of Ω.

After such a value of a had been found, it became possible by the selection of Ω to find such an orbit, which begins on the LEO and does not leave the vicinity of libration center for more than 2 years.

As an example of the transfer trajectory we shall describe the orbit with the SC launch from LEO at altitude 185 km and inclination 65 degrees on April 7 1991, 0 hour ET. Fig. 9 shows the projections of the orbit to the planes EXY and EXZ of the following coordinate system: the center is the Earth center E, EX axis is directed out from the Sun, EZ axis is directed to the ecliptic pole and EY completes the system.

The limited orbit obtained after transfer trajectory is characterized by the following parameters:

- the maximum distance from EXZ plane is about 800,000 km;
- the maximum distance from the ecliptic plane is about 150,000 km;
- the Sun-SC-Earth angle between 2.7° and 35°;
- the period of revolution around L_2 center is about 180 days.

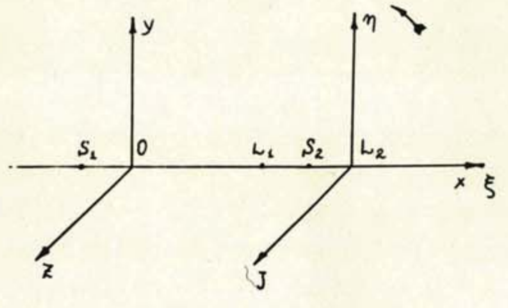


Figure 1. The coordinate system

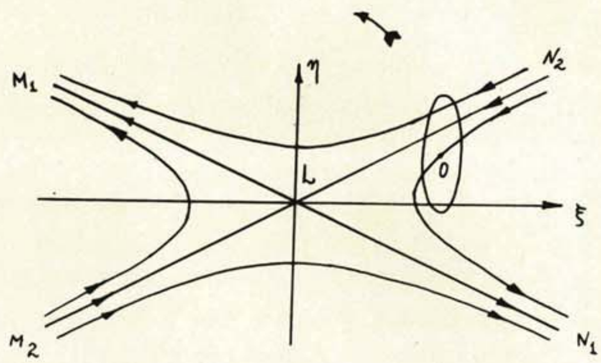


Figure 2. Kinematic sence of orbit parameters

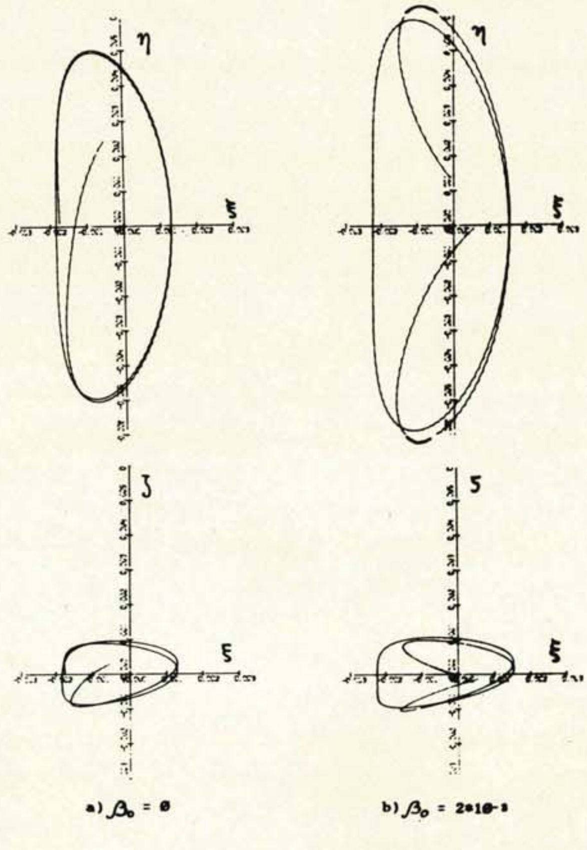


Figure 3. Projections of orbits without correction on the coordinate planes $L\xi\eta$ and $L\xi\eta$, $0 \le \vartheta \le 6.6$, $a_0 = 5 \cdot 10^{-3}$, $\varphi_0 = \pi$, $d_0 = 1 \cdot 10^{-3}$, $\psi_0 = 0$.

