ANGULAR-RATE ESTIMATION USING QUATERNION MEASUREMENTS

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Abstract

This paper presents algorithms for estimating the angular-rate vector of satellites using quaternion measurements. Two approaches are compared, one that uses differentiated quaternion measurements to yield coarse rate measurements which are then fed into two different estimators. In the other approach the raw quaternion measurements themselves are fed directly into the two estimators.

The two estimators rely on the ability to decompose the non-linear rate dependent part of the rotational dynamics equation of a rigid body into a product of an angular-rate dependent matrix and the angular-rate vector itself. This decomposition, which is not unique, enables the treatment of the nonlinear spacecraft dynamics model as a linear one and, consequently, the application of a Pseudo-Linear Kalman Filter (PSELIKA). It also enables the application of a special Kalman filter which is based on the use of the solution of the State Dependent Algebraic Riccati Equation (SDARE) in order to compute the Kalman gain matrix and thus eliminates the need to propagate and update the filter covariance matrix. The replacement of the elaborate rotational dynamics by a simple first order Markov model is also examined.

In this paper a special consideration is given to the problem of delayed quaternion measurements. Two solutions to this problem are suggested and tested. Real Rossi X-Ray Timing Explorer (RXTE) satellite data is used to test these algorithms, and results of these tests are presented.

Key words: Quaternion, Spacecraft, Angular-rate, filtering, non-linear filtering.

Introduction

In most spacecraft (SC) there is a need to know the SC angular-rate. Precise angular-rate is required for attitude determination, and a coarse rate is needed for attitude control damping. Classically, angular-rate information is obtained from gyro measurement. These days, there is a tendency to build smaller, lighter and cheaper SC, therefore the inclination now is to do away with gyros and use other means and methods to determine the SC angular-rate. The latter is also needed even in gyro equipped satellites when performing high rate maneuvers whose angular-rate is out of range of the SC gyros.

There are several ways to obtain the angular-rate in a gyro-less SC. When the attitude is known, one can differentiate the attitude in whatever parameters it is given and use the kinematics equation that connects the derivative of the attitude with the satellite angular-rate in order to compute the latter[1]. Since SC usually utilize vector measurements for attitude determination, the differentiation of the attitude introduces a considerable noise component in the computed angular-rate vector. To overcome this noise, the computed rate components can be filtered by a passive low pass filter. This, however, introduces a delay in the computed rate[1]. When using an active filter, like a Kalman filter (KF), the delay can be eliminated[2,3].

Another approach may also be adopted to the problem of angular-rate computation where the vector measurements themselves are differentiated. This approach was used by Natanson[4] for estimating attitude from magnetometer measurements, and by Challa, Natanson, Deutschmann and Galal[5] to obtain attitude as well as rate. Similarly, Challa, Kotaru and Natanson[6] used derivatives of the earth magnetic field vector to obtain attitude and rate.

All these methods use the derivative of either the attitude parameters or of the measured directions which normally determine the attitude parameters. Another approach is that of using the attitude

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parameters, or the measured directions themselves, as measurements in some kind of a KF. In this case the kinematics equation that connects the attitude parameters, or the directions, with their derivatives are included in the dynamics equation used by the filter thereby, as will be shown in the ensuing, *the need for differentiation is eliminated*[7,8].

New sensor packages have been introduced lately that yield the SC attitude in terms of the attitude quaternion[9]. Therefore it is possible to use the quaternion supplied by such sensors as measurements and, as mentioned before, eliminate the need for differentiation. In this paper we investigate this possibility.

As mentioned, in the ensuing we will apply two special KFs which make use of the SC angular dynamics model; therefore, by way of introduction, in the next section we present the development of the SC dynamics model, and in Section III we present the two filters. For comparison purposes, in Section IV we treat the approach where the angular-rate is still extracted from derivative but here we pass the resultant noisy quaternion through the two active rather than through a passive filter as was done in Ref. 2. The other approach, where the raw quaternion measurements themselves are fed into the filter, requires the addition of the quaternion to the state vector which is comprised of the angular-rate vector. This is treated in Section V. In Section VI we consider the case where the filter dynamics is drastically simplified by reducing the dynamics equation of the SC to a first order Markov process. The issue of quaternion normalization is presented in Section VII, and in Section VIII we solve the problem of measurement delay. The last section of this work is the Conclusion section.

