A Simple Attitude Control Law Satisfying Slew-Rate And Torque Constraints

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Abstract

A control law is devised which steers a rigid spacecraft from rest to rest between prescribed attitudes, taking into account arbitrary actuator configurations, slew-rate constraints and torque constraints. The control law is extremely simple, requiring no recourse to numerical methods whatsoever, and hence is suitable for implementation in an on-board attitude control system. Execution of the maneuver is achieved within a finite time which can be explicitly written down as a function of the maneuver input data. Redundancy in the actuator system can be exploited to obtain small maneuver durations. The proposed algorithm is applied to two test cases taken from the literature, with rather favorable results.

Key words: Attitude Control, Slew-Rate and Torque Constraints, Actuator Redundancy.

Introduction

The *attitude* or *orientation* of a spacecraft (modeled as a rigid body) is the matrix $g \in \mathrm{SO}(3)$ whose rows are the directions of the body's principal axes with respect to some reference coordinate system. Let us denote by I_1, I_2, I_3 the moments of inertia, by $\omega_1, \omega_2, \omega_3$ the angular velocities and by T_1, T_2, T_3 the exerted torques about the principal axes. Then the attitude kinematics of the spacecraft are described by the equation

$$\dot{g}(t) \;=\; L\big(\omega(t)\big)\, g(t) \tag{1}$$

where

$$L(\omega) \;:=\; \begin{bmatrix} 0 & \omega_3 & -\omega_2 \\ -\omega_3 & 0 & \omega_1 \\ \omega_2 & -\omega_1 & 0 \end{bmatrix} \tag{2}$$

whereas the dynamics are governed by Euler's equations

$$\begin{aligned} I_1\dot{\omega}_1(t) &= (I_2 - I_3)\,\omega_2(t)\,\omega_3(t) \,+\, T_1(t), \\ I_2\dot{\omega}_2(t) &= (I_3 - I_1)\,\omega_3(t)\,\omega_1(t) \,+\, T_2(t), \\ I_3\dot{\omega}_3(t) &= (I_1 - I_2)\,\omega_1(t)\,\omega_2(t) \,+\, T_3(t). \end{aligned} \tag{3}$$

The actuators which produce the control torques are usually not aligned with the principal axes. Assume that there are N actuators such that the k-th actuator produces a torque in the body-direction $b^{(k)} = (b_1^{(k)}, b_2^{(k)}, b_3^{(k)})^T$ (where $\|b^{(k)}\| = 1$). Then the $(3\times N)$-matrix C whose columns are $b^{(1)}, \ldots, b^{(N)}$ will be called the *actuator configuration matrix* of the spacecraft. We assume that C has maximal rank (which means that torques about all three axes can be exerted). Then the torque vector $T(t) := \big(T_1(t), T_2(t), T_3(t)\big)^T \in \mathbb{R}^3$ is related to the control vector $\tau(t) := \big(\tau_1(t), \ldots, \tau_N(t)\big)^T \in \mathbb{R}^N$ (whose entries are the actuator torques) by the equation

$$T(t) \;=\; C\,\tau(t)\,. \tag{4}$$

Our task will now be to solve the following problem: Given the spacecraft principal moments of inertia I_1, I_2, I_3, the actuator configuration matrix C of the spacecraft, an initial attitude g_0 and a target attitude g_1, find a control law $t \mapsto \tau(t)$ which steers the spacecraft from rest to rest between the attitudes g_0 and g_1 in finite time while satisfying prescribed constraints on the angular rates and the torques.

Main Results

Let us start by specifying exactly the type of maneuver we want to carry out and let us fix some notation for the maneuver data.

Maneuver Data. A maneuver will be sought to steer a spacecraft from rest to rest between given attitudes g_0 and g_1. With the spacecraft are associated the principal moments of inertia I_1, I_2 and I_3 and an actuator configuration matrix $C \in \mathbb{R}^{3\times N}$. We write $\gamma := g_1 g_0^{-1}$ and $\alpha := \arccos\big((\mathrm{tr}[\gamma] - 1)/2\big)$ where tr denotes the trace of a matrix; moreover, we let

$$c \;:=\; \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} \;:=\; \frac{\alpha}{2\sin\alpha} \begin{bmatrix} \gamma_{23} - \gamma_{32} \\ \gamma_{31} - \gamma_{13} \\ \gamma_{12} - \gamma_{21} \end{bmatrix}. \tag{5}$$

Also, we let

$$\begin{array}{ll} A_1 := I_1 c_1, & B_1 := (I_2 - I_3) c_2 c_3, \\ A_2 := I_2 c_2, & B_2 := (I_3 - I_1) c_3 c_1, \\ A_3 := I_3 c_3, & B_3 := (I_1 - I_2) c_1 c_2, \\ A := \sqrt{A_1^2 + A_2^2 + A_3^2}, & B := \sqrt{B_1^2 + B_2^2 + B_3^2}. \end{array} \quad (6)$$

Finally, we define $\alpha, \beta \in \mathbb{R}^N$ by the equation

$$\begin{bmatrix} \alpha_1 & \beta_1 \\ \vdots & \vdots \\ \alpha_N & \beta_N \end{bmatrix} := C^T (C C^T)^{-1} \begin{bmatrix} A_1 & B_1 \\ A_2 & B_2 \\ A_3 & B_3 \end{bmatrix}. \quad (7)$$

To have available some test cases, we give the inertia tensors for three satellite shapes (assuming in each case a constant mass density ρ).

