PERIODIC ORBITS AROUND NATURAL ELONGATED BODIES

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Abstract

In this paper we consider the gravitational field of a massive straight segment rotating in the space as an approximation to the gravitational field created by an elongated celestial body. For this body, we express the potential function in closed form. In a synodic reference frame, we find stationary points and their stability. Besides, we compute families of periodic orbits. These libration points and periodic orbits may be taken into consideration for close survey of natural elongated bodies in rotation.

Key words: Straight segment potential, equilibria, periodic orbits.

Introduction

Space agencies have included missions to small celestial bodies in their current programs, like the NEAR and ROSETTA missions, just to mention but a few. The missions consider the flying of a spacecraft around an asteroid and even the landing on its surface. Classically, for bodies that looks like spheroids, the gravitational potential is expanded into series of spherical harmonics, and the convergence of such series is fast enough and only some of the first terms of the expansion are taken into account. However, when the shape is irregular, which happens in many of the celestial objects (asteroids, comet nuclei or planets' satellites like Phobos), these series hardly converge in the vicinity of the body, hence, new models that fit better the main shape features of the body must be used instead.

When irregular shaped bodies are considered, such as the asteroids Eros, Ida, Amaltea (J5), etc., we found the body elongation as their main shape feature. This elongated shape makes pseudo-spherical approach to the gravitational field of this mass distribution far from the true effect. Indeed, the series expansion of the gravitational potential has its convergence guaranteed outside any sphere centered at the center of mass of the body and radius such that it encloses completely the mass of the body; thus, in the cases of elongated bodies there is a gap when the representation of the field force is uncertain.

This is the reason why some alternative models to the expansion in spherical harmonic have already been proposed. For instance, Werner[l3] use the potential and force of an homogeneous polyhedron close in shape to the asteroid. Prieto and Gómez-Tierno[8] model this type of bodies by a massive dipole; they find also that an axial symmetric body can be replaced by a massive wire lying in the axis of symmetry with appropriate mass distribution. Halamek[3] and Riaguas et al.[10] also studied the gravitational field of a massive straight segment.

In this paper, we consider the gravitational field originated by a massive straight segment rotating uniformly about an axis perpendicular to it. For this body, we express the potential function in closed form, and also an analysis of the linear stability of the equilibria (in a svnodic frame) is made. We found families of periodic orbits around the equilibria by means of a generalization[5,6] of the method of numerically continuation of periodic orbits witli respect to a parameter[1].

Equations of motion

Let us consider a straight segment of length $2l$ and mass M. The gravitational potential per unit mass created by this one dimensional body at a certain point P in the space is given by the line integral

$$U(P) = -G\int_L \frac{dm}{r}$$

where G stands for the Gaussian constant. Assuming the linear mass density (σ) to be constant, this quadrature may be solved in closed form, and its value is[10]

$$U(P) = -\frac{GM}{2l}\log\left(\frac{r_1 + r_2 + 2l}{r_1 + r_2 - 2l}\right), \qquad (1)$$

equation that depends only on the distances: the length of the segment $2l$, and the distances r_1 and r_2 of the particle to the end points of the segment.

Asteroids and planetary satellites belongs to the class of natural elongated bodies that are in pure rotation. They are old objects in the solar system and have reached the state of lowest energy for a given angular momentum, i.e., pure rotation about the principal axis of highest moment of inertia; any primeval nutation faded away because nutation induces time-varying internal stresses that dissipate mechanical energy through hysteresis cycles.

Since in our model we approximate the elongated body by a straight segment, we shall assume that the segment uniformly rotates about an axis z (perpendicular to it and fixed in the space) with angular velocity $\boldsymbol{\omega}$. With this, we define a synodic reference frame *Oxyz*. with origin at the center of mass *O*, and such that the segment lies on the axis *Ox*. In this synodic system the equations of motion of a point mass are

$$\ddot{\boldsymbol{x}}+2\boldsymbol{\omega}\times\dot{\boldsymbol{x}}+\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\boldsymbol{x})+\dot{\boldsymbol{\omega}}\times\boldsymbol{x}=-\nabla_{\boldsymbol{x}}\boldsymbol{U}(\boldsymbol{x}),$$

and inasmuch as we assume that the rotation is uniform ($\dot{\boldsymbol{\omega}}=0$) the equations of motion become

$$\begin{aligned}\ddot{x}-2\omega\dot{y}&=\omega^2x-U_x,\\ \ddot{y}+2\omega\dot{x}&=\omega^2y-U_y,\\ \ddot{z}&=-U_z,\end{aligned}$$

where the potential U is the one (1) above obtained.