II. Filter Dynamics Model

The main dynamics model is that which describes the propagation of the SC angular velocity, $\underline{\boldsymbol{\omega}}$. The angular dynamics of a constant mass SC is given in the following equation[10]

$$\dot{\underline{\boldsymbol{\omega}}}\mathrm{I} + \dot{\mathbf{h}} + \underline{\boldsymbol{\omega}} \times (\mathrm{I}\underline{\boldsymbol{\omega}} + \mathbf{h}) = \mathbf{T} \tag{1}$$

where $\underline{\boldsymbol{\omega}}^T = [\omega_x, \omega_y, \omega_z]$, I is the SC inertia tensor, $\mathbf{h}$ is the momentum of the momentum wheels, and $\mathbf{T}$ is the external torque operating on the SC. The components ω_x, ω_y and ω_z are the three components of the sought angular-rate vector, $\underline{\boldsymbol{\omega}}$, of the SC body with respect to inertial space when resolved in the body coordinates. Eq. (1) can be written as

$$\dot{\underline{\boldsymbol{\omega}}} = \mathrm{I}^{-1}[(\mathrm{I}\underline{\boldsymbol{\omega}} + \mathbf{h})\times]\underline{\boldsymbol{\omega}} + \mathrm{I}^{-1}(\mathbf{T} - \dot{\mathbf{h}}) \tag{2}$$

where $[(\mathrm{I}\underline{\boldsymbol{\omega}} + \mathbf{h})\times]$ is the cross product matrix of the vector $(\mathrm{I}\underline{\boldsymbol{\omega}} + \mathbf{h})$. Define

$$\mathrm{F}(\underline{\boldsymbol{\omega}}) = \mathrm{I}^{-1}[(\mathrm{I}\underline{\boldsymbol{\omega}} + \mathbf{h})\times] \tag{3}$$

and

$$\mathbf{u}(t) = \mathrm{I}^{-1}(\mathbf{T} - \dot{\mathbf{h}}) \tag{4}$$

then Eq. (2) can be written in the form

$$\dot{\underline{\boldsymbol{\omega}}} = \mathrm{F}(\underline{\boldsymbol{\omega}})\underline{\boldsymbol{\omega}} + \mathbf{u}(t) \tag{5}$$

As was shown in Ref. 2, there are 8 primary models, and infinite linear combinations of them, which express Eq. (1) in the form of Eq. (5). Eq. (5) describes the SC correct dynamics; however, we usually do not know the exact values of I, T, $\mathbf{h}$ and its derivative, therefore we do not know the exact relationship between $\dot{\underline{\boldsymbol{\omega}}}$ and these elements. We express our lack of knowledge by adding a stochastic process to the dynamics equation of Eq. (1). We assume that this stochastic process, $\mathbf{w}(t)$, is a zero mean white noise process. The resulting model which is used by the estimator is

$$\dot{\underline{\boldsymbol{\omega}}} = \mathrm{F}(\underline{\boldsymbol{\omega}})\underline{\boldsymbol{\omega}} + \mathbf{u}(t) + \mathbf{w}'(t) \tag{6}$$

If we denote $\underline{\boldsymbol{\omega}}$ by $\mathbf{x}$, then Eq. (6) can be written as

$$\dot{\mathbf{x}} = \mathrm{F}(\mathbf{x})\mathbf{x} + \mathbf{u}(t) + \mathbf{w}'(t) \tag{7}$$

where obviously

$$\mathrm{F}(\mathbf{x}) = \mathrm{I}^{-1}[(\mathrm{I}\mathbf{x} + \mathbf{h})\times] \tag{8}$$

For the time being we assume that we measure the angular-rate; that is, $\mathbf{x}$, therefore the measurement equation is

$$\mathbf{z}_k = \mathrm{H}\mathbf{x}_k + \mathbf{v}_k \tag{9}$$

where

$$\mathrm{H} = \mathrm{I}_3 \tag{10}$$

is a zero mean white measurement noise, and I_3 is $\mathbf{v}_k$ the third dimensional identity matrix.

III. Angular-rate Estimation

As mentioned in the introduction section, we use two filtering algorithms to estimate the angular-rate. These algorithms are described next.

The dynamics equation presented in Eq. (7) is a nonlinear differential equation due to the term $F(\mathbf{x})\mathbf{x}$. A standard filter for this case is the Extended Kalman Filter (EKF). One can also apply the Extended Interlaced Kalman filter[3] where three linear KFs are run in parallel. Other possibilities which are applicable to the form of non-linearity presented in Eq. (7) are the Pseudo-Linear Kalman (PSELIKA) filter and the State Dependent Algebraic Riccati Equation (SDARE) filter which were used successfully in Ref. 2. In view of their performance, the latter two filters are used in this work too.

III.1 The Pseudo-Linear Kalman (PSELIKA) Filter

The PSELIKA filter algorithm disregards the non-linearity and treats the dynamics system as if it were just a time varying system, consequently, the ordinary KF algorithm is applied. First, the continuous differential equation (7) expressing the SC dynamics is discretized and then the KF algorithm is applied as follows. First evaluate:

$$W'_k = E\{\mathbf{w}'(t_k)\mathbf{w}'(t_k)^T\} \tag{11}$$

$$R_k = E\{\mathbf{v}_k\mathbf{v}_k^T\} \tag{12}$$

and choose an approximate value for, $\hat{\mathbf{x}}_0$, the initial estimate of the rate vector. In the absence of such initial estimate, choose $\hat{\mathbf{x}}_0 = 0$. Next, determine P_0, the initial covariance matrix of the estimation error according to the confidence in the choice of $\hat{\mathbf{x}}_0$. The recurrence algorithm is then as follows.

