FAST NUMERICAL INTEGRATION OF INTERPLANETARY ORBITS

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ABSTRACT

Mathematical considerations on the stability of the numerical integration of interplanetary orbits lead to a special step-size control for the time that minimizes the computation time. These controls are derived from the eigenvalues of the Jacobian of the right-hand sides of Newtons equations of motion. It is shown that the implementation of this control into a multistep method of constant step-size by means of a time regularisation has certain disadvantages. They can be avoided by integrating the equations of motion by means of a multi-step method with variable step-sizes like the one due to Nordsieck. Initialisation problems and the computation of the coefficients of the difference equations for higher order methods are addressed.

Keywords: Fast numerical integration, interplanetary orbits, step-size control, stability, eigenvalues, Nordsieck-method.

1. INTRODUCTION

The orbit of the nucleus of comet Halley, the first target of the GIOTTO S/C, has been determined altogether more than 300 times at ESOC during the last two years. The GIOTTO flight dynamics team envisaged frequent runs of the relevant orbit determination software because it knew that

- the first interplanetary mission of ESA could require more checks in the navigation process than other routine projects;
- different models for the nongravitational force and for the observation biases had to be compared from time to time and last not least
- some observers expected a quick assessment of the quality of their individual astrometric data in order to accommodate their equipment (telescope, plate, filter), the exposure time and the measuring and reduction of plates to the the comet specific problems.

The above facts indicated to seek for a fast orbit determination method.

The numerical integration of state and variational equations becomes the dominant time consumer in the method if one has to process observations from several apparitions of a comet. This was the case for comet Halley and will be the case for the next potential target of GIOTTO, comet Grigg-Skjellerup. The following considerations on a fast numerical integration of interplanetary orbits were therefore the prerequisite for the development of the ESOC orbit determination software. It was supposed to contain a generator for states and state-transition matrices that needs a minimum amount of integration steps per orbital revolution and a minimum number of evaluations of the right hand sides of the equations of motion and partials in each integration step.

2. INTGRATION ALGORITHM AND STEP-SIZE CONTROL

It is well known that an approximation of the 6-dimensional state s(t) of a probe moving in space, e.g. of its position and velocity or its orbital elements and angular variable, and of the partials $\partial s/\partial s_o$ of this state w.r.t. a given initial state s_o at the epoch t_o, i.e. of its 6x6 state transition matrix $\Phi(t)$, can be determined as a function of time t by means of a numerical integration of the equations of motion

$$\frac{ds}{dt} = f(s,t) \qquad (1)$$

and of the variational equations

$$\frac{d\Phi}{dt} = \frac{\partial f}{\partial s}(s,t)\cdot\Phi \qquad (2)$$

The variational equations (2) are linear in Φ since s(t) and hence $\partial f(s,t)/\partial s$ can be assumed to be a known function of t for s(t) can be determined independent of Φ from equations (1).

Any integration algorithm provides approximations $\bar{s}_i$, $\bar{\Phi}_i$ of s(t) and $\Phi(t)$ at discrete times t_i, i=0,... . It is well known that the step-sizes

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$h_i = t_{i+1} - t_i$ are subject to restraints if one wants to achieve some prescribed accuracy of the approximations. The step-sizes must be 'controlled'.

A suitable control can be derived from the local and global truncation errors of the algorithm and in particular for finite step-sizes from an analysis of its stablity e.g. w.r.t. round-off errors. The latter fact is sometimes overlooked although it often imposes more stringent constraints on the step-sizes than the limitation of truncation errors.

The step-size control depends on the integration algorithm and on the right hand sides of the differential equation. The following simple and well known example illustrates once more some of the above statements.

Let us assume our right hand side f(s,t) contains a component that contributes noticeable accelerations $\delta(t)$ of the proble only in the samll time interval $(t_\delta - \Delta t/2,\ t_\delta + \Delta t/2)$, $\Delta t \ll$ orbital revolution period. Such accelerations typically occur during a close fly-by at a small planet and they may cause remarkable deflections of the trajectory.