Before continuing, a choice of units is in order. In words of Meyer,[7] scaling and changing units are essentially the same thing, hence, let us take a scaling of the Lagrangian, as the equations of the motion are not modified when the whole Lagrangian function is multiplied by a constant.

By defining an *effective* potential

$$W=U(x,y,z)-\frac{\omega^2}{2}(x^2+y^2) \qquad (2)$$

the Lagrangian function is

$$L=\frac{1}{2}(\dot{x}^2+\dot{y}^2)+\omega(x\dot{y}-y\dot{x})-W. \qquad (3)$$

We make the following scaling

$$r\to 2lr,\quad t\to t/\omega$$

which is equivalent to choose $2l$ the length of the segment as the unit of length, and $P/2\pi$ (with P the period of the rotation of the segment) as the unit of time. After this scaling, the Lagrangian (3) is converted into

$$\begin{aligned}L=\omega^2(2l)^2\,[\,&\frac{1}{2}(\dot{x}^2+\dot{y}^2)+(x\dot{y}-y\dot{x})\\ &+\frac{1}{2}(x^2+y^2)+k\log\left(\frac{s+1}{s-1}\right)]\end{aligned}$$

where $k=GM/(\omega^2(2l)^3)$. The dimensionless parameter k is the ratio of the gravitational acceleration to centrifugal acceleration. $k<1$ means fast rotation of the segment, whereas $k>1$ means slow rotation.

Now, the equations of motion are

$$\begin{aligned}\ddot{x}-2\dot{y}&=-W_x=x\left(1-\frac{2k}{sp}\right),\\ \ddot{y}+2\dot{x}&=-W_y=y\left(1-\frac{2ks}{(s^2-1)p}\right),\\ \ddot{z}&=-W_z=-\frac{2kzs}{(s^2-1)p},\end{aligned} \qquad (4)$$

where s, d and p are the auxiliary functions defined by

$$s=r_1+r_2,\; d=r_1-r_2,\; p=r_1r_2.$$

The system (4) admits the Jacobian integral

$$C=2W(x,y,z)-(\dot{x}^2+\dot{y}^2+\dot{z}^2), \qquad (5)$$

and since $(\dot{x}^2+\dot{y}^2+\dot{z}^2)\geq 0$, there results that $C-2W\geq 0$, that is to say, for a given set of initial conditions, the orbit must be inside the region determined by the zero velocity surface $C=2W(x,y,z)$. These surfaces are precisely the level contour surfaces of the hypersurface $f(x,y,z,C)=2W(x,y,z)-C=0$.

In this paper, we deal exclusively with motions on the plane *xy*. Studies about motions on other coordinate planes or in the space will appear elsewhere.[9]

Restricting ourselves to motions on the plane in which the straight segment rotates, the function $W(x,y)$ is represented in Fig. 1, and its zero velocity curves, that is to say, the contour levels of this surface, appear in Fig. 2. As the later figure suggests, there are four equilibria, two of the elliptic type on the *y*-axis, and two of the hyperbolic type on the *x*-axis, but the figure corresponds to a particular value of the parameter k, in this case, $k=1$.

Figure 1. The effective potential $W(x, y, 0)$.

The equilibria

Let us find the equilibria of the equations (4). Obviously the third equation vanishes only when $z = 0$, hence the possible equilibria will be located on the xy-plane. The other possibilities are $x = 0$ and $2ks = (s^2 - 1)p$, or $y = 0$ and $2k = sp$. Let us determine the position of these equilibria.

Equilibria on the x-axis, that will be denoted as *collinear* points are those that satisfy the conditions $y = 0$ and $2k = sp$. Firstly, let us assume that the point is outside the segment and let us denote by ζ the distance from the equilibria to one of the end points of the segment. In this situation, the condition $2k = sp$ is converted into

$$2\varsigma^3 + 3\varsigma^2 + \varsigma - 2k = 0\,, \qquad (6)$$

either the equilibria is the positive or negative semi-axis. This cubic has only one positive root, and it is zero (that is to say, is exactly at the end point of the segment) for $k = 0$ (no rotation). Hence, there are two collinear equilibria, symmetric with respect to the origin.