***time propagation*: -**

Let A_k be the discrete dynamics matrix obtained when $F(\mathbf{x})$ of Eq. (8) is discretized, and let $\mathbf{u}_k$ be the discrete deterministic input signal, then propagate the state estimate according to:

$$\hat{\mathbf{x}}_{k+1/k} = A_k\hat{\mathbf{x}}_{k/k} + \mathbf{u}_k \tag{13.a}$$

and the covariance matrix according to:

$$P_{k+1/k} = A_k P_{k/k} A_k^T + W'_k \tag{13.b}$$

***measurement update*: -**

Compute the Kalman Gain as follows:

$$K_{k+1} = P_{k+1/k}H^T[HP_{k+1/k}H^T + R_{k+1}]^{-1} \tag{13.c}$$

Update the estimate according to:

$$\hat{\mathbf{x}}_{k+1/k+1} = \hat{\mathbf{x}}_{k+1/k} + K_{k+1}[\mathbf{z}_{k+1} - H\hat{\mathbf{x}}_{k+1/k}] \tag{13.d}$$

and update the covariance matrix using:

$$P_{k+1/k+1} = [I - K_{k+1}H]P_{k+1/k}[I_3 - K_{k+1}H]^T + K_{k+1}R_{k+1}K_{k+1}^T \tag{13.e}$$

III.2 The State Dependent Algebraic Riccati Equation (SDARE)

The continuous-discrete-time SDARE filter which was used in Ref. 2 was based on the work of Cloutier, D'Souza and Mracek[11,12], Pappano and Friedland[13], and Mracek, Cloutier and D'Souza[14]. That continuous-discrete-time filter for the continuous-time dynamics and the discrete-time measurement is as follows (see Ref. 2).

As with the PSELIKA filter, choose an approximate value for the initial estimate of the rate vector. In the absence of such initial estimate, choose again $\hat{\mathbf{x}}_0 = 0$.

- *time propagation*:

Propagate the state estimate according to:

$$\hat{\mathbf{x}}_{k+1/k} = A_k\hat{\mathbf{x}}_{k/k} + \mathbf{u}_k \tag{14}$$

- *measurement update*:

At the measurement updating time, t_{k+1}, solve the following algebraic Riccati equation for P_{k+1}:

$$A(\hat{\mathbf{x}}_{k+1/k})P_{k+1} + P_{k+1}A^T(\hat{\mathbf{x}}_{k+1/k}) - P_{k+1}H^T R_{k+1}^{-1} H P_{k+1} + W'_{k+1} = 0 \tag{15.a}$$

and compute the gain matrix:

(15.b) $$K_{k+1} = P_{k+1} H^T R^{-1}_{k+1}$$

Finally compute the updated state estimate:

(15.c) $$\hat{\mathbf{x}}_{k+1/k+1} = \hat{\mathbf{x}}_{k+1/k} + K_{k+1}[\mathbf{z}_{k+1} - H\hat{\mathbf{x}}_{k+1/k}]$$

IV. The Filtered Quaternion-Rate Approach

As mentioned before, it is possible to derive $\underline{\boldsymbol{\omega}}_r$, a crude estimate of $\underline{\boldsymbol{\omega}}$ using the quaternion first time-derivative[1,2]; however, the resultant estimate is noisy. If $\underline{\boldsymbol{\omega}}_r$ is passed through a passive low-pass filter the noise may be filtered out at the expense of a delay[1]. Here we investigate the quality of the filtered rates when the two active filters described before are used to filter $\underline{\boldsymbol{\omega}}_r$. First we show how $\underline{\boldsymbol{\omega}}_r$ is derived from $\dot{\mathbf{q}}$, the differentiated quaternion. As is well known [see e.g. Ref. 10], the quaternion dynamics equation is

(16) $$\dot{\mathbf{q}} = \tfrac{1}{2}\Omega\mathbf{q}$$

where

(17) $$\Omega = \begin{bmatrix} 0 & \omega_z & -\omega_y & \omega_x \\ -\omega_z & 0 & \omega_x & \omega_y \\ \omega_y & -\omega_x & 0 & \omega_z \\ -\omega_x & -\omega_y & -\omega_z & 0 \end{bmatrix}$$

It is also known [see e.g. Ref. 1] that Eq. (16) can be written as

(18) $$\dot{\mathbf{q}} = \tfrac{1}{2}Q\boldsymbol{\omega}$$

where

(19) $$Q = \begin{bmatrix} q_4 & -q_3 & q_2 \\ q_3 & q_4 & -q_1 \\ -q_2 & q_1 & q_4 \\ -q_1 & -q_2 & -q_3 \end{bmatrix}$$

Define the pseudo inverse

(20.a) $$Q^{\#} = (Q^T Q)^{-1} Q^T$$

where T denotes the transpose. Note that

(20.b) $$Q^T Q = I_3$$

where I_3 is the fourth dimensional identity matrix. From Eqs. (18) and (20.a) it is easily seen that a rough estimate of the rate vector can be computed as follows

(21.a) $$\underline{\boldsymbol{\omega}}_r = 2Q^{\#}\dot{\mathbf{q}}$$

which in view of Eqs. (20) can be written simply as

(21.b) $$\underline{\boldsymbol{\omega}}_r = 2Q^T\dot{\mathbf{q}}$$

The dynamics equation for the estimator was introduced in Section II (see Eq. 7); thus, in view of Eq. (21), like Eq. (10), the measurement equation which corresponds to that dynamics model is

(22.a) $$\underline{\boldsymbol{\omega}}_r = H_\omega \underline{\boldsymbol{\omega}} + v_\omega$$

where

(22.b) $$H_\omega = I_3$$

and $\mathbf{v}_\omega$ is a zero mean white noise.