The local truncation error of an algorithm of order n is proportional to h^n. One can therefore often find for a $h > \Delta t$ an order n such that the local truncation error drops below a prescribed or acceptable limit. This means that when the step-size control is derived from truncation errors alone (which is often done) the integration algorithm may take no notice of the noticeable perturbation of the trajectory due to the short term or short periodic acceleration $\delta(t)$.

The step-size control depends on the integration method. We have chosen at ESOC an implicit multi-step difference scheme as integrator for the following reasons:

- it does not require more than two evaluations of the right hand sides of (1) during the iterative solution of the implicit difference equations (predictor-corrector method) independent of the order of the method;
- the most time consuming part of this evaluation, the retrieval of the positions of the perturbing planets from an ephemeris file, and the computation of the partials on the right hand side of (2) need not be done more than once if the time t is the independent variable.

More details on the algorithm used will be given in chapter 4.

Those reasons imply that the number of evaluations of the right hand sides in (1) and (2) is independent of the order of the algorithm. Hence, the step-size control can be made rather independent of the local truncation error, i.e. the permissible step-sizes and hence the computation time is mainly defined by stability considerations.

An analysis of the full stability problem in the case of the large step-sizes we are interested in is possible only for model equations approximating the true differental equations, i.e. one can derive first order conditions from an analysis of linearized difference equations for the truncation errors. For details of this method we refer to Ref. 1.

Let $\lambda_k(t)$, k=1...6, be the eigenvalues of the Jacobian $\partial f/\partial s$ of the right hand sides f(s,t) of system (1). They are the natural frequencies of the system. We notice that they are identical to the eigenvalues of the Jacobian of the right hand side of the second system (2) for the state transition matrix because

$$\frac{\partial}{\partial \phi}\left[\frac{\partial f}{\partial s}(s,t)\cdot\phi\right] \quad \text{is proportional to} \quad \frac{\partial f}{\partial s}(s,t).$$

It can be shown after some lengthy analysis that the step-sizes of the multi-step methods must fulfil the following stability conditions

$$\max_k C\cdot|\lambda_k|\cdot h = C\cdot\Lambda(t)\cdot h < 1 \qquad (3)$$

and

$$\text{Real}\ (\lambda_k) > 0 \qquad , k = 1...6, \qquad (4)$$

if one does not want the truncation errors to grow beyond any limit with increasing number of integration steps. We will call $\Lambda(t)$ the 'maximum eigenvalue'. These are the conditions that depend on the system to be integrated. There are other, well known purely method depending stability and consistency conditions. They must always be fulfilled by a proper choice of the coefficients of the difference scheme. They do not depend on the step-sizes (see Ref. 3) and therefore do not influence the step-size control.

The coefficient C in condition (3) also depends on the method and the order selected. C for instance increases with increasing order (see Ref. 1). This must be taken into consideration when determining the optimum order from a trade off between local truncation error and stability condition.

We would like to mention in this context that the first condition (3) is tidily related to the sampling theorem of the theory of signal processing. It states that an error free reconstruction of a signal from samples, i.e. in our case from discrete values of the right hand side f(s,t), requires a certain minimum amount of samples: There must be taken at least 2 samples within one revolution period of the signal component with highest frequency. This is equivalent to the condition $\max|\lambda_k|/\pi\cdot h < 1$.

The solution of our system will contain terms proportional to $e^{\lambda_k(t-t_o)}$. Hence, small truncation errors can build up to such an extent with increasing time t that they will eventually swamp the required solution if $\text{Real}(\lambda_k) > 0$. Condition (4) prevents this type of instability. It does not depend on the step-size h and has therefore no influence on the step-size control.

The above considerations and the requirement to make the number of integration steps per revolution a minimum led us finally to the following step-size control in the numerical integration of the orbit and partials.

