STATION KEEPING OF A QUASIPERIODIC HALO ORBIT USING INVARIANT MANIFOLDS

C Simó G Gómez J Llibre & R Martínez

Dept. de Matemàtica Aplicada i Anàlisi, Facultat de Matemàtiques, Univ. de Barcelona.

Dept. de Matemàtiques, E.T.S.E.I.B. Univ. Politècnica de Catalunya.

Dept. de Matemàtiques, Facultat de Ciències, Univ. Autònoma de Barcelona.

ABSTRACT

We study the station keeping of quasiperiodic orbits near a periodic halo orbit of the Earth+Moon-Sun system. The full solar system is considered as a perturbation of the RTBP. For the RTBP the halo orbits near the libration points L_1 ot L_2 are well known but they are unstable. In the full solar system the periodic orbits are lost but the system still has quasiperiodic solutions relatively close to the previous halo orbits. They are approximately obtained analytically and refined numerically. These orbits can be taken as nominal orbits for halo missions. They inherit the unstable character of the halo orbits. Despite they are no longer periodic we can still introduce their invariant manifolds, which are approximated by analytical and numerical solutions of the variational equations. Simple geometrical considerations allow us to develop a very cheap station keeping algorithm.

Keywords: Halo orbits, quasiperiodic orbits, instability, invariant manifolds, station keeping.

1. INTRODUCTION AND GENERAL OVERVIEW

As it is well known since Euler and Lagrange epoch, the three body problem and, in particular, the restricted three body problem,RTBP, has 5 libration points. The RTBP describes the motion of a particle, of very small mass (the spacecraft), under the gravitational attraction of two massive bodies (Sun and Earth+Moon barycenter), that are in circular motion around their center of masses (Ref. 14). Synodical coordinates are introduced to keep fixed both main bodies. This is accomplished by introducing axes which rotate with the same angular velocity as that of the main bodies.

The libration points are equilibrium points in these coordinates. At them the attraction of the massive bodies is exactly cancelled by the centrifugal force. Three of them are in the line joining both primaries and the one between primaries is called L_1. The two remaining libration points are the equilateral ones. They will not be discussed here.

The previous situation is quite idealistic. However, real world can be considered as a not too large perturbation of this ideal behaviour in many cases. We will focus on the Sun-Barycenter system for which we modify slightly the mass of the Sun to satisfy Kepler's third law, and we add the following terms:
(a) Due to the real motion of the Barycenter and to the previously skipped part of the mass of the Sun,
(b) Due to the fact that instead of the Barycenter there is the Earth-Moon system,
(c) The effect of planets, mainly Venus and Jupiter,
(d) The solar radiation pressure.
In this field of forces the dynamical libration points do not subsist. However we can introduce geometrical libration points, with respect to Sun and Barycenter, given by the same relations that are obtained in the RTBP. A particle placed on them moves slowly, provided it is not too far from them.

Due to these nice properties of the libration points, they are suitable for space missions. The L_1 point of the Sun-Barycenter system is an ideal site for a solar observatory. The Sun's surface is always available, the Earth is far enough to have low noise and near enough to allow for good communications. However, L_1 is not suitable because the signals from the spacecraft disappear in the solar noise. Some angular deviation from the Sun is required. There are periodic orbits (the so called halo orbits) in the RTBP which do exactly what we want: they are not too far from L_1 and the angular distance to the Sun is big enough. For previous work on these orbits see Ref. 3, 4, 5, 7. In this work we deal with this type of orbits and we study how they are modified when the perturbations (a) to (d) are considered. The definition of the nominal orbit can be strict or not, that is, we can force the spacecraft to follow the nominal orbit closely, or we can only ask to the spacecraft to be not too far from a given path. For the ISEE-3 mission the second procedure was adopted (Ref. 7). In this work we propose to follow the orbit closely. This means a substantial reduction in the expected fuel consumption for station keeping. In the real world periodic orbits no longer subsist. They are replaced by quasiperiodic ones (Ref. 1). A quasiperiodic motion can be seen as superposition of harmonics with different incommensurable frequencies. A halo orbit should be replaced by a nearby quasiperiodic orbit, and the possible escaping components should be avoided because the halo orbits of small and medium size are unstable. A particle starting at a distance d from a halo orbit, leaves this orbit as d exp(mt). For the L_1 case considered, and t in days, a typical value of m is 0.042. In one year an initial error will be multiplied by more than $3\cdot10^6$. An important fact is that, in the range of interest only one unstable direction appears. For the basic definitions and

Proceedings of the Second International Symposium on Spacecraft Flight Dynamics, Darmstadt, FR Germany, 20–23 October 1986 (ESA SP-255, Dec. 1986)

properties of invariant stable/unstable manifolds see Ref. 12. The basic idea to cope with the escaping direction is to perform on/off manoeuvres to annihilate the unstable component of the motion. This should be done in the most effective way.