The Pseudo-Linear Kalman Filter (PSELIKA) and the State Dependent Algebraic Riccati Equation (SDARE) filter were used to obtain the angular-rate from quaternion observations using the Quaternion-Rate approach. The data which was used to test this approach was real measurements downloaded from the RXTE satellite, which was launched on Dec. 30, 1995. We chose a segment of data starting Jan. 4, 1996 at 21 hours, 30 minutes, and 1.148 sec. The quaternion which was used was based on the SC attitude as determined by its star trackers. Fig. 1 presents $\underline{\boldsymbol{\omega}}$, the nominal angular-rate and Fig. 2 presents the error between $\underline{\boldsymbol{\omega}}_r$, the raw angular-rate, and $\underline{\boldsymbol{\omega}}$, the nominal rate. In order to quantify the error, a single figure of merit (FM) is computed. First the average square error of each component is computed as follows $\overline{e_i^2} = \frac{1}{T-t_0}\int_{t_0}^{T} e_i^2 dt$ i=x, y, z. This computation yields $\overline{e_x^2}, \overline{e_y^2}$ and $\overline{e_z^2}$. Then the FM is computed as $FM = \sqrt{\overline{e_x^2} + \overline{e_y^2} + \overline{e_z^2}}$. In order to exclude the transients we set $t_0 = 100\,\text{sec}$. It was found that $FM(2) = 7.3998 \cdot 10^{-3}$ deg/sec where FM(2) is the FM of Fig. 2. Fig. 3 presents the estimation error when the PSELIKA filter was applied to $\underline{\boldsymbol{\omega}}_r$. It was found that $FM(3) = 1.5311 \cdot 10^{-3}$ deg/sec. Finally, Fig. 4 shows the same when the SDARE filter was used and it was found that $\mathrm{FM}(4) = 1.4550 \cdot 10^{-3}$ deg/sec. As indicated by FM(2), the computed angular-rate, $\underline{\boldsymbol{\omega}}_{\mathrm{r}}$, particularly its x component, was rather noisy. When either the PSELIKA or the SDARE filter were applied to $\underline{\boldsymbol{\omega}}_{\mathrm{r}}$, other than a few spikes, the resulting $\hat{\underline{\boldsymbol{\omega}}}$ was smoother. In this example there was no real difference between the performance of the two filters (see FM(3) and FM(4)).

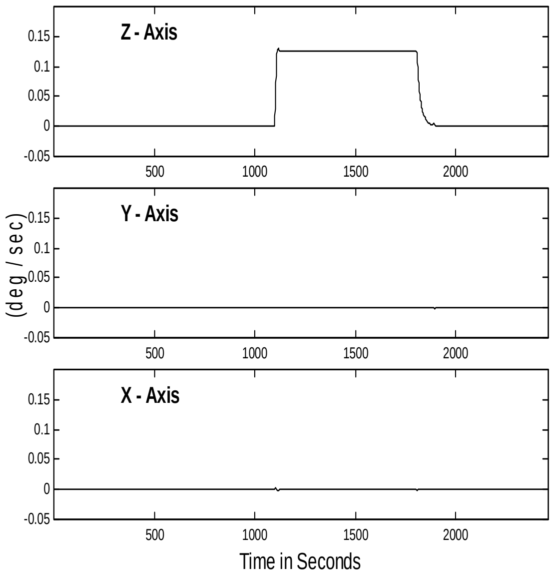

Fig. 1: The Nominal Angular-Rate.

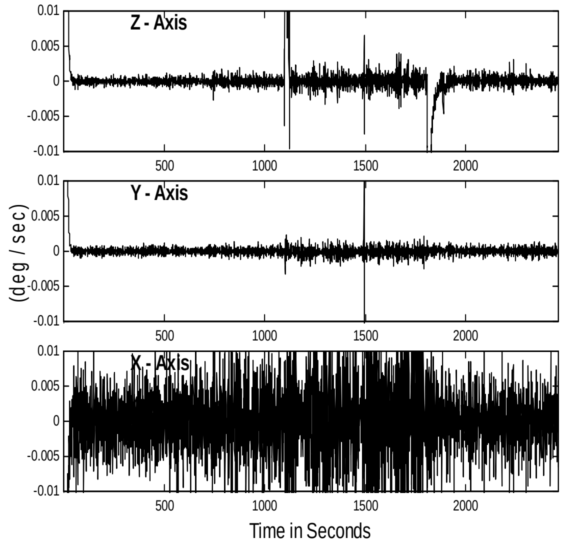

Fig. 2: The Error Between the Raw Angular Rate, $\underline{\boldsymbol{\omega}}_{\mathrm{r}}$, and the Nominal Angular-Rate.