• For a box with side lengths a, b and c, the inertia tensor in the principal axes system is

$$I = \rho \cdot \frac{abc}{12} \begin{bmatrix} b^2 + c^2 & 0 & 0 \\ 0 & c^2 + a^2 & 0 \\ 0 & 0 & a^2 + b^2 \end{bmatrix}. \quad (8)$$

• For an ellipsoid with semiaxes a, b and c, the inertia tensor in the principal axes system is

$$I = \rho \cdot \frac{4\pi abc}{15} \begin{bmatrix} b^2 + c^2 & 0 & 0 \\ 0 & c^2 + a^2 & 0 \\ 0 & 0 & a^2 + b^2 \end{bmatrix}. \quad (9)$$

• For a cylinder with height h and ellipsoidal cross-section with semiaxes a and b, the inertia tensor in the principal axes system is

$$I = \rho \cdot \frac{\pi abh}{12} \begin{bmatrix} h^2 + 3b^2 & 0 & 0 \\ 0 & h^2 + 3a^2 & 0 \\ 0 & 0 & 3a^2 + 3b^2 \end{bmatrix}. \quad (10)$$

Our first theorem gives a torque law which effects an eigenaxis slew steering the spacecraft from rest to rest between two specified attitudes during a specified time interval.

Theorem 1. *Given a time interval* $[t_0, t_1]$, *choose a function* $q : (t_0, t_1) \to (0, \infty)$ *with* $q(t) \to \infty$ *for* $t \to t_0$ *and* $t \to t_1$, *let* Q *be an antiderivative of* $1/q$ *and let* $\rho := Q(t_1) - Q(t_0)$. *Then a slew about the axis* $\mathbb{R}c$ *which steers the spacecraft from rest to rest between the attitudes* $g(t_0) = g_0$ *and* $g(t_1) = g_1$ *is characterized by the following data.*

Attitude evolution:

$$g(t) = \exp\left(\frac{Q(t) - Q(t_0)}{\rho} L(c) \right) g_0 \quad (11)$$

Angular velocities about the principal axes:

$$\omega(t) = \begin{bmatrix} \omega_1(t) \\ \omega_2(t) \\ \omega_3(t) \end{bmatrix} = \frac{1}{q(t)\,\rho} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} \quad (12)$$

Torques about the principal axes:

$$T(t) = \begin{bmatrix} T_1(t) \\ T_2(t) \\ T_3(t) \end{bmatrix} = \frac{-1}{q(t)^2 \rho^2} \begin{bmatrix} A_1 \rho \dot{q}(t) + B_1 \\ A_2 \rho \dot{q}(t) + B_2 \\ A_3 \rho \dot{q}(t) + B_3 \end{bmatrix} \quad (13)$$

Control torques produced by the actuators:

$$\tau(t) = C^T (C C^T)^{-1} T(t) + u(t) \quad (14)$$

where u *is an arbitrary function with values in the kernel of* C. *The choice* $u \equiv 0$ *yields the solution which minimizes* $\|\tau(t)\|_2$ *for each* $t \in [t_0, t_1]$.

Proof. The attitude evolution (11) satisfies $g(t_0) = g_0$ and $g(t_1) = g_1$, where the latter equation is a consequence of Rodrigues' formula and the choice of c. Moreover, $\dot{g}(t) = \big(q(t)\rho\big)^{-1} L(c) g(t)$ so that the underlying angular velocity evolution is given by $L\big(\omega(t)\big) = \big(q(t)\rho\big)^{-1} L(c)$ which is (12). (Note that the angular velocity vector is always aligned with the vector c and that $\omega(t_0) = \omega(t_1) = 0$ due to the singularities of q at times t_0 and t_1.) Plugging $\omega(t) = \big(q(t)\rho\big)^{-1} c$ into Euler's equations (3) immediately yields the torque law (13), and (14) is the general solution for (4) once the torque $t \mapsto T(t)$ is given. Finally, we observe that, for each $t \in [t_0, t_1]$, the vectors $C^T (C C^T)^{-1} T(t)$ and $u(t)$ are orthogonal, which implies

$$\|\tau(t)\|_2^2 = \|C^T (C C^T)^{-1} T(t)\|_2^2 + \|u(t)\|_2^2. \quad (15)$$

■

It can be shown (see [3]) that the attitude evolution described in Theorem 1 minimizes the integral $\int_{t_0}^{t_1} q(t) \big(\omega_1(t)^2 + \omega_2(t)^2 + \omega_3(t)^2\big)\, dt$, but this optimality property is irrelevant for the purposes of the present paper. We will now associate with an arbitrary weight function defined on the normalized time interval $(0,1)$ a family of maneuvers parametrized by the duration D of the maneuver.

Theorem 2. *Choose an absolutely continuous function $w : [0,1] \to \mathbb{R}_0^+$ with $w(0) = w(1) = 0$ and a vector-valued function $v : [0,1] \to \ker C$ with values in the kernel of C. Write $\|w\|_1 := \int_0^1 w(x)\,\mathrm{d}x$ and let*

$$z(x) := \frac{w'(x)}{\|w\|_1} \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_N \end{bmatrix} - \frac{w(x)^2}{\|w\|_1^2} \begin{bmatrix} \beta_1 \\ \vdots \\ \beta_N \end{bmatrix}. \tag{16}$$