The on/off control is asked to satisfy the following requirements:
(a) The manoeuvres should not be too small,
(b) The manoeuvres should not be too frequent,
(c) Allow for delays or advances in the execution of the manoeuvres with small additional cost,
(d) If the unstable component is bigger than a given value a manoeuvre should be done,
(e) Between too small and too large controls, a manoeuvre can be done if the time since last manoeuvre is big enough and it is checked that the unstable component increases as $\exp(mt)$,
(f) It should be possible to continue the control for long mission durations. This is specially critical for the Moon case where the time scale is short. A typical mission, 6 years long, using the L_2 halo orbits for the Earth-Moon system, means near 150 revolutions. It seems feasible that the total cost of the station keeping for this mission can be reduced to less than 25 m/sec.

The study of the station keeping of halo orbits is divided in two main steps: (i) The determination of the nominal path, (ii) The definition and implementation of the control strategy. The RTBP is used as first approach. Periodic halo orbits have been computed analytically and numerically having good agreement between both computations. The study of the behaviour near these orbits has also been done by analytical and numerical integration of the variational equations. A first idea of the on/off control can be given considering the real situation as a perturbation of the restricted problem. This gives insight on the robustness of the control parameters and shows the optimality of the (x,y)-control in front of the other ones. The next step has been the obtention of the equations of motion, accounting for the different forces, in a suitable reference system, in which they are seen as a perturbation of the RTBP.

The quasiperiodic orbits are obtained using an extension of the Lindstedt-Poincaré method (Ref. 9) as we did for the halo orbit. Next, the definitive nominal orbit in the full real solar system is produced. A parallel shooting method (Ref. 13) is used with 'starting' orbit the quasiperiodic one computed in the previous step. Then the control strategy can be implemented for the real case. A large amount of control simulations have been done using different values of tracking errors, errors in the execution of manoeuvres and random errors in the radiation pressure.

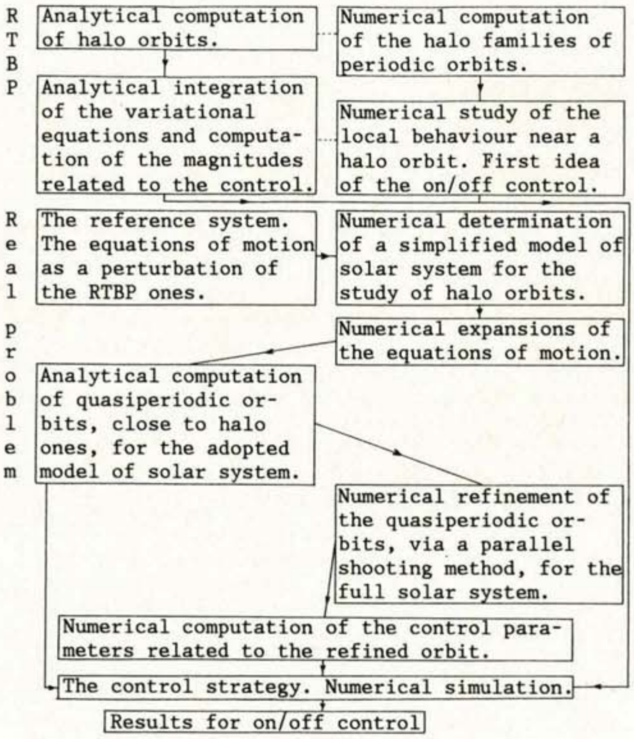
In this paper we summarize the main methods and results. Full details appear in Ref. 6. This work was done under ESA contract. The technical supervision of the late Dr. E.A.Roth and Dr. J.Rodríguez-Canabal is acknowledged. We are also indebted to Fundació Empresa i Ciència.

2. PERIODIC HALO ORBITS

We consider the equations of motion of the RTBP with mass parameter μ, in synodical coordinates (Ref.14):

$$\ddot{x} - 2\dot{y} = \Omega_x \ , \ \ddot{y} + 2\dot{x} = \Omega_y \ , \ \ddot{z} = \Omega_z, \tag{1}$$

where $\Omega = \frac{1}{2}(x^2+y^2) + \frac{1-\mu}{r_1} + \frac{\mu}{r_2}$, $r_1^2 = (x-\mu)^2+y^2+z^2$,



Block diagram

$r_2^2 = (x-\mu+1)^2 + y^2 + z^2$.

