TIME TRANSFORMATION AND LINEARIZATION ON RADIAL INTERMEDIARIES IN THE ZONAL EARTH ARTIFICIAL SATELLITE THEORY

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ABSTRACT

In artificial satellite theory, even including all zonal coefficients, Cid-Deprit's radial intermediary defines a system with only one degree of freedom describing the spacecraft's radial motion. In the case of small eccentricities, appropriate truncation of negligible terms produces a system integrable in closed form by elliptic integrals, giving r as a harmonic oscillator, provided the independent variable be changed to either a true or an eccentric anomaly. The solution is simple and concise enough that it could serve as the basis of an ephemeris generator running in real time on board a satellite.

Keywords: Artificial Satellite, Radial Intermediary, Time transformation.

1. INTRODUCTION

In previous papers (Refs. 1, 2) we have made a new treatment of the artificial satellite theory. Departing from Brouwer's approach and following the ideas of Refs. 3, 4, our first step consists of the transition to an intermediary. After the elimination of the perigee, we get the Cid-Deprit's radial intermediary (see Eq. 2) which is an integrable Hamiltonian with one degree of freedom.
Here we consider the case of orbits with small eccentricity, assuming that $\varepsilon^2 e^2 = O(\varepsilon^3)$ where $\varepsilon = J_2$ and e is the eccentricity, removing terms having $\varepsilon^2 e^2$ as a factor. Thus the intermediary reduces to

$$J = \frac{1}{2}\left(R^2 + \frac{\Theta^2}{r^2}\right) - \frac{b_1}{r} - \varepsilon\frac{b_3}{r^3} - \varepsilon^2\frac{b_2}{r^2} \quad . \tag{1}$$

with

$$b_1 = \mu,$$
$$b_2 = -\frac{3\Theta^2}{16p^4}\left(s^2(4s^2 - 3) + 16\sum_{n=2}^{\infty}\frac{J'_{2n}}{p^{2(n-2)}}B^0_{2n}\right),$$
$$b_3 = \mu\left(\frac{1}{2} - \frac{3}{4}s^2\right),$$

where $J'_{2n} = J_{2n}/J_2^2, p = \Theta^2/\mu, c = N/\Theta = \cos I, s^2 = 1-c^2$, I stands for the inclination of the satellite's orbital plane, and B^0_{2n} is a function of Θ and N. Then, instead of solving the problem by a Delaunay normalization (see Refs. 5, 6), which would lead to an approximate solution in powers of ε, we look for a closed form solution of the problem. Our aim is an ephemeris program that can be held in the memory of presently available microcomputers to be put on board satellites. To this effect we propose to integrate the differential equations after a transformation of the type $r = F(\rho)$, $dt = g(r)d\tau$, as in the Kepler problem, i.e. introducing 'anomalies' such that $\rho = \rho(\tau)$ is an harmonic oscillator. These transformations have been applied to some types of planar motions, in particular to central force-fields. The present paper shows that motions in three dimensions, as the one defined by J, may be analyzed with the same methods.
The time transformation giving $t = t(\tau)$ as a generalized Kepler equation is integrated in closed form by means of Jacobian elliptic integrals. This equation reduces to a Kepler equation for the unperturbed problem. The variables θ and ν result then in closed form by means of Jacobian elliptic integrals. This is in contrast with what happens with others analytical theories (see Refs. 7, 8) which have proposed a generalized Sundman's transformation of the form $dt = kr^{\alpha}dr$, $k, \alpha \in R$. With those theories, its time transformation is integrated numerically. In our case, despite the fact that the transformation we propose is more complex, nevertheless we give $t = t(\tau)$ in closed form. This is because our solution is based on the Hamiltonian J, and not on the initial Hamiltonian.