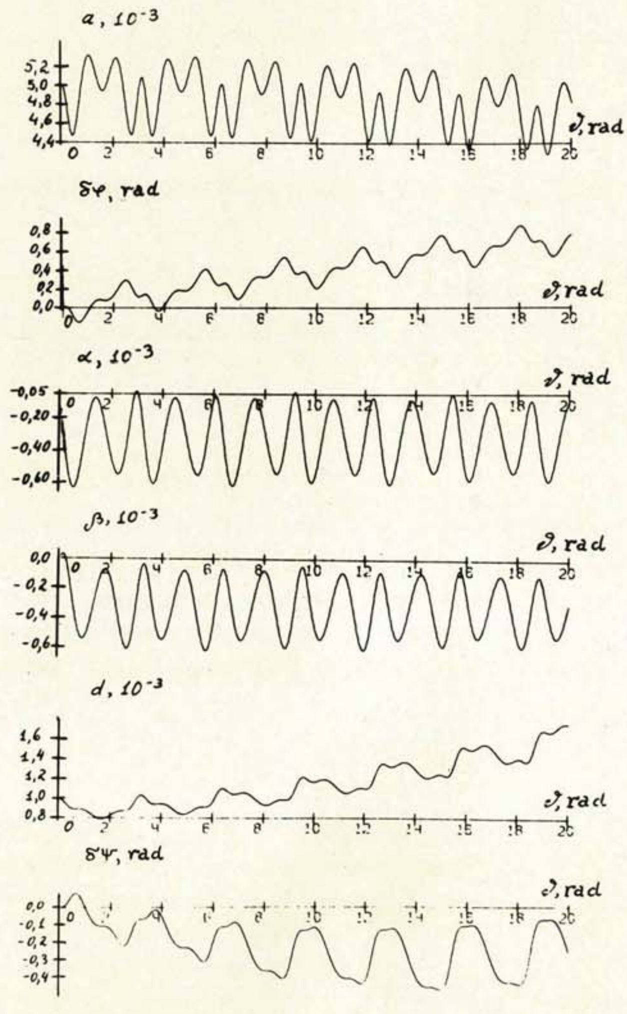


Figure 4. Graphs of change of osculating parameters (with correction) $\delta\varphi = \varphi - [\varphi_0 - \omega(\vartheta - \vartheta_0)]$, $\delta\psi = \psi - [\psi_0 + \Omega(\vartheta - \vartheta_0)]$, $a_0 = 5 \cdot 10^{-3}$, $\varphi_0 = \pi$, $\beta_0 = 0$, $d_0 = 1 \cdot 10^{-3}$, $\psi_0 = 0$.

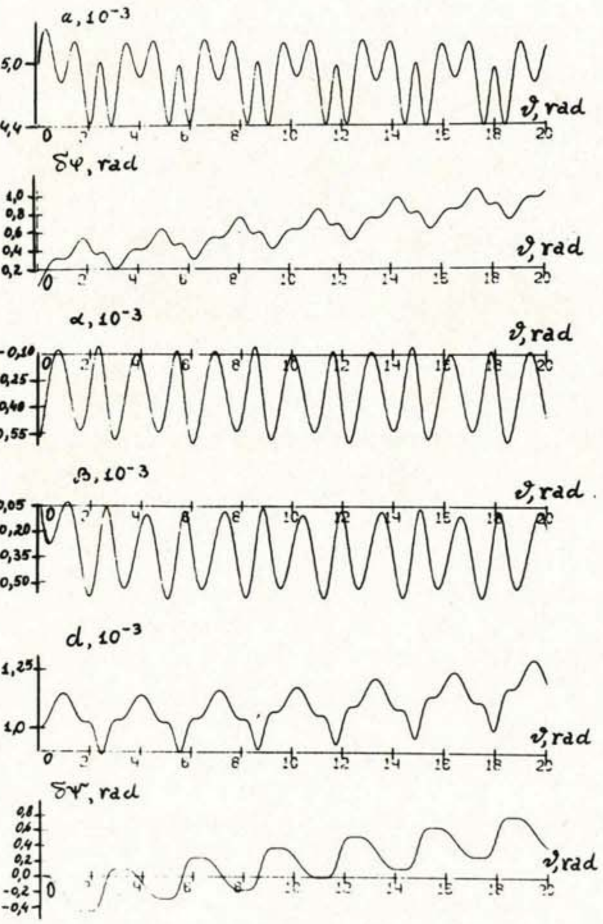
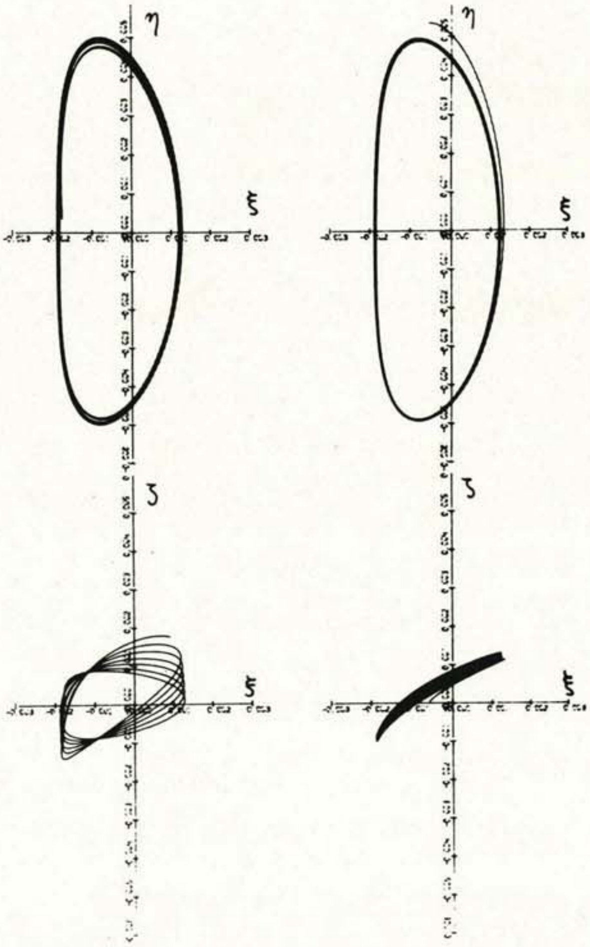