We look for periodic orbits of the spatial RTBP in synodical coordinates (the family of orbits called halo orbits), around one of the collinear libration points L_i , i=1,2. We consider a reference system with origin at the libration point, the x axis directed from the bigger primary to the smaller one, the z axis oriented as the sidereal angular momentum of the secondary around the primary, and the y axis completing a positively oriented coordinate system. Let $\gamma = d(m_2,L_i)$,i=1,2. We scale variables in such a way that γ becomes the unity. A new time, τ , is introduced according to $\tau = \gamma^{3/2}t$. The standing equations are (Refs. 10, 11).

$$w^2x''-2wy'-(1+2c_2)x = \Sigma c_{n+1}(n+1)\rho^n P_n(x/\rho), \tag{2}$$

$$w^2y''+2wx'+(c_2-1)y = \Sigma c_{n+1} y \Sigma Q_{mn} , \tag{3}$$

$$w^2z'' + \lambda^2 z = \Sigma c_{n+1} z \Sigma Q_{mn} + \Delta . z , \tag{4}$$

where $Q_{mn} = \rho^{n-1}(1+4m-2n)P_{n-2m-1}(x/\rho)$, the sums are for $n \geqslant 2$, m from 0 to $[(n-1)/2]$ and where

$$' = \frac{d}{d(\tau w)} , \ w = 1 + \sum_{i \geqslant 1} w_{2i} , \ \Delta = \lambda^2 - c_2 = \sum_{i \geqslant 1} \Delta_{2i} , \ \rho^2 = x^2+y^2+z^2,$$

λ is the positive zero of $\lambda^4+(c_2-2)\lambda^2-(c_2-1)(1+2c_2)$,

and $c_n = \gamma^{-3}[(\pm 1)^n \mu + (-1)^n (1-\mu)\gamma^{n+1}(1\mp\gamma)^{-n-1}]$ with upper sign for L_1 and lower sign for L_2. We set $\gamma = r_1$ obtained by solving Euler's quintic equation.

Taking into account the symmetries, for instance $(x,y,z,t) \longrightarrow (x,-y,z,-t)$, and using $\varphi = \exp(i\lambda w \tau)$, we look for a solution of the type

$$x = \Sigma a_{ijk}\alpha^i\beta^j\varphi^k \ ; y = \sqrt{-1}\,\Sigma b_{ijk}\alpha^i\beta^j\varphi^k \ ; z = \Sigma c_{ijk}\alpha^i\beta^j\varphi^k ,$$

$$w = 1 + \Sigma d_{ij}\alpha^i\beta^j \ , \Delta = \Sigma f_{ij}\alpha^i\beta^j ,$$

with some symmetries in the coefficients (Ref. 11) and where $i,j \geq 0$, $k \in \mathbb{Z}$ and $i+j \geq 2$ in w and Δ. The parameter λ is the modulus of the imaginary eigenvalue related to the planar equations in the neighborhood of L_i (Ref. 14), and $b_{101}=-\lambda/(\lambda^2+1-c_2)$. The quantities Δ and w appear in the Lindstedt-Poincaré method when the periodic orbit is searched (Ref. 9, 11). By substituting the expected solutions into Eqs. 2-4, and equating terms in $\alpha^i\beta^j\gamma^k$ in the left and right parts, all the coefficients are determined recurrently.

For the L_1 case in the Sun-Barycenter problem using the value $\beta=0.07$, the Table 1 gives the differences in initial conditions and period with respect to the 11th order theory when theories of order n,n=10,...3 are used. As a check the differences between the 11th and 15th order analytical theories and the numerically periodic orbit are included. If we compare with Ref. 11, 4, a notorious improvement is obtained.

Case	x(Km)	z(Km)	$\dot{y}$(mm/s)	Period(s)
A11-A10	2.499	.264	-4.544	10.874
A11-A9	2.030	1.417	-44.777	10.874
A11-A8	23.600	4.144	-42.356	49.169
A11-A7	24.526	10.728	-43.156	49.169
A11-A6	250.093	-16.874	452.638	-875.130
A11-A5	-190.220	-99.387	438.237	-875.130
A11-A4	-3817.41	-554.727	6836.76	-13108.2
A11-A3	-5442.82	-1602.12	6677.25	-13108.2
A11-Num	-0.090	-0.048	0.192	0.076
A15-Num	-0.0005	0.0009	0.0007	-0.007

1 Unit of lenght = 149597906 Km,
1 Unit of velocity = 29785898.0 mm/s,
1 Unit of time = 5022440.67 s.

Table 1

Using a continuation method for computing zeros of a system of non linear equations (Ref. 13) a program has been developed to compute numerically families of periodic orbits. Then the full family of halo orbits in the L_1 and L_2 cases for the Sun-Barycenter and the Earth-Moon problems has been produced numerically. For previous results (Ref. 2, 8).