2. TRANSITION TO A RADIAL INTERMEDIARY

In this paper the polar nodal variables ($r, \theta, \nu, R, \Theta, N$) will be used. The variable r is the radial distance from the Earth's center of mass to the satellite, θ is the argument of latitude, and ν is the argument ofthe ascending node. The variables R, Θ, N are the momenta conjugate to the coordinates r, θ, ν respectively.

It is assumed that the satellite moves in an axially symmetric gravitational field whose potential is of the form

$$V = -\frac{\mu}{r}\left(1 - \sum_{n=2}^{\infty} J_n\left(\frac{\alpha}{r}\right)^n P_n(\sin\phi)\right) \quad .$$

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where α stands for the equatorial radius (from now on we will take $\alpha = 1$), $P_n(\sin\phi)$ is the Legendre polynomial of degree n in $\sin\phi$ and ϕ is the satellite's declination.

Denoting by

$$H = \frac{1}{2}\left(R^2 + \frac{\Theta^2}{r^2}\right) + V$$

the Hamiltonian function, we make two canonical transformations of Lie type (Ref. 9). The result is Cid-Deprit's radial intermediary

$$\overline{H} = \frac{1}{2}\left(R^2 + \frac{\Theta^2}{r^2}\right) - \frac{\mu}{r} - \varepsilon\frac{\mu}{r^3}\left(\frac{1}{2} - \frac{3}{4}s^2\right)$$

$$+ \frac{1}{2}\varepsilon^2\ \frac{\Theta^2}{r^2p^4}\ \left(\frac{3}{8}s^2(4s^2-3) + 2\sum_{n=2}^{\infty}\frac{J'_{2n}}{p^{2(n-2)}}B^0_{2n}\right) \quad (2)$$

$$+ \frac{1}{2}e^2\left[\frac{3}{32}s^2(23s^2-16) + \right.$$

$$\left. + \sum_{n=2}^{\infty}\frac{J'_{2n}}{p^{2(n-2)}}B^0_{2n}\ \sum_{j=1}^{n-1}\binom{2n-1}{2j}\binom{2j}{j}\left(\frac{e}{2}\right)^{2j-2}\right]$$

which is an integrable Hamiltonian of one degree of freedom in (r, R). As one of its features let us mention that the system defined by $\overline{H}$ still contains short periodic terms due to the factors r^{-3} and r^{-2}. Yet the angles θ and ν are ignorable which results, among other things, in the orbital plane having a fixed inclination.

We consider here the case of orbits of small eccentricity, such that $\varepsilon^2e^2 = O(e^3)$. We remove those negligible terms, and then, Cid-Deprit's radial intermediary reduces to the Hamiltonian J mentioned in Eq. 1.

The differential system defined by J is

$$\frac{dr}{dt} = \frac{\partial J}{\partial R} = R, \quad \frac{dR}{dt} = -\frac{\partial J}{\partial r} \quad (3)$$

and two quadratures for the variables θ and ν

$$\theta - \theta_0 = \int\frac{\partial J}{\partial\Theta}, \quad \nu - \nu_0 = \int\frac{\partial J}{\partial N} \quad . \quad (4)$$

Although the system Eq. 3 may be solved by a quadrature using Eq. 1, we will follow the classical way introducing 'anomalies' and the corresponding 'generalized Kepler equation'. For an approximate solution of this problem see Ref. 2.

3. TIME TRANSFORMATION AND LINEARIZATION OF THE RADIAL MOTION

It is well known that, in the Kepler problem, the radial motion reduces to a harmonic oscillator when we make the time transformation $t \to E$ given by

$$dt = \frac{r}{na}dE \ ,$$

n standing for the mean motion. Then we have

$$r = a(1 - e\cos E),$$

$$\tan\frac{\theta-\theta_0}{2} = \sqrt{\frac{1+e}{1-e}}\tan\frac{E}{2} \ ,$$

$$n(t - t_0) \equiv E - e\sin E \ ,$$

where θ_0 is the argument of the perigee.