Figure 2. Zero velocity curves on the xy-plane.

One could think of obtaining some other equilibria inside the segment, albeit it has no physical meaning. Although from the equations there is a third equilibrium inside the segment for $0 \le k \le 1/8$, this solution does not exist, for inside the segment $s=1$**,** which is a singularity of the potential function W. Consequently, we conclude the two equilibria obtained as solutions of the previous cubic are the only ones on the x-axis, that will dubbed E_1 $(x > 0)$ and E_3 $(x < 0)$, symmetrical each other with respect to the y-axis.

Equilibria on the y axis, that will be denoted as equilateral points, are those that satisfy the conditions $x=z=0$ and $2ks = p(s^2-1)$. Since $x = 0$, there follows that the distance to the end points of the segment $r_1 = r_2 = r$ (hence their name of equilateral), and besides, $r > 1/2$. With this, the condition $2ks = p(s^2-1)$ reads

$$4r^3 - r - 4k = 0\,;$$

but this cubic has only one root in the interval (0.5, ∞); thus, there are two equilateral equilibria, E_2 $(y > 0)$ and E_4 $(y < 0)$, symmetrical each other with respect to the x-axis. There is possible to obtain an explicit formula for the solution of a cubic equation; in this case, the real root is

$$\frac{3^{1/3} + (36k + \sqrt{-3 + 1296k^2}\,)^{2/3}}{3^{2/3} + (36k + \sqrt{-3 + 1296k^2}\,)^{1/3}}\,.$$

Linear stability

In order to determine the stability of the equilibria above found, one needs the variational equations of the system (4). By defining a vector $\boldsymbol{\xi} = (\xi_1, \xi_2, \xi_3, \xi_4)$ as the variations

$$\xi = (\,\delta x, \delta y, \delta\dot{x}, \delta\dot{y}\,)\,,$$

the variational equations of (4) is the system

$$\dot{\boldsymbol{\xi}} = A\boldsymbol{\xi} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -W_{xx} & -W_{xy} & 0 & 2 \\ -W_{xy} & -W_{yy} & -2 & 0 \end{pmatrix} = \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \\ \xi_4 \end{pmatrix},$$

where the second derivatives are

$$W_{xx} = -1 + \frac{2k}{ps} - xk\left[\frac{d}{p^3} + x\left(\frac{2(s^2-1)}{p^2 s^3} + \frac{s^2 + d^2}{p^3 s}\right)\right],$$

$$W_{xy} = \frac{2k(\,p - s^2\,)}{p^3 s}\,xy,$$

$$W_{yy} = \frac{p + 2ks - ps^2}{p(s^2-1)} - y^2 ks\,\frac{2p(s^2+1) + (s^2-1)(s^2+d^2\,)}{p^3(s-1)^2(s+1)^2}\,,$$

(7)

that must be evaluated at the equilibria. The problem being symmetric with respect to the two coordinate axes, x and y, the stability of the point E_l is the same as its symmetric E_3, and the same happens for the points E_2 and E_4.

Stability of the collinear points E_l and E_3

The coordinates of the collinear point E_l are $(x_o, y_o) = (1/2 + \zeta,, 0)$, hence

$$s = 2\varsigma + 1, \quad d = -1, \quad p = \varsigma(1 + \varsigma)$$

With this, the second derivatives (7) of the potential W evaluated at the point E_l are

$$W_{xx} = -1 - \frac{k(1+2\varsigma)}{\varsigma^2(1+\varsigma)^2},$$
$$W_{xy} = 0,$$
$$W_{yy} = -1 + \frac{k(1+2\varsigma)}{2\varsigma^2(1+\varsigma)^2},$$

but k and ζ satisfy the equation (6), that is to say, $k = (2\varsigma^3 + 3\varsigma^2 + \varsigma)/2$. Replacing this value of k in the above derivatives, there results

$$W_{xx} = -3 - \frac{1}{2\varsigma(1+\varsigma)},$$
$$W_{xy} = 0,$$
$$W_{yy} = \frac{1}{4\varsigma(1+\varsigma)}.$$

The matrix A of the variational equations is now

$$a = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 3+2b & 0 & 0 & 2 \\ 0 & -b & -2 & 0 \end{pmatrix},$$

where we put $b = 1/(4\zeta(1 + \zeta))$. Note that $b = b(\zeta)$ is monotone in the interval $(0, +\infty)$.