Figure 5. Graphs of change of osculating parametrs (with correction) $\delta\varphi = \varphi - [\varphi_0 - \omega(\vartheta - \vartheta_0)]$, $\delta\Psi = \Psi - [\Psi_0 + \Omega(\vartheta - \vartheta_0)]$, $a_0 = 5\cdot10^{-3}$, $\varphi_0 = \pi/2$, $\beta_0 = 0$, $d_0 = 1\cdot10^{-3}$, $\Psi_0 = 0$.



a) $\varphi_0 = \pi$ b) $\varphi_0 = \pi/2$

Figure 6. Projection of orbits with correction on the coordinate planes L ξη and L ξζ. $a_0 = 5\cdot10^{-3}$, $\beta_0 = 0$, $d_0 = 1\cdot10^{-3}$, $\Psi_0 = 0$, $0 \le \vartheta \le 20$.

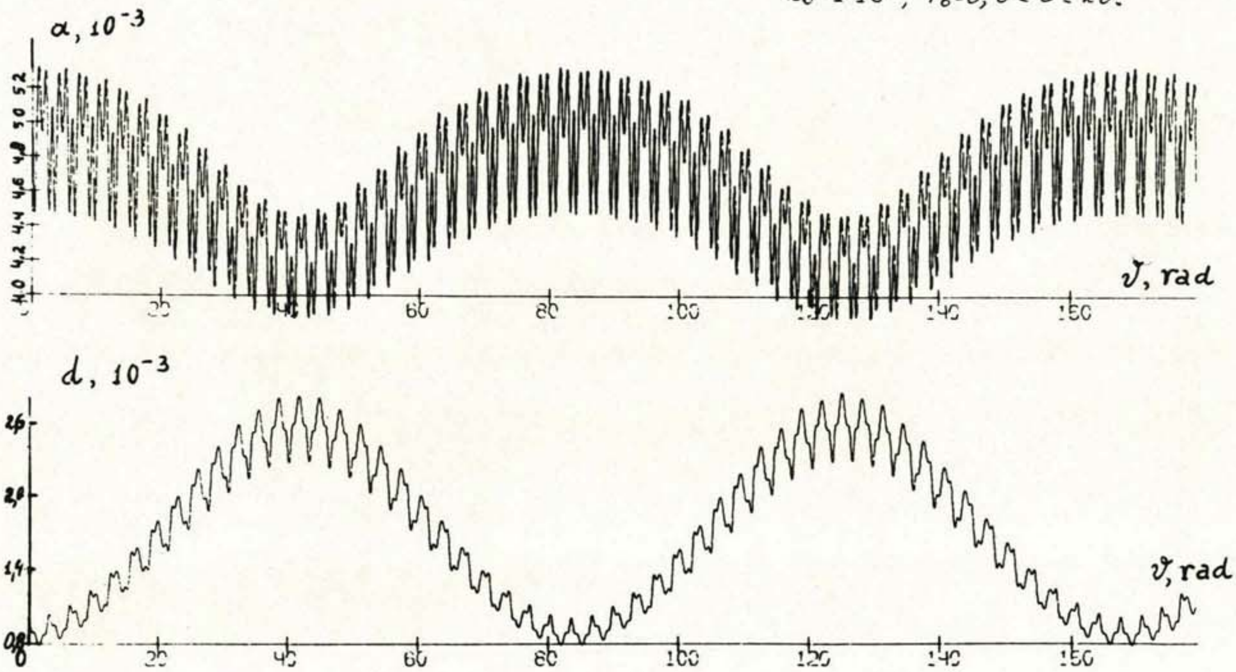


Figure 7. Graphs of change of osculating parameters a, d. $a_0 = 5\cdot10^{-3}$, $\varphi_0 = \pi$, $\beta_0 = 0$, $d_0 = 1\cdot10^{-3}$, $\Psi_0 = 0$

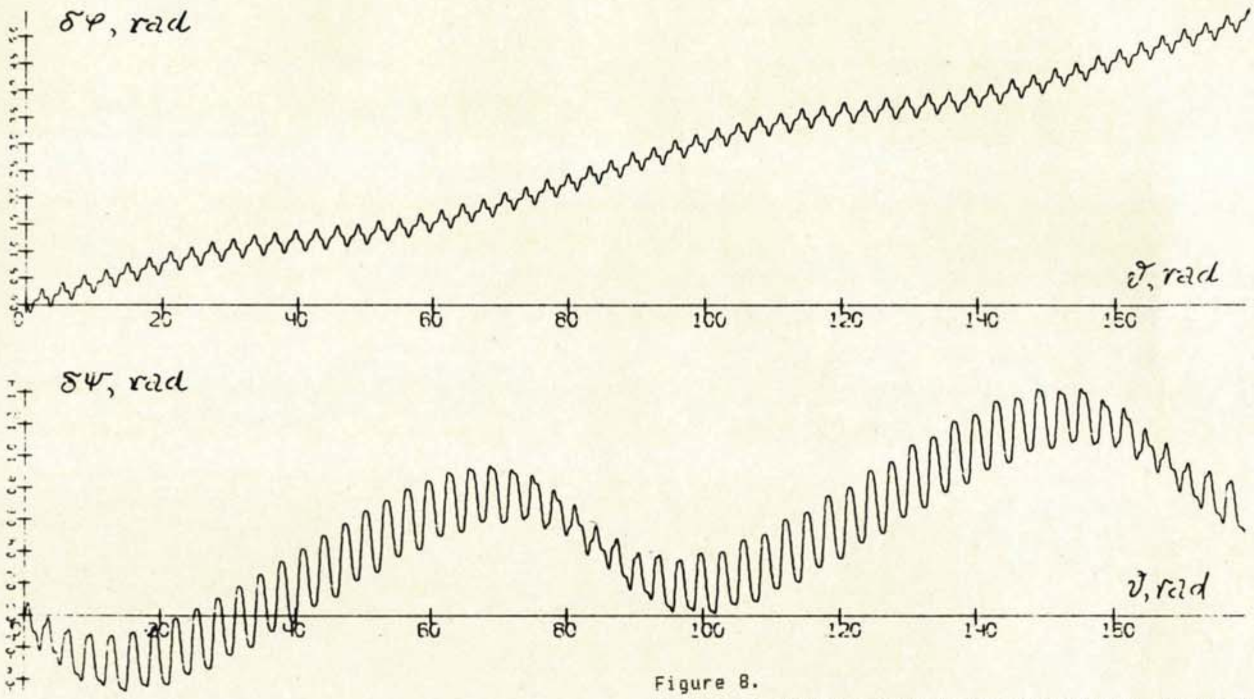
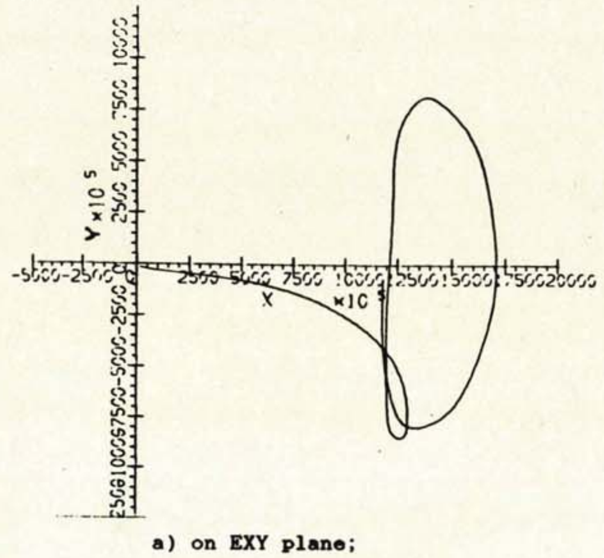
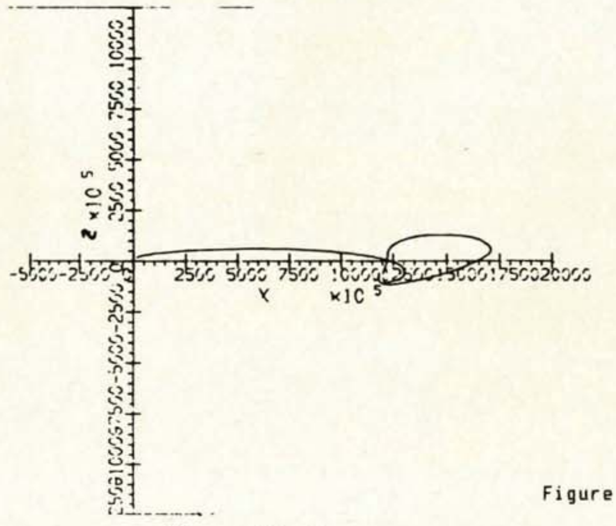