If we change not only the independent variable but also the variable r: $(r,t) \to (\rho,f)$, by means of the formulas

$$r = \frac{1}{\rho} \ , \quad dt = \frac{1}{\sqrt{\mu p}}r^2\,df \ ,$$

we face yet another harmonic oscillator in the following manner

$$\rho = \frac{1}{p}(1 + e\cos f) \ ,$$

$$\theta = \theta_0 + f \ ,$$

$$n(t - t_0) = E - e\sin E \ ,$$

$$\tan\frac{E}{2} = \sqrt{\frac{1-e}{1+e}}\tan\frac{f}{2} \quad .$$

The aim of our research is to analyze 'natural' estensions of those transformations when we consider the radial intermediary J.

Let us consider a change of variables of the form $(r,t) \to (\rho,\tau)$ given by

$$r = F(\rho) \ , \quad dt = g(r)d\tau \ , \quad (5)$$

where $F^*(\rho) = dF/d\rho \neq 0$ and $g(r) > 0$ in the domain of motion. If one denotes

$$\overline{V} = 2\left(h + \frac{b_1}{r} - \frac{\Theta^2 - \varepsilon^2 b_2}{r^2} + \varepsilon\frac{b_3}{r^3}\right) \quad ,$$

the integral of the energy may be written in the form

$$\frac{1}{2}\left(\frac{dr}{dt}\right)^2 = \frac{1}{2}\overline{V} \ .$$

Making the transformation Eq. 5 we get

$$\frac{d^2\rho}{d\tau^2} = \frac{1}{2}\frac{d}{d\rho}\left(\frac{g^2\overline{V}}{F^{*2}}\right) \ . \quad (6)$$

In particular we look for functions g and F such that

$$\frac{d^2\rho}{d\tau^2} = A\rho + B \quad , \quad (7)$$

where A and B are constants. In case $A < 0$, with ρ_p and ρ_a as the extrema of ρ, the solution of the harmonic oscillator Eq. 7 is

$$\rho = \sigma(1 - \eta\cos\gamma) \quad (8)$$

where

$$\sigma = \frac{\rho_a + \rho_p}{2} \ , \quad \eta = \frac{\rho_a - \rho_p}{\rho_a + \rho_p} \quad , \quad \gamma = \tau\sqrt{-A} + \lambda$$

with $\rho_p \le \rho_a$ and λ is a constant (see Ref. 10, 11 for details).

From Eqs. 6, 7 we get

$$g = F^*\sqrt{\frac{A\rho^2 + 2B\rho + C}{\overline{V}}} \quad , \quad (9)$$

where C is a constant.

In our problem, looking for 'elliptic' motions, the equation $\overline{V} = 0$ has three real roots $\varepsilon r_1 < r_p \le r_a$, and we may write

$$\frac{1}{2}\overline{V} = \frac{h}{r^3}(r - r_a)(r - r_p)(r - \varepsilon r_1) \ , \quad (10)$$

where r_p, $r_a > 0$ and r_1 may be ≥ 0 or ≤ 0 depending on whether b_3 is ≥ 0 or ≤ 0 respectively.

In this paper we consider the transformation Eq. 5 such that $F(\rho) = \rho$. Then from Eqs. 9, 10 we obtain

$$g(r) = \frac{r^{3/2}}{\sqrt{-2h/A}\sqrt{r - \varepsilon r_1}} \tag{11}$$

Now, we write Eq. 8 in the form

$$r = \bar{a}(1 - \bar{e}\cos\bar{E}) \tag{12}$$

with

$$\bar{E} = \tau\sqrt{-A},\quad \bar{a} = \frac{r_a + r_p}{2},\quad \bar{e} = \frac{r_a - r_p}{r_a + r_p},\quad \bar{p} = \bar{a}(1 - \bar{e}^2),$$

where we take $\bar{E} = 0$ when $\tau = 0$, i.e. $\lambda = 0$. Then, we may introduce the 'true' anomaly $\bar{f}$, such that

$$\tan\frac{\bar{E}}{2} = \sqrt{\frac{1-\bar{e}}{1+\bar{e}}}\tan\frac{\bar{f}}{2}\ . \tag{13}$$

In the polar coordinates $(r, \bar{f})$, Eqs. 12, 13, define an ellipse (ε). According to the expression of $g(r)$, a scale factor A exists between the independent variables t and τ. We chose here $A=-1$ and we have $\tau = \bar{E}$, which means that the new independent variable is the eccentric anomaly for the ellipse (ε).