3. LOCAL BEHAVIOUR. INVARIANT MANIFOLDS

To know the dynamics near a halo orbit we have studied the variational equations associated to (1) along the periodic orbit. Let $D\Phi(\tau)$ be the variational flow. The eigenvalues of the monodromy matrix, M, are $\lambda_1>1$, $\lambda_2<1$, $\lambda_3=\lambda_4=1$, $\lambda_5=\bar{\lambda}_6$, with $\|\lambda_5\|=1$ (at least for small and medium size halo orbits). The three pairs have the following geometrical meaning:
(a) The first pair (λ_1,λ_2) with $\lambda_1.\lambda_2=1$ is associated to the unstable character (hyperbolic behaviour according to (Ref. 18)) of the small and medium size halo orbits: λ_1 is the dominant eigenvalue, and the associated eigenvector, $e_1(0)$, is the most expanding direction. Then $e_1(\tau)=D\Phi(\tau).e_1(0)$ and the tangent vector to the orbit span the tangent plane to the local unstable manifold $W^{u,l}$ (Ref. 12). Following the flow forward the full unstable manifold is produced. Using the symmetry the stable manifold is obtained.
(b) For the second pair (λ_3,λ_4) there is only one vector $e_3(0)$, with eigenvalue equal to $1.e_3(\tau)$ is the tangent vector to the orbit. The other eigenvalue equal to 1 is associated to variations of the energy. The monodromy matrix restricted to the eigenspace related to the eigenvalue 1 has the form $\begin{pmatrix}1 & \epsilon\\ 0 & 1\end{pmatrix}$. ϵ is not zero due to the variation of the period when the orbit changes along the family.
(c) The monodromy matrix restricted to the plane spanned by the real and imaginary parts of the eigenvectors associated to λ_5,λ_6 is a rotation of angle Γ.

Instead of $e_i(\tau)$, i=1,...,6, we introduce the Floquet modes, $\bar{e}_i(\tau)$, as 6 periodic vectors from which $e_i(\tau)$ are easily recovered (Ref. 15). For instance, $\bar{e}_1(\tau)$ is defined as $e_1(\tau).\exp(-\tau.\ln\lambda_1/T)$, where T is the period of the halo orbit. The Floquet modes have been computed both numerically and analytically.

4. A FIRST APPROACH TO THE ON/OFF CONTROL

In this section we explain, for the RTBP, the proposed method of station keeping using on/off manoeuvres. The small modifications to account for the perturbations are technical. The basic idea is to cancel just the unstable component.

Let $\delta=(\delta x,\delta y,\delta z,\delta\dot{x},\delta\dot{y},\delta\dot{z})^T$ be the error vector, i.e., the difference between the actual coordinates and the nominal ones. At the current moment, τ, δ can be expressed in the local basis $\{\bar{e}_i(\tau)\}$ as $\delta=\Sigma c_i\bar{e}_i(\tau)$, i=1,...,6. We are interested in c_1, the component along the unstable direction. We have

$$c_1=\Pi_1(\tau)\delta x+\Pi_2(\tau)\delta y+\Pi_3(\tau)\delta z+\Pi_4(\tau)\delta\dot{x}+\Pi_5(\tau)\delta\dot{y}+\Pi_6(\tau)\delta\dot{z}$$

The magnitudes $\Pi_i(\tau)$ are called the projection factors. $\Pi_i(\tau)$ is the (signed) minor, $m_i(\tau)$, of the i-th element in the first column of the matrix of Floquet modes divided by the determinant of the Floquet basis. Using as 2σ values for the tracking errors (supposed to be normal variables with zero mean) 3,5 and 30 Km in x,y and z and 2, 2 and 6 mm/s in the related velocities, the 2σ value for c_1 due to tracking errors is, at most, 10^{-7} in adimensional units. Hence it is not useful to do a manoeuvre to cancel the unstable component if this one is too small. From some preliminary numerical explorations it is readily seen that the optimal manoeuvres should not include z component.

We define the unitary control as the jump in velocity to be applied to the spacecraft to cancel one unit of unstable (normalized) component. Let Δ_1,Δ_2 the unitary controls along the x and y directions. If we write

$$\bar{e}_1(\tau)/\|\bar{e}_1(\tau)\|+(0,0,0,\Delta_1,\Delta_2,0)^T=\sum_{j=2}^{6}\alpha_j\bar{e}_j(\tau),$$

and try to minimize $\Delta_1^2+\Delta_2^2$, it follows that:

$$\Delta_i=-\det(\tau)m_{i+3}(\tau)/((m_4(\tau)^2+m_5(\tau)^2).\|\bar{e}_1(\tau)\|),\ i=1,2.$$

The gain function is defined as $g(\tau)=\|\Delta(\tau)\|^{-1}$. Comparing the useful controls: (x,y)-plane, x-axis and y-axis, it is found that the x-axis control has an average of efficiency of 0.84 with respect to the (x,y) one, ranging from 0.72 to 0.96. For the y-axis control the average efficiency is 0.48 and ranges from 0.26 to 0.70. Therefore the x-only control is almost as good as full control and the y-only control has a cost that is roughly the double. For the full control (i.e. (x,y) control), the gain factor is rather uniform: the quotient of the maximal to the minimal values is roughly 1.43. These results are not very sensitive to β nor μ. It is natural to ask whether it is reccomendable to wait for a given manoeuvre trying to reach another epoch with a better gain. Of course the gain can increase but also the unstable component increases. We should look for the function $\exp((\ln\lambda_1/T)\tau)/g(\tau)$ in every case. It turns out that this function is always increasing. Therefore it is never good to wait for a manoeuvre.