Figure 8.
Graphs of change of osculating parameters deviations
$\delta\varphi = \varphi - [\varphi_0 - \omega(\vartheta - \vartheta_0)]$, $\delta\psi = \psi - [\psi_0 + \Omega(\vartheta - \vartheta_0)]$,
$a_0 = 5 \cdot 10^{-3}$, $\varphi_0 = \pi$, $\beta_0 = 0$, $\alpha_0 = 1 \cdot 10^{-3}$, $\psi_0 = 0$.



a) on EXY plane;



b) on EXZ plane

Figure 9. Projections of transfer trajectory on coordinate planes of EXYZ system

Appendix

The characteristics of intermediate orbit in the Sun-Earth system

#	Constant	Libration center L_1	L_2
1	μ	3.040424E-6	
2	γ	-0.01001098	0.01007824
3	B_c	4.061074	3.940522
4	B_L	-301.6699	295.6709
5	ω	2.086454	2.057014
6	λ	2.532659	2.484317
7	k	3.229268	3.187229
8	l	0.5345736	0.5452636
9	Ω	2.015211	1.985075
10	x_1	9.293999	9.039702
11	x_2	5.383825	5.201568
12	c_1	0.6257371	0.6302114
13	c_2	0.1122474	0.1137767
14	c_3	0.1737287	0.1762906
15	c_4	0.09287077	0.09612485

7. REFERENCES

1. Лидов М Л, Вашковьяк М А ,Маркеев А П 1976, Полуаналитический метод расчета движения космического аппарата в окрестности коллинеарной точки либрации, Космические исследования, том 14, 909-921.

2. Лидов М Л, Лукьянов С А 1976, Статистические оценки в задаче управления движением космического аппарата в окрестности коллинеарной точки либрации, Космические исследования, том 14, 929-937.

3. Маркеев А П 1978, Точка либрации в небесной механике, Москва, Наука.

4. Farquhar R W, Kamel A A, 1973, Quasi-Periodic Orbits about the Translunar Libration Point, Celestial Mechanics, Vol 7, 458-473.

5. Breakwell J V , Kamel A A, Rather M J 1974, Station-Keeping for translunar communication station, Celestial Mechanics, Vol 10, 357-374.

6. Farquhar R W , Muhcnen D P, Richavdson D L, 1977, Mission Design for a Halo Orbiter of Earth, Journal of Spacecraft and Rockets,Vol 14, 170-177.

7. ИКИ АН СССР Пр.995 1985, Исследование движения космического аппарата в окрестности коллинеарного центра либрации, Эльясберг П Е , Синицын Б М ,Тимохова Т А.

8. Дубошин Г Н 1964, Небесная механика.Аналитические и качественные методы, Москва, Наука.

9. Субботин М Ф 1968, Введение в теоретическую астрономию, Москва, Наука.

10. Себехей В 1968, Теория орбит (ограниченная задача трех тел), Москва, Наука.

11. ИКИ АН СССР Пр.1003, 1985, Коррекция космического аппарата в окрестности коллинеарного центра либрации, Эльясберг П Е, Тимохова Т А.

12. ИКИ АН СССР Пр.1012, 1965, Выбор и поддержание ограниченной орбиты в окрестности коллинеарного центра либрации ограниченной эллиптической задачи трех тел. Эльясберг П Е, Тимохова Т А.