Then a maneuver of duration D which steers a spacecraft from rest to rest between attitudes g_0 and g_1 is produced by the control law

$$\tau(t) := \frac{1}{D^2}\big(z(t/D) + v(t/D)\big) \tag{17}$$

which gives rise to the torque evolution

$$T(t) = \frac{1}{D^2}\left(\frac{w'(t/D)}{\|w\|_1} \begin{bmatrix} A_1 \\ A_2 \\ A_3 \end{bmatrix} - \frac{w(t/D)^2}{\|w\|_1^2} \begin{bmatrix} B_1 \\ B_2 \\ B_3 \end{bmatrix} \right) \tag{18}$$

and the angular velocity evolution

$$\omega(t) = \frac{w(t/D)}{D\|w\|_1} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}. \tag{19}$$

Writing $\overline{\omega}_i := \max_{0\le t\le D} |\omega_i(t)|$, $\overline{\omega} := \max_{0\le t\le D} \|\omega(t)\|$, $\overline{T}_i := \sup_{0\le t\le D} |T_i(t)|$ and $\overline{T} := \sup_{0\le t\le D} \|T(t)\|$, this maneuver satisfies the following equations:

$$\overline{\omega}_i = \frac{|c_i|}{D\|w\|_1} \max_{0\le x\le 1} w(x); \tag{20}$$

$$\overline{\omega} = \frac{\sqrt{c_1^2+c_2^2+c_3^2}}{D\|w\|_1} \max_{0\le x\le 1} w(x); \tag{21}$$

$$\overline{T}_i = \frac{1}{D^2\|w\|_1^2} \sup_{0\le x\le 1} \Big| \|w\|_1 w'(x) A_i - w(x)^2 B_i \Big|; \tag{22}$$

$$\overline{T} = \frac{1}{D^2\|w\|_1^2} \sup_{0\le x\le 1} \sqrt{A^2\|w\|_1^2 w'(x)^2 + B^2 w(x)^4}. \tag{23}$$

Moreover, if we choose v to be identically zero, then $\overline{\tau}_k := \sup_{0\le t\le D} |\tau_k(t)|$ is given by

$$\overline{\tau}_k = \frac{1}{D^2\|w\|_1^2} \sup_{0\le x\le 1} \Big| \alpha_k \|w\|_1 w'(x) - \beta_k w(x)^2 \Big|. \tag{24}$$

Proof. Let $[t_0, t_1] := [0, D]$ and $q(t) := w(t/D)^{-1}$ in Theorem 1; then equations (14), (13) and (12) become (17), (18) and (19), respectively. The remaining statements are immediate consequences of the above explicit formulas. ∎

Example 1. Let $w(x) := \sin(\pi x)$. Elementary calculations show that

$$\overline{\omega}_i = \frac{\pi|c_i|}{2D}; \tag{25}$$

$$\overline{\omega} = \frac{\pi\sqrt{c_1^2+c_2^2+c_3^2}}{2D}; \tag{26}$$

$$\overline{T}_i = \frac{\pi^2}{4D^2} \cdot \begin{cases} (A_i^2+B_i^2)/|B_i| & \text{if } |A_i| \le |B_i|, \\ 2|A_i| & \text{otherwise;} \end{cases} \tag{27}$$

$$\overline{T} = \frac{\pi^2}{4D^2} \cdot \max\{2A, B\}. \tag{28}$$

Moreover, if $v \equiv 0$ then

$$\overline{\tau}_k = \frac{\pi^2}{4D^2} \cdot \begin{cases} (\alpha_k^2+\beta_k^2)/|\beta_k| & \text{if } |\alpha_k| \le |\beta_k|, \\ 2|\alpha_k| & \text{otherwise}. \end{cases} \tag{29}$$

Example 2. Let

$$w(x) := \begin{cases} x/\varepsilon & \text{if } 0 \le x \le \varepsilon, \\ 1 & \text{if } \varepsilon \le x \le 1-\varepsilon, \\ (1-x)/\varepsilon & \text{if } 1-\varepsilon \le x \le 1 \end{cases} \tag{30}$$

where $\varepsilon \in (0, \frac{1}{2}]$ is a fixed parameter. Elementary calculations show that

$$\overline{\omega}_i = \frac{|c_i|}{(1-\varepsilon)\,D}; \tag{31}$$

$$\overline{\omega} = \frac{\sqrt{c_1^2+c_2^2+c_3^2}}{(1-\varepsilon)D}; \tag{32}$$

$$\overline{T}_i = \frac{(1-\varepsilon)|A_i| + \varepsilon|B_i|}{\varepsilon(1-\varepsilon)^2 D^2}; \tag{33}$$

$$\overline{T} = \frac{1}{\varepsilon(1-\varepsilon)D^2} \cdot \sqrt{A^2(1-\varepsilon^2) + B^2\varepsilon^2}. \tag{34}$$

Moreover, if $v \equiv 0$ then

$$\overline{\tau}_k = \frac{(1-\varepsilon)|\alpha_k| + \varepsilon|\beta_k|}{\varepsilon(1-\varepsilon)^2 D^2}. \tag{35}$$

It is now very simple to reverse the situation: Given prescribed limits for the angular velocities and the torques, we can find the minimum duration D for which the maneuver in Theorem 2 can be executed whilst satisfying the constraints.

Theorem 3. *Given the maneuver data as before and constraints $\omega_i^{\max}$, $\omega^{\max}$, $T_i^{\max}$, $T^{\max}$ and $\tau_k^{\max}$, a maneuver of duration D satisfying the constraints can* *be found by successively setting* $\overline{\omega}_i := \omega_i^{\max}$, $\overline{\omega} := \omega^{\max}$, $\overline{T}_i := T_i^{\max}$, $\overline{T} := T^{\max}$ *and* $\overline{\tau}_k := \tau_k^{\max}$, *solving the resulting equations in* (20) *for* D *and then choosing the maximal* D *thus obtained.*

Proof. This is an immediate consequence of Theorem 2. ∎

Optimization of the Maneuver Duration

We now want to show how the redundancy in the actuator system can be optimally exploited in order to cut down the maneuver duration while satisfying the constraints on the actuator torques.