5. THE REAL LIFE EQUATIONS

For the comparison of numerical results and for the obtention of the equations of motion in a suitable way we have introduced two systems of reference: adimensional and normalized. We denote by $\bar{e}$ and $\bar{a}$ the coordinates of a point in the ecliptical and adimensional systems. The ecliptical one is centered at the center of mass of the solar system We go from one system to the other through $\bar{e} = kC\bar{a}+\bar{b}$, where k (change of scale factor), C(orthogonal matrix) and $\bar{b}$ (translation) must be computed. Of course they depend on time. The parameters $\bar{b}$ and k are computed using that the primary is placed at $(\mu,0,0)$ and the secondary at $(\mu-1,0,0)$. To end the determination of C we require that the velocity of the secondary with respect to the primary be contained, in the (x,y) plane.

For the collinear libration points we define the normalized system to be centered at the libration point with the axes oriented as in the second section. The distance from the libration point to the nearest primary is taken as unity.

The time is changed so that the mean sidereal period of the secondary with respect to the primary be 2π. We change slightly the mass of the primary in order that Kepler's third law holds. The remaining part of the mass is seen as a perturbation.

Let $\underline{R}$, $\underline{R}_S$, $\underline{R}_E$, $\underline{R}_M$, $\underline{R}_B$ and $\underline{R}_{P_i}$ (resp. $0,S,E,M,B,P_i$) be the position vectors with respect to an inertial frame centered at the center of masses of the solar system (resp. the masses) of spacecraft, Sun, Earth, Moon, Barycenter and the i-th planet, respectively. The motion of the spacecraft is governed by

$$\ddot{\underline{R}} = \sum_{A\in\{S,E,M,P_1,\dots,P_K\}} \frac{G_1 A(\underline{R}_A-\underline{R})}{\|\underline{R}_A-\underline{R}\|^3} \qquad (6),$$

where G_1 is the gravitational constant. The position of the Sun, $\underline{R}_S$, satisfies a similar equation. We introduce $\underline{r}=\underline{R}-\underline{R}_S$ and $\underline{r}_{SA}=\underline{R}_A-\underline{R}_S$. We write the Lagrangian in ecliptical coordinates and transform to nomalized ones. In this system the coordinates of the spacecraft are denoted by $\underline{a}=(x,y,z)$ and the modulus by a. The body A is placed at $\underline{r}_A=(x_A,y_A,z_A)$ and with respect to the Sun at $\underline{r}_{SA}=(x_{SA},y_{SA},z_{SA})$. The angle from $\underline{r}_A$ to $\underline{a}$ is denoted by A_1 . So, the equations (6) around L_i for $i=1,2$ are transformed to

$$
\begin{aligned}
x'' - 2y'' - (1+2c_2)x = {} & \sum_{n\geq 2} (n+1)c_{n+1}a^n P_n(x/a) \\
& + c(0) \sum_{n\geq 2} nc_n a^{n-1} P_{n-1}(x/a) \\
& + c(1)x+c(2)y+c(3)z+c(4)x'+c(5)y'+c(7) \\
& + Kk^{-3}\gamma^{-3}\mu_{\tilde{S}}(x-x_S) \sum_{n\geq 2} \frac{a^{n-2}}{|r_S|^{n+1}} \bar{P}_n(\cos S_1) \\
& + Kk^{-3}\gamma^{-3} \sum_{A\in\{E,M,B,P_1,\dots,P_K\}} i(A)\mu_A \Big[-\frac{x_{SA}}{r_{SA}^3} + \\
& + (x-x_A) \sum_{n\geq 2} \frac{a^{n-2}}{r_A^{n+1}} \bar{P}_n(\cos A_1)\Big] , \qquad (7)
\end{aligned}
$$

$$
\begin{aligned}
y''+2x'-(1-c_2)y = {} & y \sum_{n\geq 3} c_n a^{n-2}\bar{P}_n(x/a) \\
& + c(0)y \sum_{n\geq 2} c_n a^{n-2}\bar{P}_n(x/a) \\
& +c(11)x+c(12)y+c(13)z+c(14)x'+c(15)y'+c(16)z'+c(17) \\
& + Kk^{-3}\gamma^{-3}\mu_{\tilde{S}}y \sum_{n\geq 2} \frac{a^{n-2}}{|r_S|^{n+1}} \bar{P}_n(\cos S_1) \\
& + Kk^{-3}\gamma^{-3} \sum_{A\in\{E,M,B,P_1,\dots,P_K\}} i(A)\mu_A \Big[-\frac{y_{SA}}{r_{SA}^3} + \\
& + (y-y_A) \sum_{n\geq 2} \frac{a^{n-2}}{r^{n+1}} \bar{P}_n(\cos A_1)\Big] , \qquad (8)
\end{aligned}
$$