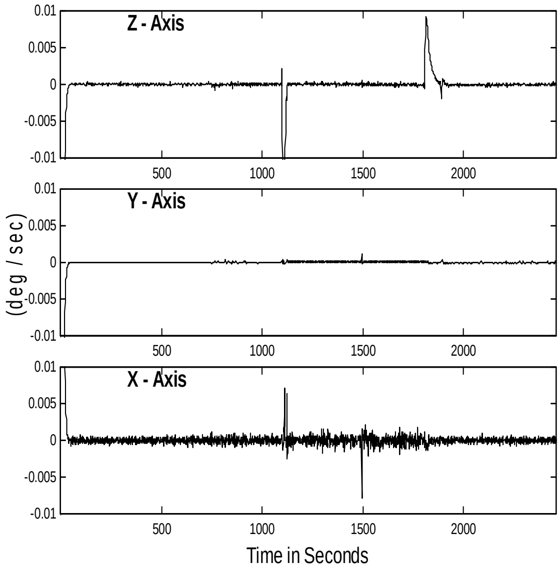

Fig. 3: The Error in the Estimated Angular-Rate, $\hat{\underline{\boldsymbol{\omega}}}$, After the PSELIKA Filter was Applied to $\underline{\boldsymbol{\omega}}_{\mathrm{r}}$.

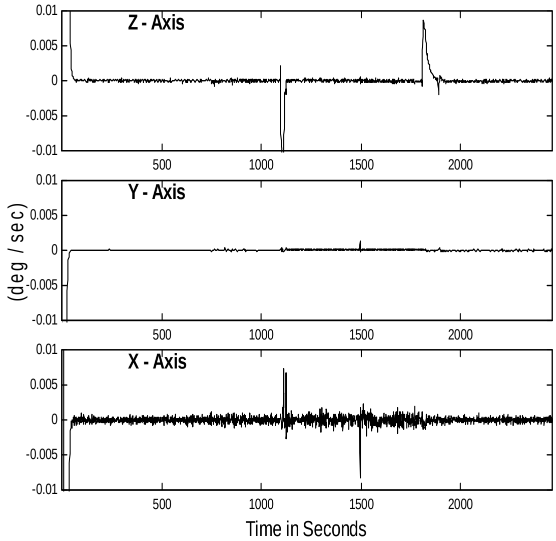

Fig. 4: The Error in the Estimated Angular-Rate, $\hat{\underline{\boldsymbol{\omega}}}$, After the SDARE Filter was Applied to $\underline{\boldsymbol{\omega}}_{\mathrm{r}}$.

As expected, the computation of $\underline{\boldsymbol{\omega}}_{\mathrm{r}}$ using Eq. (21) produced a noisy estimate due to the differentiation of the measured quaternion which was corrupted by measurement noise, and the application of the PSELIKA filter to this $\underline{\boldsymbol{\omega}}_r$ filtered out most of the noise. When the SDARE rather than the PSELIKA filter was applied to $\underline{\boldsymbol{\omega}}_r$, the filtered estimate of the angular-rate was visually identical. In other words, the effect of the application of the SDARE filter was practically identical to that of the PSELIKA filter.

V. The Quaternion Augmentation Approach

Although we also tested the Quaternion-Rate approach described in the preceding section, in this work we are mainly interested in estimating $\underline{\boldsymbol{\omega}}$ *using the measured quaternion itself* rather than its derivative. However, the quaternion is not a part of the state vector of the system (see Eqs. 6, 7). One solution to this problem was examined in the preceding section. Another solution is the augmentation of the quaternion with the angular-rate state of Eqs. (6, 7). For this we can use the quaternion dynamics equation given in Eq. (16) and obtain the following model which augments Eqs. (6) and (16)

$$\dot{\mathbf{y}} = G'(\mathbf{y})\mathbf{y} + \mathbf{e}(t) + \mathbf{g}(t) \tag{23}$$

where

$$\mathbf{y} = \begin{bmatrix} \underline{\boldsymbol{\omega}} \\ \mathbf{q} \end{bmatrix} \tag{24.a}$$

$$G'(\mathbf{y}) = \begin{bmatrix} F(\underline{\boldsymbol{\omega}}) & 0 \\ 0 & \tfrac{1}{2}\Omega \end{bmatrix} \tag{24.b}$$

$$\mathbf{e}(t) = \begin{bmatrix} \mathbf{u}(t) \\ 0 \end{bmatrix} \tag{24.c}$$

$$\mathbf{g}(t) = \begin{bmatrix} \mathbf{w}'(t) \\ 0 \end{bmatrix} \tag{24.d}$$

The measurements of the quaternion are taken at discrete time points; therefore the measurement model is a discrete one. The discrete measurement model that corresponds to the dynamics model of Eq. (23) is

$$\mathbf{q}_{m,k} = C\begin{bmatrix} \underline{\boldsymbol{\omega}} \\ \mathbf{q} \end{bmatrix}_k + \mathbf{v}_k \tag{25}$$

