THE USE OF EULER PARAMETERS IN THE ROTATION OF A SATELLITE NEAR L_4

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ABSTRACT

The rotational motion of a rigid body satellite whose center of masses is moving near the libration point L_4 is studied by using the Euler parameters. The Hamiltonian form of the equations of motion is obtained, and some inherent difficulties of the problem (singularities, triaxial complexity, etc.) do not appear.

Keywords: Rigid-body Motion, Restricted Three-Body Problem, Euler Parameters.

1. INTRODUCTION

The problem of the motion of a rigid body satellite placed in the libration point L_4, has been dealt with by several authors.
Most of these authors assume that the center of mass of the rigid body is placed at the Lagragian point (Ref. 1-4).
On the other hand, some studies have been made supposing that the center of mass of the satellite moves along an arbitrary periodic orbit (Ref. 5-8). In the two last references, under assumption that the coordinates of the center of masses of the satellite are known functions of the time, the equations of the rotational motion in Andoyer variables are integrated by using the Lie-Deprit perturbation method.
The complexity of the problem increases when the rigid body is triaxial (Ref. 8) since the potential function is composed by more terms, and besides, the use of angle-action variables introduces elliptic functions which increases the difficulty. In both cases (axisymmetric and triaxial) some cases of resonance appear. In Ref. 7, the resonant cases are reduced to a generalized Ideal Resonance Problem.
However, some of this disadvantages can be avoided by using another more convenient set of variables, the Euler parameters.
In fact, the use of Euler parameters is of particular utility in numerical applications:
1) Euler parameters have no inherent geometrical singularity.
2) The elements of the rotation matrix are simple algebraic combinations of Euler parameters.
3) The time derivatives of Euler parameters are related with the angular velocity thrugh an orthogonal transformation.
For these reasons, we formulate the problem in Euler parameters, and by means of the Hamiltonian formalism given by Maciejewski (Ref. 11), give the equations of motion, which are a differential equation system of first order which may be solved by whatever numerical method.

2. EULER PARAMETERS

It is well known that in the motion of a rigid body what is of basic importance is Euler's theorem:
" the general displacement of a rigid body with one point fixed is a rotation about an axis ".
If ω is the amplitude of the rotation, and ($\cos\alpha$, $\cos\beta$, $\cos\gamma$) are the direction cosines of the axis, the rotational motion can be represented by means of the Euler parameters (q_1, q_2, q_3, q_4) given by the expresions

$$q_1 = \sin\frac{\omega}{2}\cos\alpha \qquad q_2 = \sin\frac{\omega}{2}\cos\beta$$
$$q_3 = \sin\frac{\omega}{2}\cos\gamma \qquad q_4 = \cos\frac{\omega}{2} \tag{1}$$

These four quantities are obviously related by the condition

$$|q|^2 = q_1^2 + q_2^2 + q_3^2 + q_4^2 \tag{2}$$

The Euler parameters can be expressed in terms of the Eulerian angles (ψ, θ, ϕ) by the relations

$$q_1 = \sin\frac{\theta}{2}\cos\frac{\psi-\phi}{2} \qquad q_2 = \sin\frac{\theta}{2}\sin\frac{\psi-\phi}{2}$$
$$q_3 = \cos\frac{\theta}{2}\sin\frac{\psi+\phi}{2} \qquad q_4 = \cos\frac{\theta}{2}\cos\frac{\psi+\phi}{2} \tag{3}$$

The inverse relations are

$$\psi = \operatorname{arctg}\frac{q_1q_3+q_2q_4}{q_1q_4-q_2q_3} \qquad \phi = \operatorname{arctg}\frac{q_1q_3-q_2q_4}{q_1q_4+q_2q_3}$$

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$$\theta = \arccos\left(q_3^2 + q_4^2 - q_1^2 - q_2^2 \right) \quad (4)$$

For more details about these relations see Ref. 9.

If we have two different coordinate systems, one fixed $OXYZ = \bar{X}$ and the other mobile $Oxyz = \bar{x}$, the direction-cosines of the two sets of axes with reference to each other are given by the following expression:

$$\bar{X} = C\,\bar{x} \quad (5)$$

where the components c_{ij} of the matrix C are

$$c_{11} = q_1^2 - q_2^2 - q_3^2 + q_4^2 \qquad c_{21} = 2(q_1q_2 + q_3q_4)$$

$$c_{31} = 2(q_1q_3 - q_2q_4) \qquad c_{12} = 2(q_1q_2 - q_3q_4)$$

$$c_{22} = -q_1^2 + q_2^2 - q_2^2 + q_4^2 \quad c_{32} = 2(q_2q_3 + q_1q_4)$$

$$c_{13} = 2(q_1q_3 + q_2q_4) \qquad c_{23} = 2(q_2q_3 - q_1q_4)$$

$$c_{33} = -q_1^2 - q_2^2 + q_3^2 + q_4^2$$

Besides, the parameters $(q_1'', q_2'', q_3'', q_4'')^t = \bar{q}''$ corresponding to the result of two succesive displacements $\bar{q}'$, $\bar{q}$ are given by the equations

$$\bar{q}'' = \begin{vmatrix} q_4' & q_3' & -q_2' & q_1' \\ -q_3' & q_4' & q_1' & q_2' \\ q_2' & q_1' & q_4' & q_3' \\ -q_1' & -q_2' & -q_3' & q_4' \end{vmatrix} \bar{q} = Q_1(q')\,\bar{q}$$

$$= \begin{vmatrix} q_4 & -q_3 & q_2 & q_1 \\ q_3 & q_4 & -q_1 & q_2 \\ -q_2 & q_1 & q_4 & q_3 \\ -q_1 & -q_2 & -q_3 & q_4 \end{vmatrix} \bar{q}' = Q\ (q)\,\bar{q}' \quad (6)$$

On the other hand, if we consider the rotational motion of a rigid body around its centre of masses, the relation between the components of the angular velocity ω_1, ω_2, ω_3 (referred to the principal axes of inertia frame, namely Oxyz), and the Eulerian angles ψ, θ, ϕ, is:

$$\begin{vmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{vmatrix} = \begin{vmatrix} \sin\theta\,\sin\phi & \cos\phi & 0 \\ \sin\theta\,\cos\phi & -\sin\phi & 0 \\ \cos\theta & 0 & 1 \end{vmatrix} \begin{vmatrix} \dot\psi \\ \dot\theta \\ \dot\phi \end{vmatrix}$$

By means of the relation between Euler parameters and Euler angles, the previous equation is transformed into:

$$\frac{d}{dt}\bar{q} = \tfrac{1}{2} Q_1(\bar{\omega})\,\bar{q} \quad (7)$$

where $\bar{\omega} = (\omega_1, \omega_2, \omega_3, 0)^t$

3. THE POTENTIAL FUNCTION

Let us consider the restricted circular three-body problem, in which the primaries 0_1, 0_2, of masses m_1, m_2, rotate around their mass centre O in circular orbits, and the satellite, which does not perturb the motion of the primaries is a rigid body, so that its centre of masses is moving in a neigbourhood of the Lagragian point L_4, so that its coordinates can be regarded as known functions of the time.

The potential acting on S_3, attracted by the two primaries, given in the Mac-Cullagh form up to the second order is then (see Ref. 8)

$$V = -G \sum_{i=1}^{2} m_i \left(\frac{1}{\zeta_i} + \frac{1}{2\zeta_i^3} \left(A+B+C-3I_i \right)\right) \quad (8)$$

where A, B, C, are the principal momenta of inertia, I_i is the momenta of inertia of S_3 with respect to the axis which joint 0_3 with 0_i, and $\zeta_i = \overline{0_i0_3}$.

In order to express the function V in terms of the Eulerian parameters, we choose as a fixed plane 0_3xy the plane of motion of the primaries, the axis 0_3z is the perpendicular line to this plane, and finally, the axis 0_3x coincides with the direction $\overline{0_10_2}$ (synodic system).

Under these conditions and bearing in mind the properties of Euler parameters, we have

$$\begin{vmatrix} \alpha_i \\ \beta_i \\ \gamma_i \end{vmatrix} = C(\bar{q}'') \begin{vmatrix} 1 \\ 0 \\ 0 \end{vmatrix} = C\left(Q_1(\bar{q}')q\right) \begin{vmatrix} 1 \\ 0 \\ 0 \end{vmatrix} \quad (9)$$

where $\bar{q}$ are the Euler parameters between to the fixed and the inertial frame, and $\bar{q}'$ are those corresponding to a twist of axis 0_3Z and argument $u_i = X0_30_i$. (See Fig. 1)