$$
\begin{aligned}
z'' + c_2 z = {} & z \sum_{n\geq 3} c_n a^{n-2}\bar{P}_n(x/a) \\
& + c(0)z \sum_{n\geq 2} c_n a^{n-2}\bar{P}_n(x/a) \\
& + c(21)x+c(22)y+c(23)z+c(25)y'+c(26)z'+c(27) \\
& + Kk^{-3}\gamma^{-3}\mu_{\tilde{S}}z \sum_{n\geq 2} \frac{a^{n-2}}{|r_S|^{n+1}} \bar{P}_n(\cos S_1) \\
& + Kk^{-3}\gamma^{-3} \sum_{A\in\{E,M,B,P_1,\dots,P_K\}} i(A)\mu_A \Big[-\frac{z_{SA}}{r_{SA}^3} + \\
& + (z-z_A) \sum_{n\geq 2} \frac{a^{n-2}}{r_A^{n+1}} \bar{P}_n(\cos A_1)\Big] , \qquad (9)
\end{aligned}
$$

where ' denotes derivation with respect to the normalized time, P_n denote the Legendre polynomials, $\bar{P}_n(z)=-dP_{n-1}(z)/dz^n$ and $c(i)$ are known functions of time related to the noncircular motion of the secondary w.r.t the primary. Furthermore $K=G_1(S+E+M)$ and $\mu_A=A/(\tilde{S}+E+M)$. The mass $\tilde{S}$ is the part of the solar mass required to satisfy Kepler's third law. The remaining part and the effect of the radiation pressure are included in $\tilde{S}$. The index $i(A)$ is equal to -1 for the Barycenter and to 1 otherwise. The first line of each one of Eqs. 7-9, i.e. what is obtained skipping perturbations, reduces to the Eqs. 2-4 given in the second section.

All the time dependent functions in the equations of motion have been Fourier analyzed. To each term it is associated a weight in such a way that the global effect of the halo terms of order m has weight m. Only terms with weight less than or equal to 9 have been retained. For instance only Jupiter and Venus in circular orbit have to be retained concerning the effect of planets. For the motion of the Barycenter around the Sun some dozens of periodic terms coming from the effect of the planets must be retained, and for the Moon to keep the dominant 6 periodic terms in longitude, 4 in latitude and 4 in parallax is sufficient.

6. QUASIPERIODIC SOLUTIONS. LOCAL BEHAVIOR

According to the last expansions the non halo terms of the equations can be written as one of the following

$$c_1 x^i y^j z^k \, F(\nu t+\varphi), \text{ or } c_2 v' F(\nu t+\varphi),$$

where $i,j,k>0$, c_1 and c_2 are two real coefficients, ν and φ are the frequency and the phase, respectively, associated to the corresponding term, F means one of the trigonometric functions sinus or cosinus, and v' stands for one of the derivatives x',y' or z'. It has been useful to take only cosinus terms in the first and in the third equations, and sinus terms in the second equation. As in the second section a new time, $w\tau$, is used, where w is an unknown series in α,β introduced when the Lindstedt-Poincaré method is applied. The function w contains the halo terms plus terms appearing in the course of integration to avoid resonances. In the same way, the function $\Delta = \Delta(\alpha,\beta)$ is modified to include new terms.

The solution of Eqs. 7-9 can be written as

$$a = a_H + a_N = \sum_{n\geqslant 1} a_{Hn} + \sum_{n\geqslant h} a_{Nn},$$

where a stands for x,y or z and where x_{Hn} (y_{Hn}, z_{Hn}) contains all the terms of weight n corresponding to x (y,z) of the halo orbit, and in x_{Hn} (y_{Hn}, z_{Hn}) are included all the non halo terms of weight n of the x (y,z) coordinate. Suppose that all the terms of x, y and z are known up to weight n-1. By substitution of the coordinates x, y and z in Eqs. 7-9 we get in the right hand side some terms which have weight equal to n. These are cosinus or sinus terms of the following type

$$c\alpha^i\beta^j F(k\lambda w\tau+\sigma\tau+\varphi), \tag{10}$$

where $\sigma=\Sigma s_q \nu_{rq}$ and $\varphi=t^*\sigma+\Sigma s_q\varphi_{rq}$, ν_{rq} and φ_{rq} being some of the frequencies and phases involved in the equations, $s_q \in \{+1,-1\}$, and the parameter t^* is taken such that the angle in the halo part of the solution is equal to zero for this epoch, that is, $\tau = t-t^*$. Hence, and as $\Delta(\alpha,\beta)=\lambda^2-c_2$ gives a relation between α and β, the quasiperiodic solutions found depend on two parameters, for instance β and t^*.

Till now we have obtained analytical quasiperiodic solutions. To improve such a solution numerically we use a parallel shooting algorithm (Ref. 13). The initial data are: (1) An initial time t_0; (2) A value of z, in normalized coordinates when y (normalized) is equal to zero. The values $x_0,\dot{x}_0,\dot{y}_0,\dot{z}_0$ remain free. We remark that those data plus the sign of β determine one quasiperiodic solution. From y=0, $z=z_0$ for $t=t_0$ the parameters β_0 and t_0^* are obtained. We will denote by $x^a(\beta_0,t_0^*,t),\ldots,\dot{z}^a(\beta_0,t_0^*,t)$ the values of $x,\ldots,\dot{z}$ given by the analytical theory.