The characteristic equation of this matrix is

$$\lambda^4 + (1-b)\lambda^2 - (3+2b)b = 0,$$

which discriminant, $\Delta = 9b^2 + 10b + 1$, always is positive in the domain where b is defined.

The eigenvalues of the matrix

$$\lambda_{1,2} = \pm\frac{1}{\sqrt{2}}\left(-1+b+\sqrt{9b^2+10b+1}\right)^{1/2},$$
$$\lambda_{3,4} = \pm\frac{1}{\sqrt{2}}\left(-1+b-\sqrt{9b^2+10b+1}\right)^{1/2},$$

are such that $\lambda_{1,2}^2 > 0$ and $\lambda_{3,4}^2 < 0$. Therefore since $\lambda_1 > 0$, the collinear equilibria are always unstable.

Stability of the isosceles points E_2 and E_4.

The isosceles point E_2 is the point with coordinates $(x_o, y_o) = (0, (r^2 - 1/4)^{1/2})$, where the distance r is the real root of the equation $4r^3 - r - 4k = 0$. In this case, there results

$$s = 2r, \quad d = 0, \quad p = r^2,$$

and the second derivatives (7) of the potential W evaluated at the point E_2 are

$$W_{xx} = -1 + \frac{k}{r^3},$$
$$W_{xy} = 0,$$
$$W_{yy} = \frac{k(1-8r^2) + r^3 - 4r^5}{r^3(-1+4r^2)};$$

by replacing the value of k as a function of r in the above derivatives, there results

$$W_{xx} = -\frac{1}{4r^2},$$
$$W_{xy} = 0,$$
$$W_{yy} = -3 + \frac{1}{4r^2}.$$

Now, the matrix A of the variational equations is

$$a = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ a & 0 & 0 & 2 \\ 0 & 3-a & -2 & 0 \end{pmatrix}$$

where we called $a = 1/(4r^2)$. N.B. since $1/2 < r < \infty$ then $0 < a < 1$.

The characteristic equation is $\lambda^4 + \lambda^2 + (3-a)a = 0$, and its discriminant, $\Delta = 4a^2 - 12a + 1$ has a root at $a_c = (3 - 2\sqrt{2})/2$. Thus, for $0 < a < a_c$, $\Delta > 0$, for $a = a_c$, $\Delta = 0$, whereas for $a_c < a < 1$, $\Delta < 0$.

Since the eigenvalues are

$$\lambda_{1,2} = \pm\frac{1}{\sqrt{2}}\left(-1+\sqrt{4a^2-12a+1}\right)^{1/2},$$
$$\lambda_{3,4} = \pm\frac{1}{\sqrt{2}}\left(-1-\sqrt{4a^2-12a+1}\right)^{1/2},$$

there follows that for $0 < a < a_c$, all eigenvalues have null real part, and for $a_c < a < 1$ two eigenvalues (λ_2, λ_4) have negative real part and the other two (λ_1, λ_3) have positive real part, hence the isosceles equilibria, are unstable. In terms of r, the critical distance r_c equivalent to a_c is $r_c^2 = (3+2\sqrt{2})/2$. Thus, for $1/2 < r \leq r_c$, all eigenvalues are pure imaginary, and for $r_c < r < \infty$, the isosceles equilibria, are unstable. Alternatively, one can think of k for the critical value; in this case, from the relation $r^3 - r/4 = k$, there follows that

$$k_c = \frac{1}{8}\,(8+5\sqrt{2})\sqrt{3+2\sqrt{2}} = 4.548097039.$$

For $0 < k < k_c$, the isosceles equilibria are stable, and for $k_c < k$ the isosceles equilibria are unstable.