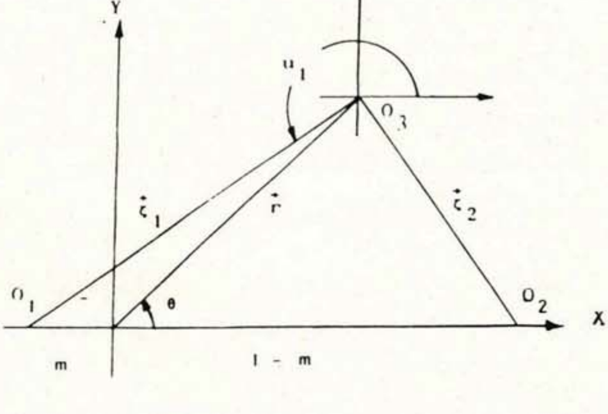

Figure 1. Geometry of the orbital motion

After some calculations, we obtain the following expressions for the direction cosines α_i, β_i, γ_i, of $\overline{0_30_i}$ (referred to the principal axes of inertia):

$$\alpha_i = (q_1^2-q_2^2-q_3^2+q_4^2)\cos u_i + 2(q_1q_2-q_3q_4)\sin u_i$$
$$\beta_i = 2(q_1q_2-q_3q_4)\cos u_i - (q_1^2-q_2^2-q_3^2+q_4^2)\sin u_i \tag{10}$$
$$\gamma_i = 2(q_3q_1+q_2q_4)$$

In Hamilton's equations of the motion, only the time derivatives of the rotational variables appears. For this reason, we can eliminate from the Hamiltonian function those terms which do not contain the rotational variables. With this assumption, the potential is

$$V = \frac{3}{2} G \sum_{i=1}^{2} \left(\frac{m_i}{\zeta_i}\right) \left((A-B)\alpha_i^2 + (C-B)\gamma_i^2\right) \tag{11}$$

where α_i^2, γ_i^2 are abtained from the Eqs. (10).

4. EQUATIONS OF MOTION

The next step consists in expressing the kinetic energy in a convenient set of canonical variables, of which the coordinates are the Euler parameters, so that the Hamiltonian equations may be used.

In 1982 Lidov (Ref. 10) gave a general method of obtaining the conjugate momenta of the Euler parameters, however, these parameters are obtained as functions of Euler angles, and their specific properties are not considered.

Recently Maciejewski (Ref. 11) has given another different way of obtaining the Hamiltonian function, based on Euler-Poinsot equations.

Essentially, the method may be summarized in the following theorem: " Let:

$$H(q,p,t) = \frac{1}{8} p^t Q(q)\tilde{I}Q(q)^t p + V(q,t) \tag{12}$$

$$\tilde{I} = \begin{vmatrix} I_1^{-1} & 0 & 0 & 0 \\ 0 & I_2^{-1} & 0 & 0 \\ 0 & 0 & I_3^{-1} & 0 \\ 0 & 0 & 0 & 0 \end{vmatrix}$$

and let $q=q(t)$, $p=p(t)$ be a solution of Hamilton's equations

$$\dot{q} = H_p^t$$
$$\dot{p} = -H_q^t \tag{13}$$

with the initial conditions:

$$q(0) = q_*, \quad p(0) = p_*, \quad |q_*|^2 = 1,$$

then, $q(t)$, $\omega(t)$, where

$$\omega(t) = P_3 \tilde{\omega}(t)$$
$$\tilde{\omega}(t) = \tfrac{1}{2} \tilde{I}\, Q(q(t))^t\, p(t)$$

$$P_3 = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{vmatrix}$$

are the solutions of Euler-Poinsot equations corresponding to an external torque given by a potential function $V=V(q,t)$ ".

With this result, we only have to integrate Hamilton's equations (Eqs. 13) corresponding to the previous Hamiltonian (Eqs. 12), $V(q,t)$ being given in the previous section (Eqs. 11).

By this procedure, the Hamiltonian function is:

$$H = \frac{1}{8}\left(\frac{T_1^2(p_i,q_j)}{A} + \frac{T_2^2(p_i,q_j)}{B} + \frac{T_3^2(p_i,q_j)}{C}\right) + $$
$$+ \sum_{i=1}^{2} \frac{3Gm_i}{2\zeta_i} \left((B-A)\beta_i^2(q_i,u_i) + (C-A)\gamma_i^2(q_i)\right)$$

where

$$T_1(p_i,q_j) = p_1q_4 + p_2q_3 - p_3q_2 - p_4q_1$$
$$T_2(p_i,q_j) = -p_1q_3 + p_2q_4 + p_3q_1 - p_4q_2$$
$$T_3(p_i,q_j) = p_1q_2 - p_2q_1 + p_3q_4 - p_4q_3$$