Let $t_0,t_1,\ldots,t_N$ be the successive time passages through y=0. From one passage to the next one we go from a point $(t_i,x_i,y_i=0,z_i,\dot{x}_i,\dot{y}_i,\dot{z}_i)=Q_i$ to a new point which is denoted by $P(Q_i)$ (to remember that this is a Poincaré map). This is done by numerical integration using the full solar system. The correct points Q_i should satisfy the matching conditions

$$F_i(Q)=P(Q_i)-Q_{i+1}=0, \quad i=0,1,\ldots,N-1 \tag{11}$$

i.e. 6N equations. We recall that the number of variables is 6N+4, the last four coming from the free initial parameters. There are several ways to choose the last 4 equations. The possibility finally chosen, showing good properties concerning condition number and physical meaning is the following:

$$F_{6N+1} = x_0-x_0^t-(x_N-x_N^t) = 0, \tag{12}$$

$$F_{6N+2} = z_N-z_N^t = 0, \tag{13}$$

$$F_{6N+3} = \Pi_4(\dot{x}_0-\dot{x}_0^t)+\Pi_5(\dot{y}_0-\dot{y}_0^t)+\Pi_6(\dot{z}_0-\dot{z}_0^t), \tag{14}$$

$$F_{6N+4} = \Pi_4(\dot{x}_N-\dot{x}_N^t)+\Pi_5(\dot{y}_N-\dot{y}_N^t)+\Pi_6(\dot{z}_N-\dot{z}_N^t), \tag{15}$$

where $x_i^t=x^a(\beta_0,t_0^*,t_i)$ with t_i choosen such that $y^a(\beta_0,t_0^*,t_i)=0$; $z_N^t=z^a(\beta_0,t_0^*,t_N)$; $\dot{x}_0^t=\dot{x}^a(\beta_0,t_0^*,t_0)$, etc., and $\Pi_{4,5,6}$ are the last three projection factors at t_0. The equation F=0 obtained by adding the equations from (11) to (15) is solved by Newton's method. As initial values $x_i^t=x^a(\beta_0,t_0^*,t_i)$, ..., $\dot{z}_i^t=\dot{z}^a(\beta_0,t_0^*,t_i)$ are taken.

As the tangent spaces at two different points of R^6 can be identified, we have used the variational matrix as if it was a monodromy matrix. In fact, the quasiperiodic orbit is not too far from the halo orbit. We have adopted the criterion of computing the eigenvalues and eigenvectors just every one "revolution", i.e., after 2 passages through y=0. From these initial vectors the equivalent to Floquet modes are obtained using the variational flow of the full solar system. From the modes the projection factors and unitary controls are computed.

7. THE STATION KEEPING METHOD. SIMULATIONS AND DISCUSSION

To do the station keeping of the spacecraft scheduled to follow the computed nominal orbit the following method is proposed:

(1) For a given time a point, X_e, is estimated by tracking. In the simulation this point has been taken as the point obtained by integration of the equations of motion plus random tracking and model errors.

(2) For that time the nominal point, X_n, is computed. Optionally this point can be taken as the point in the nominal path at minimum (local) distance from the current point. This modified nominal point corresponds to a time different from the current time, but in the simulations with numerical nominal orbit and projection factors, using on/off control, the time difference is less than 15^m in time intervals of 4 years.

(3) The residue vector, X_e-X_n, is computed. Its unstable component (UC) is obtained by inner product with the projection factors at the epoch.

(4) If the unstable component is less than some value UC_{min} (usually around 2 to 4 times 10^{-7} in adimensional units), the manoeuvre has no sense. The unstable component can be due, exclusively, to tracking errors. Some upper bound UC_{max} of the unstable component has been given such that an unconditional manoeuvre is done if $|UC| > UC_{max}$.

(5) When $|UC| \geqslant UC_{max}$ or $UC_{min} < |UC| < UC_{max}$, and (e) of §1 is satisfied, a manoeuvre should be done. The x and y components of the jump in velocity are computed using the unitary controls and the manoeuvre is done. Usually $UC_{max}/UC_{min} \sim \exp(1)$. For instance $UC_{max} = 10^{-6}$, $UC_{min} = 4.10^{-7}$ are good candidates. Typically one month, at least, has been asked between 2 consecutive manoeuvres.

To test the behavior of the proposed control a simulation program has been implemented. When a simulation is started, at a given epoch, the nominal point is computed. Then, random variations in the 6 coordinates are introduced due to tracking errors according to the given laws. Random errors are introduced at each step of integration (a one step method is used and the errors are, of course, kept fixed during one step). Usually these errors are 2σ =5% normal errors in the radiation pressure. At every step the unstable component is computed, but to this computation (not to the current point) again tracking errors are applied. Then we proceed as described in (3) to (5). Some execution errors are also introduced. They can be due to wrong direction for the manoeuvre or wrong modulus of the same.