Theorem 4. *Assume that the constraints* $|\tau_k(t)| \leq \tau_k^{\max}$ *(where* $1 \leq k \leq N$*) have to be satisfied. Let* $\Theta := \mathrm{diag}(1/\tau_1^{\max}, \ldots, 1/\tau_N^{\max})$ *and* $\zeta(x) := \Theta z(x)$ *where* z *is as in Theorem 2. Then the maneuver of the type described in Theorem 2 which has the shortest possible duration is obtained if we choose* $v(x) := \Theta^{-1}\xi_x$ *where, for each* $x \in [0,1]$*, the vector* $\xi_x \in \ker(C\Theta^{-1})$ *is chosen as to minimize* $\|\zeta(x) + \xi\|_\infty$ *over all* $\xi \in \ker(C\Theta^{-1})$.

Proof. The constraint to be satisfied is

$$\sup_{0 \leq t \leq D} \|\Theta\tau(t)\|_\infty \leq 1\,. \tag{36}$$

Now the function $v : [0,1] \to \ker C$ in Theorem 2 can be written as $v(x) = \Theta^{-1}\xi_x$ where each ξ_x is an element of $\Theta(\ker C) = \ker(C\Theta^{-1})$. Then $\tau(t) = D^{-2}\Theta^{-1}\big(\zeta(t/D) + \xi_{t/D}\big)$ and hence

$$\sup_{0 \leq t \leq D} \|\Theta\tau(t)\|_\infty = \frac{1}{D^2} \sup_{0 \leq x \leq 1} \|\zeta(x) + \xi_x\|_\infty\,. \tag{37}$$

Obviously, the duration D which ensures the constraint can be chosen the smaller the smaller the supremum on the right-hand side is. The optimal choice is obtained if we choose, for each $x \in [0,1]$, the vector ξ_x in such a way that

$$\begin{aligned} \|\zeta(x) + \xi_x\|_\infty &= \inf\{\|\zeta(x) + \xi\|_\infty \mid \xi \in \ker(C\Theta^{-1})\} \\ &= \sup\{\langle a, \zeta(x)\rangle \mid \|a\|_1 = 1, a \in \ker(C\Theta^{-1})^\perp\} \\ &= \sup\{\sum_{i=1}^N a_i\zeta_i(x) \mid \sum_{i=1}^N |a_i| = 1, Pa = 0\} \qquad (38) \\ &= \sup\{\sum_{i=1}^N (r_i - s_i)\zeta_i(x) \mid r_i \geq 0, s_i \geq 0, \\ &\qquad \begin{bmatrix} e & e \\ P & -P \end{bmatrix} \begin{bmatrix} r \\ s \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}\} \end{aligned}$$

where $e := (1, \ldots, 1) \in \mathbb{R}^{1\times N}$ and where $P \in \mathbb{R}^{(N-3)\times N}$ is any matrix whose row vectors form a basis of the kernel of $C\Theta^{-1}$. Here the transition from the first to the second line follows from elementary duality theory (cf. [1], Thm. 4.9, pp. 91-92), and the transition from the third to the fourth line (obtained by substituting $a_i = r_i - s_i$ and $|a_i| = r_i + s_i$) shows that the optimization problem addressed can be recast as a standard linear programming problem (and hence can be solved by the simplex algorithm or Karmarkar's algorithm, for example). ∎

Quaternion Representation

We will now apply our algorithm to two situations described in the literature and will compare its performance with that of other control algorithms proposed for these situations. In both cases the attitude is parametrized in terms of quaternions (q_1, q_2, q_3, q_4) so that the attitude matrix is given by

$$\begin{bmatrix} q_1^2 - q_2^2 - q_3^2 + q_4^2 & 2(q_1q_2 + q_3q_4) & 2(q_1q_3 - q_2q_4) \\ 2(q_1q_2 - q_3q_4) & -q_1^2 + q_2^2 - q_3^2 + q_4^2 & 2(q_2q_3 + q_1q_4) \\ 2(q_1q_3 + q_2q_4) & 2(q_2q_3 - q_1q_4) & -q_1^2 - q_2^2 + q_3^2 + q_4^2 \end{bmatrix}$$

where $q_1^2 + q_2^2 + q_3^2 + q_4^2 = 1$; equation (1) then reads

$$\begin{bmatrix} \dot q_1 \\ \dot q_2 \\ \dot q_3 \\ \dot q_4 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} q_4 & -q_3 & q_2 \\ q_3 & q_4 & -q_1 \\ -q_2 & q_1 & q_4 \\ -q_1 & -q_2 & -q_3 \end{bmatrix} \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix}. \tag{39}$$

First Example

As a first example, we apply our algorithm to the situation of the XTE spacecraft as described in [4]. The XTE spacecraft is equipped with four reaction wheels† whose configuration is given by the matrix

$$C = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 & 1 & -1 \\ 1 & 0 & -1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}. \tag{40}$$

† A spacecraft equipped with reaction or momentum wheels is not a rigid body; hence our theory is, strictly speaking, not applicable, and our results will slightly differ from the ones obtained after properly incorporating the wheel dynamics. However, in [4] the same rigid body model as in our paper is used so that the results can be directly compared. The incorporation of the wheel dynamics into the system equations is straightforward and requires only minor modifications.