Periodic orbits

It is known that close to both stable and unstable equilibrium solutions, there can exist periodic orbits (see, for instance, Szebehely[12] p. 231 ff.). These solutions are small ellipses in linear approximation. Similarly to the three-body problem, we will look for these kind of solutions around both collinear and equilateral points of the straight segment potential.

In order to compare with previous works, we select the parameter $k = 1$, that is the value chosen by Scheeres[11] when approximates a triaxial ellipsoid of uniform density to the main features of the asteroid Eros. Note that this minor planet is irregularly shaped with length 40.5 km, width 14.5 km and thickness 14.1 km; therefore we consider the potential of a straight segment of length 2 l = 40.5 km.

To find families of periodic orbits, we use the method of numerical continuation with respect to a parameter. The method is essentially the one given by Deprit and Henrard[1] with some additions made in Lara et al.[5,6]. The process addresses a Boundary Value Problem for the variational equations relative to conservative dynamical systems with two degrees of freedom.

Briefly, it consists of the following: starting with a set of initial conditions close to one periodic solution, we correct this initial set to obtain initial conditions for a true periodic orbit. Then, we vary the value of the parameter (the Jacobian constant C in the present case), and by calculating and refining a tangent prediction we obtain new initial conditions corresponding to a periodic orbit for the new value of the chosen parameter. In order to improve the prediction, we must numerically integrate the equations of motion and their tangent and normal variations, the variational equations associated with this solution. The main feature of this method, is that it splits the normal displacements along an orbit from the tangent ones: the later, indeed, are secular in nature. For details, the reader is addressed to Ref. 1, 5.

For convenience, in what follows we refer to the energy function $h = -C/2$ instead of the Jacobi constant C.

For the collinear points we found that conditions for the existence of small ellipses around them are fulfilled in linear approximation. Then we use that approximation of a periodic orbit to take into account the non-linearity of the problem and compute periodic orbits. Figure 3 shows the family of periodic orbits for variations of the energy function, around the collinear point E_1.

Figure 3. Family of periodic orbits around E_1.

The situation is quite different for the isosceles points where conditions for the existence of small ellipses around them are not fulfilled. We proceed in a different way.

Far away from the origin, the segment will be seen as a point and consequently the problem is approximately the two body problem. Therefore initial conditions of a circular solution of the two body problem will correspond to an orbit around the segment that is approximately periodic. The initial conditions of this solution are improved with a corrector algorithm until finding an exact periodic solution around the segment. The initial orbit of the family of periodic orbits around both isosceles points E_2 and E_4 presented in Fig. 4 was computed in that way. An analogous family was found around the center equilibria of the rotating triaxial ellipsoid approximating Eros (see Fig. 5 and 6 of Ref. 11).

Figure 4. Family of periodic orbits around the isosceles points; two graphics with different scales are plotted in order to appreciate the variation in shape of the orbit when close to the straight segment.

As well as for the ellipsoidal model we found periodic orbits in the rotating frame that are both direct and retrograde orbits (in the inertial frame). Table I provides initial conditions corresponding to the orbits plotted in the figures; in all cases $y = \dot{x} = 0$ while $\dot{y}$ is computed from the integral relation (5), where $h = -C/2$. The periodicity condition is max | $\xi(0)$ - $\xi(T)$ | < 10^{-11} where ξ means any of the coordinates or velocities.

For the orbits of each family, we computed an index of stability, namely the trace k = | $Tr(T)$ | of the matrizant of the associated Hill equation at the end of one period[1] T. As it is well known, when $k > 2$, the characteristic exponents of the orbit are of the unstable type; if $k < 2$, they are of the stable type; and $k = 2$ represents a case of indifferent stability.

The evolution of the stability index k for the collinear family is presented in Fig. 5. Note that for $h \in$ [-l.2253, -1.2194] the periodic orbits present a linearly stable character, although those orbits approach very close to the right end of the segment (closer than l/30). Out of that small interval of the energy function, the numerical integration through very few periods of the (unstable) periodic orbits tends to derail onto either escape or collision orbits despite integrating the problem by recurrent power series that is an extremely accurate and stable procedure[2]. The numerical evidence of the existence of a second integral of the motion[4] for only small regions of the configuration space[9] is responsible, we believe, to that behavior.