Several runs of the simulation program have been done (see table 2 for a small sample). As a summary, for the standard values of the different parameters (case A of table 2) at most 20 cm/s per year are required. The time interval between manoeuvres ranges form 1 to 6 months, with an average slightly less than 3 months. In the most pessimistic case (big errors in tracking, model and execution of manoeuvres) at most 50 cm/s per year are required (compare with the ISEE-3 values as given in Ref. 7). The comparison of results for different values of the amplitude of the tracking errors shows that the tracking errors are responsible (with the standard values taken) of one half of the total Δv required. Hence, it is important to reduce them as much as possible. The errors in the execution of the manoeuvres are responsible for some 10% of the total Δv . The random errors mean some 15% of the total Δv (even in our case where a large S/m is used). This means that the remaining 25% is due to inaccuracies of the model and to the numerical simulation errors. This can be seen as the action of neglected terms in the acceleration.
As a final remark, the proposed method for the nominal orbits and station keeping should be easily applied to any binary perturbed system (Sun-planet or planet-satellite) in the solar system, if the time scales of the motion are not too small.

	Man	Δv	t_{min}	t_{max}	v_{min}	v_{max}
	17	65	50	155	1.7	5.7
	18	75	44	151	2.4	5.4
A	19	74	31	142	2.1	5.7
	17	71	40	125	2.6	5.4
	17	71	46	174	2.1	6.3
	17	67	39	137	2.2	5.4
	22	200	38	91	4.6	12.2
	22	206	34	91	4.8	11.9
	24	222	36	99	6.6	11.7
B	22	192	33	114	4.2	11.6
	16	141	51	126	4.6	10.8
	20	166	34	120	4.2	10.9

Table 2

Man = number of manoeuvres in a 4 years long mission
Δv = total amount of increment of velocity (cm/s). Control done in the (x,y)plane.
t_{min}, t_{max} = minimum and maximum time interval between manoeuvres.
v_{min}, v_{max} = minimum and maximum values of the manoeuvres (cm/s).
Adopted values of the parameters:
(A) $UC_{max} = 10^{-6}$; $UC_{min} = 4.10^{-6}$; t = 60 days, desired minimum time interval between manoeuvres if possible; 2σ tracking errors 3,5 and 30 Km, 2.2 and 6 mm/s in $x,y,z,\dot{x},\dot{y},\dot{z}$; 2σ relative error in the execution of manoeuvres 5% of the modulus in x,y, 2% in z ; 1σ relative error in the size of the radiation pressure 5% , in their latitude and longitude 0.05 rad; systematic error in the size of the radiation pressure 0%.

(B) Used values 2.10^{-6}; 8.10^{-7}; 60; 6,10,60,4,4,12; 10%,4%; 20%,0.2; 10%, respectively.

8. REFERENCES

1. Arnold V 1976, Méthodes mathématiques de la mécanique classique, Moscow, Mir.
2. Breakwell J V, Brown J 1979, The halo family of three dimensional periodic orbits in the Earth-Moon restricted three body problem, Cel. Mechanics 20, 389-404.
3. Farquhar R W 1970, The control and use of libration point satellites, NASA TR R-346.
4. Farquhar R W, Muhonen D P, Richardson D 1977, Mission design for a halo orbiter of the Earth, J. Spacecraft 14, 170-177.
5. Farquhar R W, Muhonen D P, Neumann C R, Heuberger H 1980, Trajectories and orbital manoeuvres for the first libration-point satellite, J. Guidance and Control 3, 549-554.
6. Gómez G, Llibre J, Martínez R, Simó C, 1985, Station keeping of libration point orbits, ESA Final Report.
7. Heuberger H 1977, Halo orbit station keeping for ISEE-3, AIAA Paper.
8. Howell K C 1984, Three dimensional periodic halo orbits, Cel. Mechanics 32, 53-72.
9. Poincaré H 1893, Les Méthodes Nouvelles de la Mécanique Celeste, Vol II, París, Gauthier-Villars.
10. Richardson D L 1980, A note on a Lagrangian formulation for motion about the collinear points, Cel. Mechanics 22, 231-236.
11. Richardson D L 1980, Analytic construction of periodic orbits about the collinear points, Cel. Mechanics 22, 241-253.
12. Smale S 1967, Differentiable dynamical system, Bull. Amer. Math. Soc. 73, 747-817.
13. Stoer J, Bulirsch R 1980, Introduction to Numerical Analysis, New York, Springer.
14. Szebehely V 1967, Theory of orbits, New York, Academic Press.
15. Wiesel W, Shelton W 1983, Model control of an unstable periodic orbit, J. Astronautical Sci. 31, 63-76.