$\beta_i(q_i,u_i)$, $\gamma_i(q_i)$ are given by Eqs. (10), ζ_i, u_i are known functions of the time and the motion equations are:

$$\dot{q}_1 = \frac{1}{4}\left(\frac{T_1q_4}{A} - \frac{T_2q_3}{B} + \frac{T_3q_3}{C}\right)$$

$$\dot{q}_2 = \frac{1}{4}\left(\frac{T_1q_3}{A} + \frac{T_2q_4}{B} - \frac{T_3q_1}{C}\right)$$

$$\dot{q}_3 = \frac{1}{4}\left(\frac{T_2q_1}{B} + \frac{T_3q_4}{C} - \frac{T_1q_2}{A}\right)$$

$$\dot{q}_4 = \frac{1}{4}\left(-\frac{T_1q_1}{A} - \frac{T_2q_2}{B} - \frac{T_3q_3}{C}\right)$$

$$\dot{p}_1 = \frac{1}{4}\left(\frac{T_1p_4}{A} - \frac{T_2p_3}{B} + \frac{T_3p_2}{C}\right) -$$
$$- \sum_{i=1}^{2} \frac{3Gm_i}{\zeta_i}\left((B-A)\beta_i\frac{\partial\beta_i}{\partial q_1} + 2(C-A)\gamma_i q_3\right)$$

$$\dot{p}_2 = \frac{1}{4}\left(\frac{T_1p_3}{A} + \frac{T_2p_4}{B} - \frac{T_3p_1}{C}\right) -$$
$$- \sum_{i=1}^{2} \frac{3Gm_i}{\zeta_i}\left((B-A)\beta_i\frac{\partial\beta_i}{\partial q_2} + 2(C-A)\gamma_i q_4\right)$$

$$\dot{p}_3 = \frac{1}{4}\left(-\frac{T_1p_2}{A} + \frac{T_2p_1}{B} + \frac{T_3p_4}{C}\right) -$$
$$- \sum_{i=1}^{2} \frac{3Gm_i}{\zeta_i}\left((B-A)\beta_i\frac{\partial\beta_i}{\partial q_3} + 2(C-A)\gamma_i q_1\right)$$

$$\dot{p}_4 = \frac{1}{4}\left(-\frac{T_1p_1}{A} - \frac{T_2p_2}{B} - \frac{T_3p_3}{C}\right) -$$
$$- \sum_{i=1}^{2} \frac{3Gm_i}{\zeta_i}\left((B-A)\beta_i\frac{\partial\beta_i}{\partial q_4} + 2(C-A)\gamma_i q_2\right)$$

where

$$\frac{\partial \beta_i}{\partial q_1} = 2q_2 \cos u_i - 2q_1 \sin u_i$$

$$\frac{\partial \beta_i}{\partial q_2} = 2q_1 \cos u_i + 2q_2 \sin u_i$$

$$\frac{\partial \beta_i}{\partial q_3} = -2q_4 \cos u_i + 2q_3 \sin u_i$$

$$\frac{\partial \beta_i}{\partial q_4} = -2q_3 \cos u_i - 2q_4 \sin u_i$$

The system of Equations (13) consists in one differential system of 8 equations of first order, and may be solved by a classical numerical method.

A numerical study of this system and a comparison of the method above exposed and the given by Elipe (Ref. 8) will appear elsewhere.

5. CONCLUSIONS

The use of Euler parameters in the study of the rotation of a satellite near a Lagragian point has been introduced. This has some interesting advantages; the resonant cases are eliminated, equations of motion are differential equations of 1^{st} order with simple algebraic combinations of the variables; there is not more complexity in the triaxial case in relation with the axisymetric.

6. ACKNOWLEDGEMENTS

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7. REFERENCES

1. Kane T R & Marsh E 1971, *Cel Mech* 4, 78.

2. Robinson W J 1973, *Cel Mech* 8, 323.

3. Barkin Y V & Markov Y G 1980, *Kosm Issled* 18, 832.

4. Sidlichowky M 1982, *Bull Astron Inst Czechol* 33, 75.

5. Gamarnik N I 1981, *Sov Astron* 25, 753.

6. Krasilnikov P S 1982, *Sov Astron* 26, 90.

7. Elipe A & Férrer S 1986, *Bull Astron Inst Czechol* 37, 12.

8. Elipe A 1985, *Astrodynamics 1985* Vol 58 II, *Advances in the Astronautical Sciences*, 781, Kaufman et al. (Eds.), San Diego.

9. Whittaker E T 1965, A treatise on the Analytical Dynamics of Particles and Rigid Bodies, Dover Pub. Inc., New York.

10. Lidov M L 1982, *Kosm Issled* 20,2.

11. Maciejewski A J 1985, *Cel Mech* 37, 47.