Figure 6 presents the evolution of k along the direct isosceles family. This family shows a stable behavior except for values of h in the approximate interval [-1.61140, -1.41161], where the values of k grow highly due to the strong instability of the periodic orbits. The family around the isosceles points made of retrograde orbits (in the inertial frame) is always stable.

Figure 5. Family of periodic orbits around the E_l collinear point: stability index k versus the energy function h. The dashed line (k = 2) separates stability from instability. Note the small stable region in the down right corner of the figure.

Figure 6. Evolution of k along the isosceles family. Below: magnification of the termination of the family. Dashed lines (k = 2) separates stable and unstable regions.

Conclusions

The gravitational field of a very elongated celestial body is modeled by a massive straight segment rotating in the space. For this logarithmic potential we found equilibria, their stability and also we found families of periodic orbits.

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REFERENCES

1 Deprit, A., and Henrard, J., Natural Families of Periodic Orbits, *Astronomical Journal*, **72**:158-172, 1967.

2 A. Deprit and R.M.V. Zahar, Numerical Integration of an Orbit and Its Concomitant Variations, *Z. Angew. Math. Phys.* **17**:425-430, 1966.

3 Halamek, P., *Motion in the potential of a thin bar.* Ph.D. dissertation. Univ. of Texas. Austin, 1988.

4 Hénon, M. and Heiles, C., The Applicability of the Third Integral of Motion: Some Numerical Experiments. *Astronomical Journal* **69**:73-79, 1964.

5 Lara, M., Deprit, A., Elipe, A., Numerical continuation of frozen orbits for the zonal problem. *Celestial Mechanics & Dyn. Astron.*, **62**:167-181, 1995.

6 Lara, M., On numerical continuation of families of periodic orbits in a parametric potential *Mechanics Research Communications*, **23**:291-298. 1996.

7 Meyer, K.R., Scaling Hamiltonian systems. *SIAM J. Math. Anal*. **15**:877-889, 1984.

8 Prieto-Llanos, T., Gómez-Tierno, M.A., Station keeping at libration points of natural elongated bodies, *Journal of Guidance, Control and Dynamics*, **14**:787-794, 1994.

9 Riaguas, A. Ph. D. dissertation (in preparation).

10 Riaguas, A., Elipe, A. and Lara, M., Periodic orbits around a massive straiaht segment. *Celestial. Mechanics & Dyn. Astron.* (to appear), 1999.

11 Scheeres, D.J. Dynamics about Uniformly Rotating Triaxial Ellipsoids: Application to Asteroids, *Icarus* **110**:225-238, 1994.

12 Szebehely, V. *Theory of Orbits.* Academic Press, 1967.

13 Werner, R. A., The gravitational potential of a homogeneous polyhedron or don't cut corners, *Celestial Mechanics & Dynamical Astronomy*, **59**:253-278, 1994.

Table 1

Initial conditions of the orbits of the collinear (above) and isosceles (below) families presented in Fig. 3 and 4. The reference value of the energy function for the collinear family is h_c = -1.550740055311294.

h-h_c	x	T	k
0.335	1.792182810836383	7.155750267372269	4.2776
0.331	1.765929688280791	6.978643504804138	1.7248
0.325	1.739208382339637	6.806226616848227	2.2766
0.311	1.695847754541601	6.544080468680613	8.7342
0.281	1.633231253652561	6.209587612224323	16.0027
0.239	1.569880626706139	5.933802211977488	22.2754
0.149	1.460681999181959	5.609609930635675	35.4615
0.000	1.243708008046054	5.336073142540486	68.1469

h	x	T	k
-1.4111	1.878546858604925	19.31279120369169	0.176
-1.4117	1.865586848357960	19.01847708393285	3.329
-1.4230	1.808819329973669	17.96397712790325	174.969
-1.4410	1.760197734862181	17.20650488604653	306.521
-1.4600	1.720952599384964	16.61779843262338	335.352
-1.4800	1.686422190168124	16.06906977490361	297.563
-1.5100	1.645215087497171	15.24572057735213	183.967
-1.5400	1.622353538964723	14.28477902249745	76.138
-1.5700	1.638486553450859	13.07315509037703	20.468
-1.6200	1.786910290595456	11.06837493868003	0.813