where $\mathbf{q}_{m,k}$ is the measurement at time t_k,

$$C = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{26}$$

and $\mathbf{v}_k$ is the measurement noise at that time. An inspection of the matrices $G'(\mathbf{y})$ of Eq. (24.b) and C of the last equation reveals that even when $\underline{\boldsymbol{\omega}}$ is constant this pair is deterministically unobservable. This problem can be overcome though using the fact that Eq. (16) can be written as Eq. (18) which can also be written as

$$\dot{\mathbf{q}} = [\tfrac{1}{2}Q \,|\, 0]\begin{bmatrix} \underline{\boldsymbol{\omega}} \\ \mathbf{q} \end{bmatrix} \tag{27}$$

therefore Eq. (23) can be transformed into

$$\dot{\mathbf{y}} = G(\mathbf{y})\mathbf{y} + \mathbf{e}(t) + \mathbf{g}(t) \tag{28.a}$$

where

$$G(\mathbf{y}) = \begin{bmatrix} F(\underline{\boldsymbol{\omega}}) & 0 \\ \tfrac{1}{2}Q & 0 \end{bmatrix} \tag{28.b}$$

We note that the measurement equation (see Eq. 25) is unchanged although the dynamics matrix of the system changes from $G'(\mathbf{y})$ to $G(\mathbf{y})$. Unlike the pair $G'(\mathbf{y})$ and C, the pair $G(\mathbf{y})$ and C is not necessarily deterministically unobservable. In fact, the results which are presented in Fig. 5 show that the pair is observable even when $\underline{\boldsymbol{\omega}}$ is time varying. Moreover, in the computation of Ω which is needed in Eq. (24.b) we use our best estimate of $\underline{\boldsymbol{\omega}}$. At least initially, this estimate may be way off yielding a wrong **Ω** and, consequently, a wrong $G'(\mathbf{y})$. On the other hand, in the computation of $G(\mathbf{y})$, given in Eq. (27.b), we use Q rather than Ω, and since Q is based on the computed **q** which is fairly accurate, we obtain a pretty accurate $G(\mathbf{y})$. In other words, not only is the pair $\{G(\mathbf{y}),\ C\}$ observable, the use of $G(\mathbf{y})$ yields a more accurate filter model than does $G'(\mathbf{y})$. The FM of Fig. 5 was found to be $FM(5) = 6.1839 \cdot 10^{-4}$ deg/sec. When comparing the FM of Fig. 5 to those of Figs. 3 and 4, it is realized that the addition of **q** to the state vector yields a better filter. It is noted that the level of the spikes present in Figs. 3 and 4 was reduced when this filter was used.

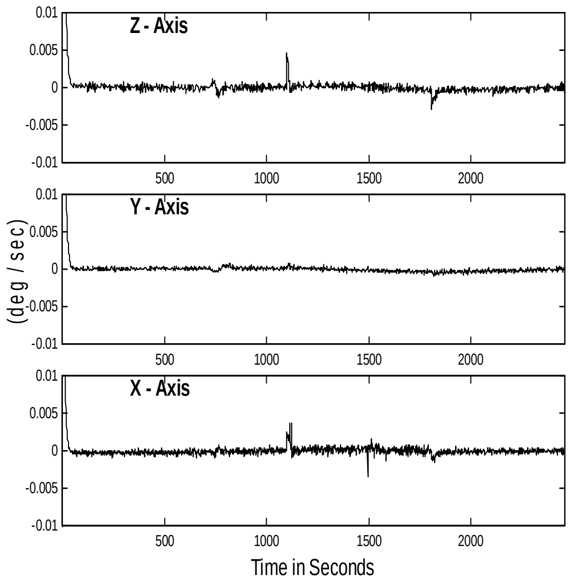

Fig. 5: The Estimated Angular-Rate, $\hat{\underline{\boldsymbol{\omega}}}$, After Applying the PSELIKA Filter to the Augmented Model.

While $\mathbf{v}_k$, the measurement noise vector, can be assumed to be statistically independent over time, its components are correlated with one another; moreover, it cannot be assumed that $\mathbf{v}_k$ has a constantly zero mean, consequently we model the measurement noise as

$$\mathbf{v}_k = \mathbf{v}_{1,k} + \mathbf{v}_{2,k} \quad (29)$$

where between the measurement points , k-1, k, k+1, the noise component, $\mathbf{v}_1$, changes according to

$$\dot{\mathbf{v}}_1 = -\mathrm{N}\mathbf{v}_1 + \underline{\boldsymbol{\mu}}_1 \quad (30)$$

It is further assumed that $\mathbf{v}_{2,k}$ is a zero mean white noise process whose covariance matrix contains, in general, non-zero off diagonal elements. As usual, the covariance matrix of the white noise vector, $\underline{\boldsymbol{\mu}}_1$, which drives $\mathbf{v}_{1,k+1}$, is selected[15] to fit the covariance matrix of $\mathbf{v}_{1,k+1}$. That matrix too may have non-zero off diagonal elements in order to generate the correct covariance between the components of $\mathbf{v}_{1,k+1}$.