The maximum torque level of each reaction wheel is $\tau_k^{\max} = 0.3$ Nm, and the maximum slew-rate is given as $\omega_i^{\max} = 0.2$ deg/s. Moreover, the moments of inertia are $I_1 = 6292$ kg m^2, $I_2 = 5477$ kg m^2 and $I_3 = 2687$ kg m^2. A maneuver shall be executed to steer the spacecraft from the initial attitude quaternion

$$(0.2652, 0.2652, -0.6930, 0.6157) \tag{41}$$

to the target quaternion $(0, 0, 0, 1)$ which means that $\gamma = g_1 g_0^{-1}$ equals

$$\gamma = \begin{bmatrix} -0.101163 & -0.712698 & -0.694134 \\ 0.994022 & -0.101163 & -0.041000 \\ -0.041000 & -0.694134 & 0.718673 \end{bmatrix}. \tag{42}$$

Straightforward calculations show that

$$C^T (CC^T)^{-1} \;=\; \frac{1}{2\sqrt{2}} \begin{bmatrix} 1 & 2 & 0 \\ -1 & 0 & -2 \\ 1 & -2 & 0 \\ -1 & 0 & 2 \end{bmatrix} \tag{43}$$

and that $\ker C \;=\; \mathbb{R}\,(1,1,1,1)^T$. In the notation of Theorem 4, we have

$$\zeta(x) = \frac{w'(x)}{0.3\,\|w\|_1} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{bmatrix} - \frac{w(x)^2}{0.3\,\|w\|_1^2} \begin{bmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \\ \beta_4 \end{bmatrix}, \tag{44}$$

and since $\ker(C\Theta^{-1}) = \mathbb{R}(1,1,1,1)^T$ the vector ξ_x can be written in the form $\xi_x = \sigma(x)\,(1,1,1,1)^T$ with a scalar function $\sigma : [0,1] \to \mathbb{R}$. The optimal choice of σ is the one which minimizes, for each $x \in [0,1]$ for which ζ is defined, the expression

$$\left\| \begin{bmatrix} \zeta_1(x) + \sigma(x) \\ \zeta_2(x) + \sigma(x) \\ \zeta_3(x) + \sigma(x) \\ \zeta_4(x) + \sigma(x) \end{bmatrix} \right\|_\infty = \max_{1 \le i \le 4} |\zeta_i(x) + \sigma(x)|. \tag{45}$$

From a sketch of the numbers 0 and $\zeta_i(x)$ (where $1 \le i \le 4$) on the real line it is evident that we have to choose

$$\sigma^\star(x) \;:=\; -\frac{1}{2}\left(\max_{1 \le i \le 4} \zeta_i(x) + \min_{1 \le i \le 4} \zeta_i(x) \right) \tag{46}$$

and that for this choice the expression (45) is given by

$$\begin{aligned} &\frac{1}{2}\left(\max_{1 \le i \le 4} \zeta_i(x) - \min_{1 \le i \le 4} \zeta_i(x) \right) \\ &= \frac{1}{2} \max_{1 \le i < j \le 4} |\zeta_i(x) - \zeta_j(x)|. \end{aligned} \tag{47}$$

From (36) und (37) we see that a maneuver of duration D is possible within the given actuator constraints if and only if

$$D^2 \;\ge\; \frac{1}{2} \sup_{0 \le x \le 1} \left(\max_{1 \le i \le 4} \zeta_i(x) - \min_{1 \le i \le 4} \zeta_i(x) \right). \tag{48}$$

Once D is determined, the optimal choice for u in the control law (14) is

$$u(t) \;:=\; \frac{0.3}{D^2} \sigma^*(t/D) \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}. \tag{49}$$

Up to this point the weight function w has not yet been specified. We now choose the function (30) from Example 2 above. Then

$$\zeta_i(x) = \frac{(1-\varepsilon)\alpha_i \begin{Bmatrix} 1 \\ 0 \\ -1 \end{Bmatrix} - \varepsilon\beta_i \begin{Bmatrix} x^2/\varepsilon^2 \\ 1 \\ (1-x)^2/\varepsilon^2 \end{Bmatrix}}{0.3\varepsilon(1-\varepsilon)^2} \tag{50}$$

where the case-by-case distinctions refers to the cases $0 \le x < \varepsilon$, $\varepsilon < x < 1-\varepsilon$ and $1-\varepsilon < x \le 1$. From the special form of the ζ_i it is clear that to determine the supremum in (48) it is enough to consider the points $x = 0$, $x = 1$, $x = \varepsilon$ and $x = 1-\varepsilon$ (left- and right-sided limit). After some algebra condition (48) becomes

$$\begin{aligned} &D \;\ge\; \max_{1 \le i < j \le 4} \varphi_{ij}(\varepsilon) \quad \text{where} \\ &\varphi_{ij}(\varepsilon) := \sqrt{\frac{1}{2} \frac{(1-\varepsilon)|\alpha_i - \alpha_j| + \varepsilon|\beta_i - \beta_j|}{0.3\,\varepsilon\,(1-\varepsilon)^2}}. \end{aligned} \tag{51}$$

This is the condition that D has to satisfy if the constraints on the control torques are to be satisfied. In addition, in view of the slew-rate constraints, we must also satisfy condition (31) which in our case reads $D \ge 900\,|c_i|/\big(\pi(1-\varepsilon)\big)$. Hence we must have

$$D \;\ge\; \max_{1 \le i \le 3} f_i(\varepsilon) \quad \text{where} \quad f_i(\varepsilon) := \frac{900|c_i|}{\pi\,(1-\varepsilon)}. \tag{52}$$