- upgrade the step-size at each integration step i according to

$$h_{i+1} = h_i \cdot \Lambda(t_i)/\Lambda(t_{i+1}) \qquad (5)$$

in order to make $C \cdot h \cdot \Lambda(t) = \text{constant}$.

- Choose an initial step-size h_o at the epoch t_o such that
  - -- the stability conditions $C \cdot h_o \cdot \Lambda(t_o) < 1$ is fulfilled for h_o,
  - -- one gets a sufficient number N of integration steps per orbit with the above step-size control that assures negligible truncation errors. ESOC used N = 1200 steps per orbit in the orbit generator for comet Halley. It is advisable to optimise N by means of test runs.

The above rules obviously guarantee that the stability condition (3) is fulfilled throughout the integration and that one gets a minimum amount of steps.

All we need now is to determine the value of $\Lambda(t)$ for the equations of motion of an interplanetary probe.

3. EIGENVALUES

We use at ESOC the position vector $\bar{x}$ and the velocity vector $\bar{v}$ of a massless probe (comet, spacecraft) in the EME50 (= Earth Mean Equator and Equinox of 1950.0 system) as state s in the orbit generation. The centre of that system is in the barycentre of the solar system. The equations of motion due to Newton then read

$$\frac{ds}{dt} = \frac{d}{dt}\begin{pmatrix}\bar{x}\\ \bar{v}\end{pmatrix} = \begin{pmatrix}\bar{v}\\ g(\bar{x},t)+p(\bar{x},\bar{v},t)\end{pmatrix} \qquad (6)$$

with

$$g(\bar{x},t) = -\sum_{j=0}^{N} \frac{\mu_j}{|\bar{x}-\bar{x}_j(t)|^3}\,(\bar{x}-\bar{x}_j(t)). \qquad (7)$$

The $\bar{x}_j(t)$ are the known positions of the sun ($j=0$) and of N perturbing bodies (planets, moons, etc.) to be retrieved from an ephemeris file; the μ_j are the relevant gravity constants and $p(\bar{s},\bar{v},t)$ stands for other small perturbing accelerations due to general relativity and non-gravitational forces, such as out-gassing forces on comets.

The computation of the Jacobian $G(\bar{x},t) = \partial g(\bar{x},t)/\partial x$ of the gravity terms is a straightforward task. The small partials of p w.r.t. $\bar{x}$ and $\bar{v}$ are neglected in our system. We will therefore get only an approximation for $\emptyset$ from the integration of the resulting truncated transition system (2). But we know from experience that this approximation is sufficiently accurate in the parameter estimation process of the orbit determination software and for the computation of the step-size control.

Using cartesian coordinates as in equations (6) and (7) respectively has one advantage over equations of motion using elements that is worthwhile mentioning. All calculations necessary in the evaluation of the right hand sides involve very simple algebraic manipulations and square-roots. This makes the iteration software fast.

The Jacobian of system (6) has a special form,

$$\begin{pmatrix} 0 & I \\ G(\bar{x},t) & 0 \end{pmatrix} \qquad (8)$$

One can show (see Ref. 2) that its eigenvalues are the 6 square-roots of the 3 eigenvalues $\omega_{1,2,3}$ of the symmetric 3x3-matrix $G(\bar{x},t)$, i.e. square-roots of the zeroes of the 3rd-order polynomial $\det(G(\bar{x},t) - \omega I)$.

These zeroes can easily be computed at each integration step. There do not exist more than two different absolute values for the λ_k since $G(x,t)$ is a special symmetric 3x3 matrix. Figure 1 shows them for comet Halley's motion around its perihelion in 1835. The graph gives an idea of how fast the step-sizes may vary in the vicinity of the perihelion.