Since the measurement noise has a non-white component, one needs to augment the non-white state with the existing state vector to form a new augmented state. The resultant model is then as follows

$$\dot{\mathbf{x}} = \mathrm{F}\mathbf{x} + \mathbf{f} + \mathbf{w} \quad (31)$$

where

$$\mathrm{F} = \begin{bmatrix} \mathrm{F}(\underline{\boldsymbol{\omega}}) & 0 & 0 \\ \frac{1}{2}\mathrm{Q} & 0 & 0 \\ 0 & 0 & -\mathrm{N} \end{bmatrix} \quad (32.b) \qquad (32.a)\ \ \mathbf{x} = \begin{bmatrix} \underline{\boldsymbol{\omega}} \\ \mathbf{q} \\ \mathbf{v}_1 \end{bmatrix}$$

$$(32.d)\ \ \mathbf{w} = \begin{bmatrix} \mathbf{w}' \\ \mathbf{0} \\ \boldsymbol{\mu}_1 \end{bmatrix} \qquad (32.c)\ \ \mathbf{f} = \begin{bmatrix} \mathbf{u} \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix}$$

Since

$$\mathbf{q}_{m,k} = \mathbf{q}_k + \mathbf{v}_{1,k} + \mathbf{v}_{2,k} \quad (33.a)$$

then the corresponding discrete measurement equation is

$$\mathbf{z}_{k+1} = \mathrm{H}\mathbf{x}_{k+1} + \mathbf{v}_{2,k+1} \quad (33.b)$$

where

$$\mathrm{H} = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{bmatrix} \quad (33.c)$$

VI. A Simplified Filter Model

The dynamics models which were used in the preceding section can be drastically simplified by exchanging the SC non-linear dynamics model with a simple first order Markov model. This approach, which is common practice in target tracking, was applied recently to attitude determination[7]. The simplified filter dynamics equation takes the form

$$\mathrm{F}_s = \begin{bmatrix} -\mathrm{T}^{-1} & 0 & 0 \\ \frac{1}{2}\mathrm{Q} & 0 & 0 \\ 0 & 0 & -\mathrm{N} \end{bmatrix} \quad (34.a)$$

The dynamics model is then

$$\dot{\mathbf{x}}_s = \mathrm{F}_s\mathbf{x}_s + \mathbf{f} + \mathbf{w}_s \quad (34.b)$$

where

(34.c) $$\mathbf{x}_s^T = [\underline{\boldsymbol{\omega}}_s^T \,|\, \mathbf{q}_s^T \,|\, \mathbf{v}_1^T]$$

f is as before and

(34.d) $$\mathbf{w}_s^T = [\mathbf{w}_s^T \,|\, \mathbf{0}^T \,|\, \boldsymbol{\mu}_1^T]$$

The covariance matrix of $\mathbf{w}_s$ has to be computed[15] and tuned. When the quaternion measurements are used to update the filter every second there is almost no visible difference between the use of the elaborate rotational dynamics model and the simplified Markov model. However if the updates occur at longer intervals there is a remarkable difference between the two cases. Fig. 6 presents the angular-rate estimation error when the elaborate angular dynamics is used and the PSELIKA filter, which is used to estimate the rates, is updated every 30 sec. The FM computation of the error presented in Fig. 6 results in $\mathrm{FM}(6) = 1.7975 \cdot 10^{-3}$ deg/sec. When the elaborate model is replaced by the Markov model, the error in the resulting estimated rate is unacceptable. This is seen in Fig. 7 where the angular-rate estimation errors for this case are presented. This is also indicated by the large FM of this case where $\mathrm{FM}(7) = 3.9136 \cdot 10^{-2}$ deg/sec. It should be noted that in the computation FM(6) and FM(7) we set $t_0 = 200$ sec. Again, this was done in order to avoid the transients.

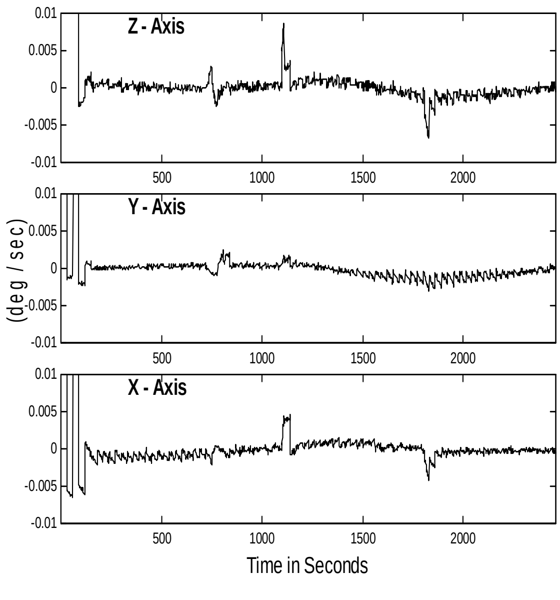

Fig. 6: The Estimated Angular-Rate Error After Applying the PSELIKA Filter to the Augmented Model with Sparse Measurements.