4. GENERALIZED KEPLER EQUATION

From Eq. 11 and the previous paragraph we have

$$dt = \frac{r^{3/2}}{\sqrt{-2h(r - \varepsilon r_1)}}\, d\bar{E}\quad ,$$

where r is given by Eq. 12. Making the change $\cos\bar{E} = y$ and integrating, we may write

$$t - t_0 = -\frac{\bar{a}\,\bar{e}}{\sqrt{-2h}}\int_{y_0}^{y}\frac{(y-1/\bar{e})^2}{(y-1/\tilde{e})(y-1/\bar{e})(1-y)(1+y)}\,dy, \tag{14}$$

where

$$\tilde{e} = \frac{\bar{a}\,\bar{e}}{\bar{a} - \varepsilon r_1}$$

and we assume that $r(t_0) = r_p$, thus $\bar{E}(t_0) = 0$ and $y_0 = y(t_0) = 1$. We consider here the case : $r_1 < 0, (\tilde{e} < \bar{e})$. According to Ref. 12, Formula 253.24, Eq. 14 takes the form

$$\frac{\sqrt{-2h}}{\bar{a}}(t - t_0) = C_1F(\bar{f}/2,k) + C_2E(\bar{f}/2,k) + C_3\Pi(\bar{f}/2,\alpha^2,k) + C_4\,\frac{\sin\bar{f}\,\sqrt{1-k^2\sin^2(\bar{f}/2)}}{1-\alpha^2\sin(\bar{f}/2)} \tag{15}$$

In this formula,

$$\alpha^2 = \frac{2\bar{e}}{1+\bar{e}}\ ,\quad k^2 = \frac{2(\bar{e}-\tilde{e})}{(1-\tilde{e})(1+\bar{e})}\ ,\quad k^2 < \alpha^2$$

$$C = \bar{a}(1+\bar{e})\{\bar{a}(1-\bar{e}) - \varepsilon r_1\}^{-1/2}$$

and

$$C_1 = -C\,a(1 - \bar{e}^2),$$

$$C_2 = C(1+\bar{e})\{\bar{a}(1-\bar{e}) - \varepsilon r_1\}\ ,$$

$$C_3 = C(1-\bar{e})(2\bar{a} + \varepsilon r_1)\ ,$$

$$C_4 = C\bar{e}\{-\bar{a}(1-\bar{e}) + \varepsilon r_1\}\ ,$$

where F, E and Π are the normal elliptic integrals of first, second and third kind respectively. It is worth to notice that the generalized Kepler equation is given explicitly in terms of $\bar{f}$ instead of the eccentric $\bar{E}$, as in the Kepler problem. In the case $r_1 > 0$ analogous expressions can be obtained.

5. THE QUADRATURES

5.1 The position of $\bar{r}$ in the orbital plane.

From the Eq. 4 we have

$$\frac{d\theta}{dt} = \frac{P_1}{r^2} + \varepsilon\frac{P_2}{r^3}\quad ,$$

where

$$P_1 = \Theta - \varepsilon^2\frac{u^4}{32\Theta^7}\{12(12s^4 - 14s^2 + 3) + J_4'(525s^4 - 690s^2 - 192)\}\ ,$$

$$P_2 = -\frac{3\mu(1 - s^2)}{2\Theta}$$

and we have only retained J_4' in order to simplify the expressions. From Eq. 4, applying Eq. 3 and integrating we get