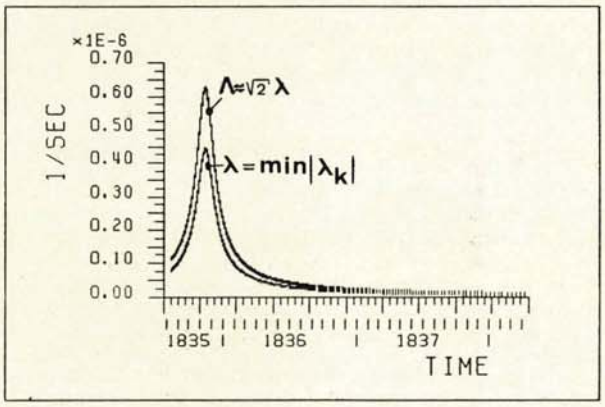

Figure 1. Absolute values of eigenvalues of Jacobian for comet Halley around perihelion

One can furthermore prove that one of the eigenvalues is positive. This fact has some consequence.

- The stability condition (4) is always violated for at least Newtons equation of motion, i.e. small truncation errors may build up and destroy the solution.

This fact is unfortunately true and rather well known in general for all Kepler motions regardless of the formulation of the equations of motion. Round-off errors will lead to small differences between the true and the approximated semi-major axis of Kepler orbits and hence to secular or 'unbounded' along-track errors in the integration process. We have to live with this phenomenon.

The computation of the eigenvalues from the 3rd order polynomial is time consuming. We have therefore approximated the maximum eigenvalue Λ by a simple analytical expression. For N=0, i.e. for the pure two-body motion of a probe about the sun, one can determine the following exact solution with $r = |\bar{x}|$.

$$\lambda_{1,2} = i\sqrt{\mu_o}\, r^{-3/2},\ \lambda_{3,4} = -\lambda_{1,2},\ \lambda_5 = \sqrt{2\mu_o}\, r^{-3/2},$$
$$\lambda_6 = -\lambda_5$$

and hence $\Lambda = \sqrt{2\mu_o}\, r^{-3/2}$.

This solution can be gained easily by means of the fact that the eigenvalues of the Jacobian must be independent of a rotation of the co-ordinate system. Rotating the position vector x into the radius-vector direction gives x = (r,0,0) and the characteristic polynomial of G(x,t) assumes the simple form

$$(-\frac{\mu_o}{r^3} - \omega)^2 \cdot (\frac{2\mu_o}{r^3} - \omega) \qquad (9)$$

and thus provides the above eigenvalues " $\lambda = \sqrt{\omega}$ ". This indicates to use either

$$\Lambda_a = \left[\sum_{j=0}^{N} \frac{2\mu_j}{|\bar{x}-\bar{x}_j(t)|^3}\right]^{1/2} \qquad (10)$$

or

$$\Lambda_b = \sum_{j=0}^{N} \left[\frac{2\mu_j}{|\bar{x}-\bar{x}_j(t)|^3}\right]^{1/2} \qquad (11)$$

as approximations of Λ . Λ_a is the simpler by-product of the evaluation of $G(\bar{x},t)$ and it is the better approximtion for the maximum eigenvalue in the case of comet Halley. Figures 2 and 3 show the errors of the 2 approximations around the two perihelion passages of the comet in 1835 and 1910 that are most important for the orbit determination.

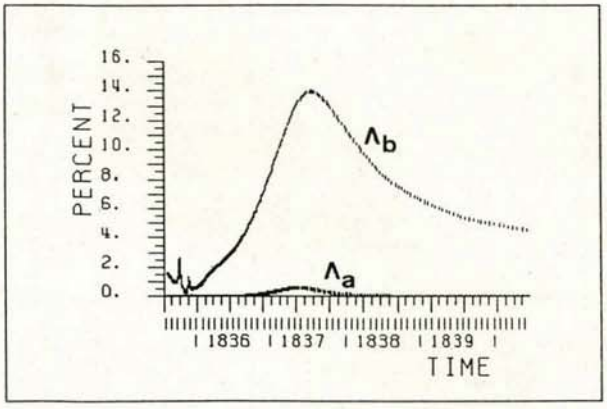

Figure 2. Errors in approximations Λ_a (see formula 10) and Λ_b (see formula 11) of eigenvalues of Jacobian for comet Halley: 1835 - 1840.