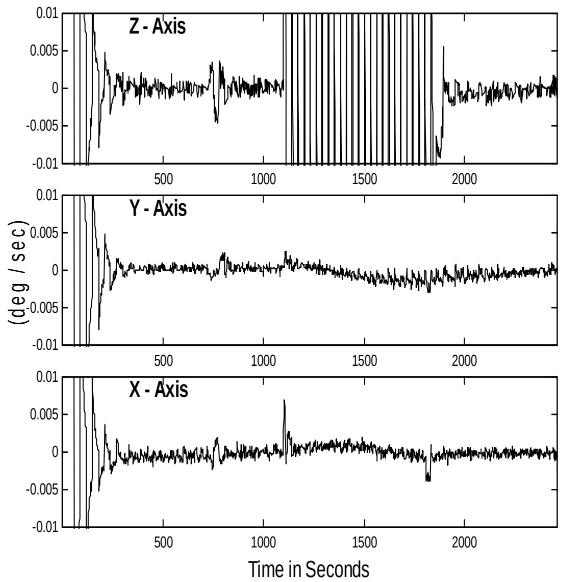

Fig. 7: The Estimated Angular-Rate Error After Applying the PSELIKA Filter to the Simplified Model with Sparse Measurements.

VII. Quaternion Normalization

Since quaternions are inherently normal, the quaternion which is estimated using the above algorithms has to be normal; however, these estimation algorithms do not assure normalization; therefore, occasionally, the estimated quaternion has to be normalized. Several algorithms were suggested for it in the past[16,17] which were compatible with the KF estimator. The accuracy achieved when using those algorithms was about the same for all of them. In this work we chose to apply the Magnitude Pseudo-Measurement (MPM) normalization algorithm[17] for the ease of its implementation. This algorithm is presented next.

The states which constitute the four elements of the quaternion are x_4, x_5, x_6 and x_7 therefore the sum $\hat{x}_4^2 + \hat{x}_5^2 + \hat{x}_6^2 + \hat{x}_7^2$ has to be equal to 1. In order to assure it we assume the existence of a “magnitude measuring device” that “measures” 1; that is,

$$z_{norm,k+1} = 1 \tag{35.a}$$

On the other hand we assume that the corresponding measurement model is

$$z_{norm,k+1} = [0, 0, 0, \hat{x}_{4,k+1}, \hat{x}_{5,k+1}, \hat{x}_{6,k+1}, \hat{x}_{7,k+1}, 0, 0, 0, 0]\, \mathbf{x} + \mathbf{v}_{norm,k+1} \tag{35.b}$$

which can be written in the form

$$\mathbf{z}_{norm,k+1} = H_{norm}\, \mathbf{x} + \mathbf{v}_{norm,k+1} \tag{35.c}$$

where, obviously

$$H_{norm} = [0, 0, 0, \hat{x}_{4,k+1}, \hat{x}_{5,k+1}, \hat{x}_{6,k+1}, \hat{x}_{7,k+1}, 0, 0, 0, 0] \tag{35.d}$$

It is possible now to perform an ordinary measurement update where the filter is fed with the “measurement” 1, and where the measurement matrix is given in Eq. (35.d). The value of $r_{norm,k+1}$, the variance of the “measurement” error $\mathbf{v}_{norm,k+1}$, can be adjusted to yield satisfactory results. We note that indeed this algorithm forces normality on the estimated quaternion without violating the KF rules.

IX. Conclusions

In this paper we examined algorithms for estimating the angular-rate vector of satellites using quaternion measurements without differentiation. The notion examined in this work is based on the ability to obtain quaternion measurements directly from a cluster of star trackers. For the sake of comparison we also examined the approach of extracting the angular-rate from quaternion differentiation. Both approaches utilize a Kalman filter. In fact two filters were examined. One was the PSEudo-Linear KAlman (PSELIKA) filter and the other was a special Kalman filter which was based on the use of the solution of the State Dependent Algebraic Riccati Equation (SDARE) in order to compute the Kalman gain matrix and thus eliminate the need to propagate and update the filter covariance matrix. The two filters relied on the ability to decompose the non-linear rate dependent part of the rotational dynamics equation of a rigid body into a product of an angular-rate dependent matrix and the angular-rate vector itself. This non-unique decomposition enabled the treatment of the nonlinear spacecraft dynamics model as a linear one and, consequently, the application of the PSELIKA filter. It also enabled the application of the SDARE filter.

When using the quaternion measurements to obtain angular-rate without differentiation, the kinematics equation of the quaternion has to be incorporated into the filter dynamics model. This can be done in two ways. It was shown that only one way can be used because only this way yields an observable system.

Real spacecraft data was used to test the suggested algorithms. As expected, when rate determination was based on quaternion differentiation, the resulting angular-rate was noisy. When either one of the filters was used, the noise was suppressed without causing delays in the estimated angular-rate components.

The replacement of the elaborate rotational dynamics by a simple first order Markov model was also examined. It was found that while the use of such a simple model was sufficient when frequent measurement updates were possible, it was totally inadequate when only sparse quaternion measurements were available.

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