Consequently, the shortest possible duration is

$$D^* := \max\{ f_k(\varepsilon), \varphi_{ij}(\varepsilon) \mid 1 \le k \le 3, 1 \le i < j \le 4 \}. \tag{53}$$

The specified maneuver data lead to

$$c_1 = c_2 = 0.611, \quad c_3 = -1.596 \tag{54}$$

and

$$\begin{aligned} \alpha_1 &= 3724.673 & \beta_1 &= 1523.796 \\ \alpha_2 &= 1673.992 & \beta_2 &= 746.807 \\ \alpha_3 &= -1006.867 & \beta_3 &= -3447.501 \\ \alpha_4 &= -4391.798 & \beta_4 &= 1176.899 \end{aligned} \tag{55}$$

where all figures are rounded to three decimal places. Plotting the functions f_k and φ_{ij} (of which f_1 and f_2 coincide for the given maneuver data) we obtain the following diagram (Figure 1) which yields the shortest possible maneuver duration as a function of the parameter ε (which can be freely chosen by the user). The optimal value $\varepsilon^* = 0.061$ is the one at which the graphs of f_3 and φ_{14} intersect; the optimal maneuver duration is then $D^* = f_3(\varepsilon^*) = \varphi_{14}(\varepsilon^*) = 486.959$ seconds. This compares favorably to the result obtained in [4] where an ideal duration of 522 seconds is reported and where simulation results (with the control law in closed-loop form) yield a maneuver with a duration of about 600 seconds.

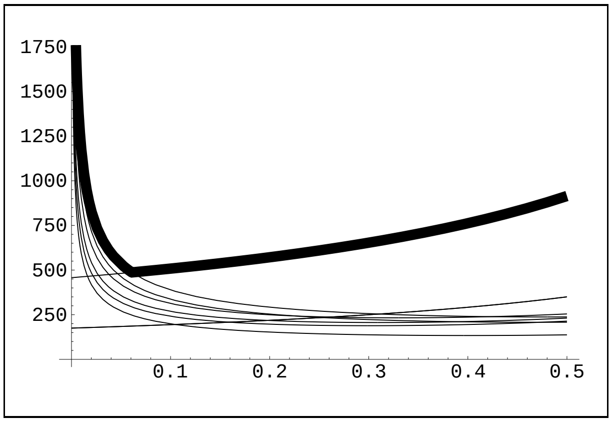

Fig. 1: Maneuver duration (in s) as function of the parameter ε.

Figure 2 shows the actuator torques implementing the optimal maneuver, whereas Figure 3 is a plot of the function $\|g(t) - g_1\|$ (where $\|a\| := \sqrt{\text{tr } a^T a}$) which measures the deviation between the current attitude $g(t)$ and the target attitude g_1 during the maneuver.

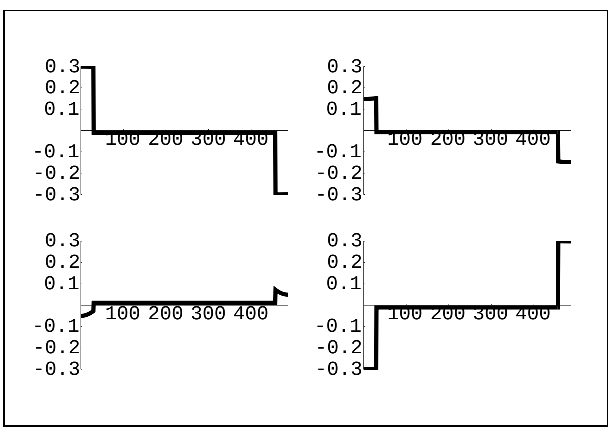

Fig. 2: Control torques (in Nm) which yield the shortest possible maneuver of the type considered.

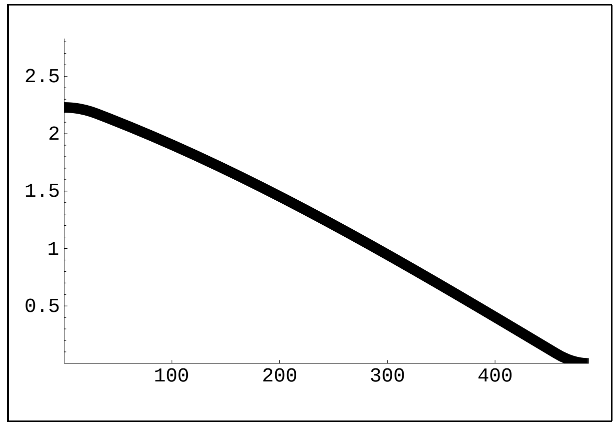

Fig. 3: Deviation between current attitude and target attitude as function of time (in seconds).

It is interesting to contrast these results with the ones obtained if only the constraints on the control torques are imposed, but not the ones on the slew-rates.

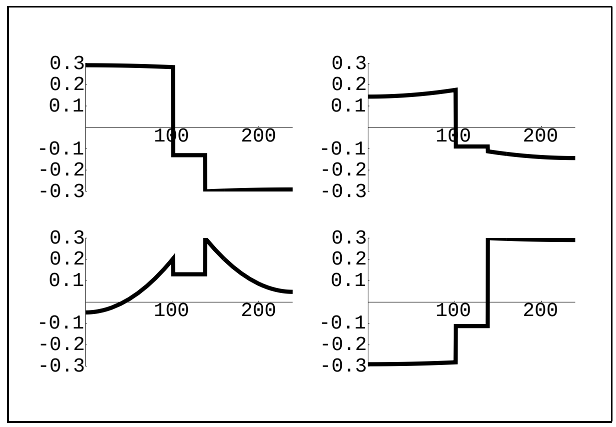

Fig. 4: Control torques (in Nm) which yield the shortest possible maneuver of the type considered if no slew-rate constraints are taken into account.