$$\theta - \theta_0 = \frac{1}{\sqrt{-2h}}(P_1I_0 + \varepsilon P_2I_1), \tag{16}$$

where

$$I_m = \int_{a_2}^{r}\frac{dr}{r^m\sqrt{(a_1 - r)(r - a_2)(r - a_3)(r - a_4)}}\ ,\quad m = 0,1$$

with $a_1 \geq r > a_2 > a_3 > a_4$. We consider here the case $r_1 < 0$. Then $a_1 = r_a$, $a_2 = r_p$, $a_3 = 0$ and $a_4 = \varepsilon r_1$. From Ref. 12, Formula 331.01, the solution of Eq. 16 is given by

$$\theta - \theta_0 = L_1F(\bar{f}/2,k) + L_2E(\bar{f}/2,k)$$

where

$$L = \frac{2}{\sqrt{-2hr_a(r_p - \varepsilon r_1)}}\ ,\quad L_1 = L(P_1 + \frac{P_2}{r_1}),\quad L_2 = LP_2\left(-\frac{1}{r_1} + \frac{\varepsilon}{r_p}\right) \tag{17}$$

and the modulus k is given in Section 4. For the case $r_1 > 0$ we get similar expressions.

5.2 The motion of the node

From Eq. 4 we have

$$\frac{d\nu}{dt} = \varepsilon\left(\frac{Q_2}{r^3} + \varepsilon\frac{Q_1}{r^2}\right)$$

where

$$Q_1 = -\frac{\Theta c}{16p^4}\{6(8s^2 - 3) + J_4'(105s^2 - 60)\}\ ,$$

$$Q_2 = 3\mu c/2\Theta.$$

Considering again the case $r_1 < 0$, the solution is similar to Eq. 16, substituting ε^2Q_1 and Q_2 for P_1 and P_2 respectively. The result is

$$\nu - \nu_0 = L_1^*F(\bar{f}/2,k) + L_2^*E(\bar{f}/2,k) \tag{18}$$

where

$L_1^* = L(\varepsilon^2 Q_1 + \frac{Q_2}{r_1})$, $L_2^* = LQ_2(-\frac{1}{r_1} + \frac{\varepsilon}{r_p})$

and L is given by Eq, 17.

6. THE PARTICULAR CASE $r_1 = 0$

When $b_3 \to 0$, which means that $\tan I \to \sqrt{2}$, we arrive to the case $r_1 \to 0$ which needs a particular analysis, because some previous expressions lose their meaning. After a development in k of the elliptic integrals, the formulas Eqs. 15, 16 and 18 reduce to the following ones

$$\sqrt{\mu/a^3}(t - t_0) = \overline{E} - \overline{e}\sin\overline{E} ,$$
$$\theta - \theta_0 = \frac{1}{\sqrt{-2hr_a r_p}}\left[(P_1 + \varepsilon P_2 \frac{1}{\overline{p}})\overline{f} + \varepsilon P_2 \frac{\overline{e}}{\overline{p}} \sin\overline{f}\right],$$
$$\nu - \nu_0 = \varepsilon\frac{1}{\sqrt{-2hr_a r_p}}\left[(Q_2\frac{1}{\overline{p}} + \varepsilon Q_1)\overline{f} + Q_2\frac{\overline{e}}{\overline{p}} \sin\overline{f}\right].$$

6. CONCLUSION

For a long time, intermediaries have been viewed in satellite theory as providing a simpler way of developing an accurate ephemeris than the conventional Delaunay normalizations. Recently it has been suggested that they can also serve as orbit generators over long arcs. In this regard, they have the marked advantage of being so concise and so fast that they may be contained in very small microcomputers embarked on a satellite. An orbit generator based on Cid-Deprit's intermediary includes the long term perturbations of first and second order; it covers not only J_2 but zonal harmonics coefficients to an arbitrary order. It is probably the first one to be so extensive. The present paper took care of all details needed to convert the algorithm into an efficient computer code.

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