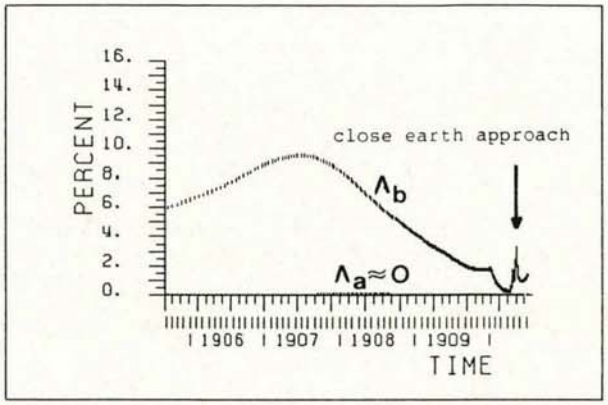

Figure 3. Errors in approximations Λ_a (see formula 10) and Λ_b (see formula 11) of eigenvalues of Jacobian for comet Halley: 1905 - 1910.

4. THE INTEGRATION ALGORITHM

A multi-step method with constant step-sizes is the most common algorithm for the numerical integrations of ordinary differential equations. One can implement a step-size control derived from the maximum eigenvalues $\Lambda(t)$ of the Jacobian into those algorithm by means of the so-called regularisation of the equations of motion (see also Ref. 4). Making the time t a dependent variable of a new indpendent auxiliary variable ξ through

$$\frac{dt}{d\xi} = \frac{1}{\Lambda(t)}$$

one can control the steps in time t as requested above with constant steps in the new variable ξ This method, however, has some very undesirable side-effects:

- One has to integrate 7 instead of 6 equations of motion.
- The right hand sides of (2) and the ephemerides of the planets must be recomputed at each step of the solution of the implicit difference equations since t will vary as a dependent variable. Both the effects mentioned will increase the computation time.
- Events like observations or manoeuvres are a function of time. In particular the implementation of manoeuvres requires an unnecessary complex logic and a very time-consuming re-initialisation of the integration process.

We therefore used a multistep method with variable step-size for the integration of all differential equations.

These methods exist and the equivalent of the most stable predictor-corrector method with constant step-sizes due to Adams-Bashford-Moulton is the Nordsieck-method (see Ref. 3). The n-th order Nordsieck-algorithm computes in each integration step the desired approximation of the solution together with its derivatives up to order n. It is not a simple task to compute the coefficients of the difference scheme for higher order methods we are interested in (see Ref. 3).

computed them ourselves by means of a formula-manipulator. They are available on request. The operational orbit generator works with an 8th order method.

The next problem we faced was the initialisation of the algorithm, a well-known task in multi-step methods. We went two different ways for two types of application.

The first method improves iteratively some given rough initial values by means of a forward-backward integration within the integrator proper. The initial values can be taken from a two-body Kepler motion. For considerations on the convergence of the method we refer to Ref. 5. This method may fail if the intial values are bad which will typically happen when the start point of the integration is close to a perturbing planet, e.g. at departure of a probe from earth.

In this latter case we use the single-step method of Everhart (see Ref. 6) as starter of the Nordsieck integrator. It provides in each integration step exactly what we need, the solution and its derivatives. But it is a bit more time consuming than the iterative initialisation and it requires more software.

5. CONCLUSION

We have shown that a few rather classical considerations on numerical integration methods and their stability led to a special design of the orbit integrators in the ESOC orbit determination software for interplanetay flights. This software was e.g. used for the frequent determinations of the orbit of comet Halley as mentioned in the introduction. It needed about 1 min of CPU time on the Siemens computer for one iteration in the least-squares process if one processed data from 3 apparitions. This compares favorably to the 30 CPU-minutes that were consumed by the corresponding software developed on behalf of the IHW at JPL when it was running on the SIEMENS on request of the GIOTTO project for comparison purposes.

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