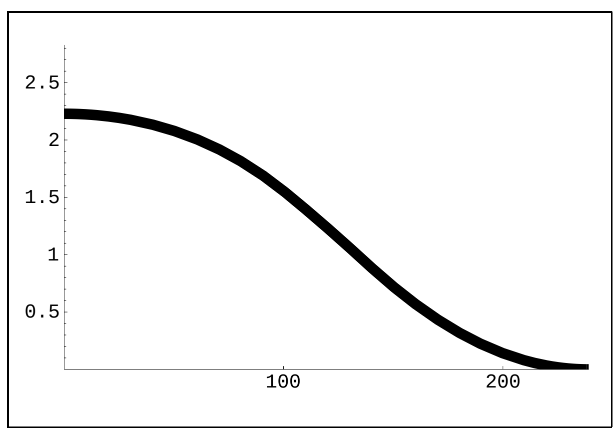

Fig. 5: Deviation between current attitude and target attitude as function of time (in seconds).

In this case Figure 1 shows that the optimal value $\varepsilon^* = 0.423$ is the one at which the graphs of φ_{14} and φ_{13} intersect; the optimal maneuver duration is then $D^* = \varphi_{14}(\varepsilon^*) = \varphi_{13}(\varepsilon^*) = 239.106$ seconds (which is less than half of the duration previously obtained). The actuator torques implementing the optimal maneuver (given in Figure 4) differ dramatically from the ones obtained before, and also the deviation between the current attitude and the target attitude (plotted in Figure 5) changes rather differently during the maneuver.

It is also interesting to compare the above results with the ones obtained by choosing $u \equiv 0$ in the control law (14). This choice (which is not optimal, but easier to implement and, in view of Theorem 1, certainly reasonable) leads to a maneuver duration of 489.921 seconds which is very close to the optimal solution. However, if only the torque constraints are considered, the choice $u \equiv 0$ leads to a duration of 272.487 seconds which appreciably differs from the optimal duration of 239.106 seconds obtained before.

Second Example

We now apply our algorithm to the situation of the space station ALPHA as described in [2]. Here the constraints $\|T(t)\| := \sqrt{T_1(t)^2+T_2(t)^2+T_3(t)^2} \leq 1000$ Nm and $\|\omega(t)\| := \sqrt{\omega_1(t)^2+\omega_2(t)^2+\omega_3(t)^2} \leq 0.05$ deg/s are imposed. The inertia matrix is given by

$$\widehat{I} = \begin{bmatrix} 124.5544 & -2.80367 & -8.76338 \\ -2.80367 & 110.7526 & -0.140927 \\ -8.76338 & -0.140927 & 199.0598 \end{bmatrix} \cdot 10^6 \mathrm{kgm}^2.$$

In this example the inertia tensor is not yet expressed in the principal axes system. Diagonalization yields

$$I := Q^T \widehat{I} Q = \begin{bmatrix} 200.077 & 0 & 0 \\ 0 & 124.124 & 0 \\ 0 & 0 & 110.165 \end{bmatrix} \cdot 10^6 \mathrm{kgm}^2 \tag{56}$$

where

$$Q = \begin{bmatrix} 0.115338 & 0.972175 & -0.203895 \\ -0.002053 & -0.205031 & -0.978753 \\ -0.993324 & 0.113306 & -0.021652 \end{bmatrix}. \tag{57}$$

Denoting by $\widehat{g}$, $\widehat{\omega}$ and $\widehat{T}$ the expression of the attitude, the angular velocity vector and the torque with respect to the originally given body system, we let

$$g := Q^T \widehat{g}, \quad \omega := Q^T \widehat{\omega}, \quad T := Q^T \widehat{T}. \tag{58}$$

Then the system equations $(\mathrm{d}/\mathrm{d}t)\widehat{g} = L(\widehat{\omega})\widehat{g}$ and $\widehat{I}\big((\mathrm{d}/\mathrm{d}t)\widehat{\omega}\big) = (\widehat{I}\widehat{\omega}) \times \widehat{\omega} + \widehat{T}$ take the form $\dot{g} = L(\omega)g$ and $I\dot{\omega} = (I\omega) \times \omega = T$ which is just the form given in equations (1) and (3). Now a maneuver shall be executed to steer the spacecraft from the initial attitude quaternion

$$(0.041996, 0.591724, 0.654368, 0.468936) \tag{59}$$

to the target quaternion $(0, 0, 0, 1)$ which means that $\widehat{\gamma} = \widehat{g}_1 \widehat{g}_0^{-1}$ equals

$$\widehat{\gamma} = \begin{bmatrix} -0.556670 & 0.663413 & -0.500000 \\ -0.564013 & 0.140077 & 0.813797 \\ 0.609923 & 0.735024 & 0.296197 \end{bmatrix}$$

which is tantamount to saying that $\gamma := g_1 g_0^{-1} = Q^T \widehat{g}_1 \widehat{g}_0^{-1} Q = Q^T \widehat{\gamma} Q$ equals

$$\gamma = \begin{bmatrix} -0.556669 & 0.663412 & -0.499998 \\ -0.564013 & 0.140077 & 0.813795 \\ 0.609922 & 0.735023 & 0.296196 \end{bmatrix}. \tag{60}$$

The constraints on the maneuver duration D are determined from equations (21) and (23) and take the form

$$\begin{aligned} \frac{\sqrt{c_1^2+c_2^2+c_3^2}}{D\|w\|_1} \max_{0 \leq x \leq 1} w(x) &\leq 0.05 \cdot \frac{\pi}{180}, \\ \frac{\sup\limits_{0 \leq x \leq 1} \sqrt{A^2 \|w\|_1^2 w'(x)^2 + B^2 w(x)^4}}{D^2 \|w\|_1^2} &\leq 1000. \end{aligned} \tag{61}$$

If we use again a weight function of the form (30), these conditions become

$$\begin{aligned} D &\geq \frac{180\sqrt{c_1^2+c_2^2+c_3^2}}{0.05\pi(1-\varepsilon)} =: f(\varepsilon), \\ D &\geq \sqrt{\frac{\sqrt{(1-\varepsilon)^2 A^2 + \varepsilon^2 B^2}}{1000\varepsilon(1-\varepsilon)^2}} =: g(\varepsilon). \end{aligned} \tag{62}$$

The shortest possible duration (for a given ε) is then $D = \max\{f(\varepsilon), g(\varepsilon)\}$, und the value for ε which minimizes this duration is $\varepsilon^* = 0.0394915$ (yielding the duration $D^* = 2583.41$ seconds), as can be seen from Figure 6 below. Again, this compares favorably to the result given in the literature, where a maneuver duration of about 3200 seconds is found ([2], Fig. 2).

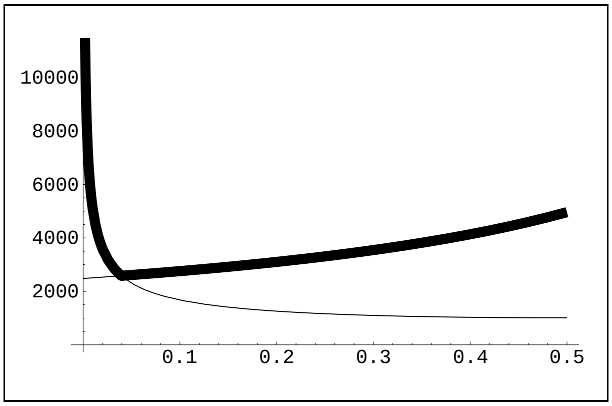

Fig. 6: Maneuver duration (in s) as a function of the parameter ε.

The following figures show the overall angular velocity, the overall torque and the deviation from the target attitude during the maneuver.

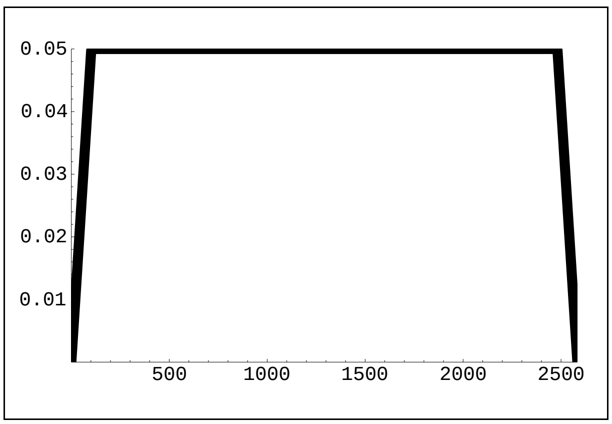

Fig. 7: Angular velocity (in deg/s) during the maneuver.

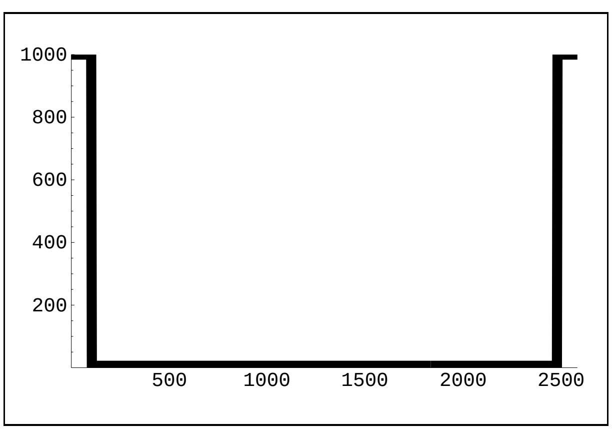

Fig. 8: Torque (in Nm) during the maneuver.

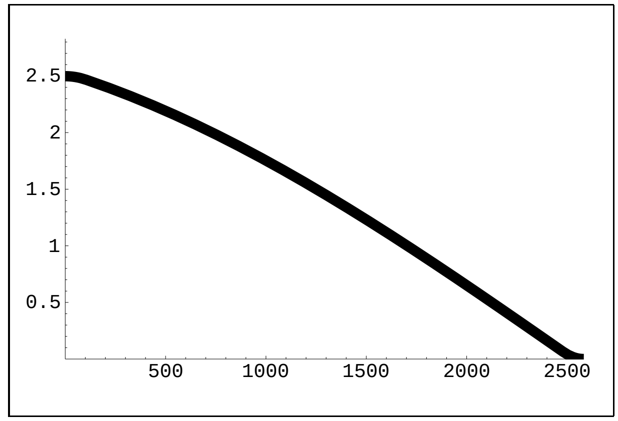

Fig. 9: Deviation between current attitude and target attitude as function of time (in seconds).

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