A LINEAR SOLUTION FOR THE ARTIFICIAL SATELLITE'S ATTITUDE OPTIMAL CONTROL

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Abstract

A first order analytical model for a general problem of artificial satellite's attitude corrections maneuvers submitted to gravity gradient torque is presented in this paper. It is assumed that the satellite is a rigid body, with cylindrical or spherical symmetry and its orbit can be elliptical or circular. The problem of optimization is formulated as a Mayer problem and the control torques are provided by a power limited propulsion system. The state is defined by Andoyer's variables and the control by the components of non-conservative external torques in the artificial satellite's axes of inertia. The Pontryagin Maximum Principle is applied to the problem and the optimal torques are given explicitly in Andoyer's variables and their adjoints. The problem of optimal attitude corrections given by the linearized Hamiltonian around the reference attitude is also analyzed, considering the mean Hamiltonian related with the gravity gradient torque. The complete first-order analytical solutions for the problem with fixed duration are gotten by simple quadratures. A law of the optimal control is proposed and the required optimal consumption is presented.

Key words: Attitude corrections, minimal fuel consumption, Mayer problem, Andoyer's variables.

INTRODUCTION

The satellite's attitude represents how the satellite is oriented in space. The attitude expresses a relation between two coordinate systems and can be represented by the Euler angles. In this paper, we assumed a 3-1-3 sequence of three consecutive rotations about the satellite axes for the Euler angles (ϕ , θ , ψ), to define the relation between the system of the artificial satellite's principal axes of inertia (Oxyz) and the system OXYZ (with axes parallel to the axes of the Earth equatorial system).

The dynamic system associated to the satellite's attitude maneuvers will be expressed in terms of the Andoyer's variables[1,2] ℓ_i , L_i , i = 1,2,3 , shown in the figure 1. The angular variables ℓ_i , i = 1,2,3 are angles, which are related to the coordinate systems Oxyz and OXYZ. The metric variables are defined as: L_2 is the modulus of the total angular momentum, L_1 and L_3 are respectively the projection of $\vec{L}_2$ on the z-axis's principal axis system of inertia and on the inertial Z-axis.

The transformation between Andoyer's variables and the Euler angles is well defined[1,3], using properties of spherical trigonometric associated with the spherical triangle $\mathbf{N_1N_2N_3}$, shown in the figure 2**.** It is possible to prove that this transformation is canonical[2].

The rotational motion of the artificial satellite with cylindrical symmetry, considering the non-conservative external torques and Andoyer's variables, can be expressed by[3]:

$$\frac{d\ell_i}{dt} = \frac{\partial H}{\partial L_i} + P_i$$
$$\frac{dL_i}{dt} = -\frac{\partial H}{\partial \ell_i} + S_i \quad ,$$
$$i = 1,2,3 , \qquad (1)$$

where:

$$H = \frac{1}{2A} L_2^2 + \frac{1}{2}\left[\frac{1}{C} - \frac{1}{A}\right] L_1^2 \ , \qquad (2)$$

where A and C are the satellite's moments of inertia on the axis Ox and Oz, respectively, and P_i and S_i depend on the components of non-conservative external torque[3] in the system Oxyz.

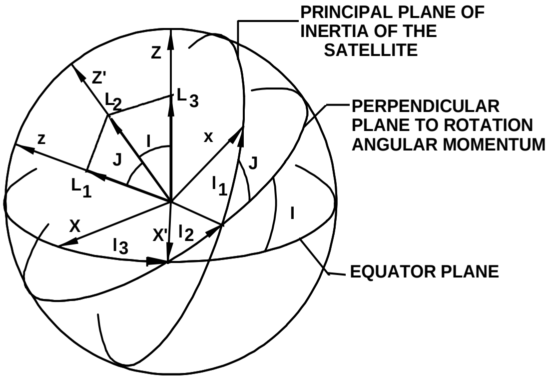

Figure 1: Andoyer's Variables

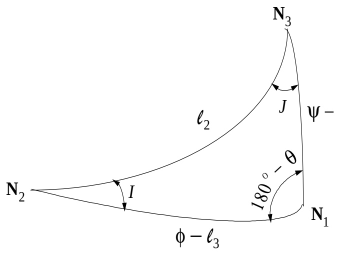

Figure 2. Spherical Triangle $\mathbf{N_1N_2N_3}$ with Andoyer's variables and Euler angles

The problem of optimization is formulated, with the dynamic system describing the rotational motion of the satellite given by eqs. (1), considering the torques provided by a power limited propulsion system and the gravity gradient torque.

OPTIMIZATION OF THE ATTITUDE'S CONTROL

The optimization problem of attitude control is initially introduced by Lagrange formulation, with the dynamical system describing the rotational motion of the satellite. The torques provided by a power limited propulsion system and the gravity gradient torque are included. The state is defined by Andoyer's variables ℓ_i, L_i, $i = 1,2,3$, and the control by the components of non-conservative external torques in the artificial satellite's principal axes of inertia (Oxyz). The Mayer problem, without constraints on control variables and fixed initial time t_o and fixed final time t_f, is also analyzed here.

The state equations are the equations of the rotational motion of the satellite, including external torques, in the extended canonical form. For a satellite with cylindrical symmetry, they are given by :

$$\frac{d\ell_1}{dt} = \left(\frac{1}{C}-\frac{1}{A}\right)L_1 + f_1(\ell_i, L_i) + C_{1x}Q_x + C_{1y}Q_y$$
$$\frac{d\ell_2}{dt} = \frac{1}{A}L_2 + f_2(\ell_i, L_i) + C_{2x}Q_x + C_{2y}Q_y + C_{2z}Q_z$$
$$\frac{d\ell_3}{dt} = f_3(\ell_i, L_i) + C_{3x}Q_x + C_{3y}Q_y + C_{3z}Q_z$$
$$\frac{dL_1}{dt} = Q_z$$
$$\frac{dL_2}{dt} = f_5(\ell_i, L_i) + C_{5x}Q_x + C_{5y}Q_y + C_{5z}Q_z$$
$$\frac{dL_3}{dt} = f_6(\ell_i, L_i) + C_{6x}Q_x + C_{6y}Q_y + C_{6z}Q_z, \qquad (3)$$

where Q_x, Q_y and Q_z are the components of propulsive torque $\mathbf{Q}$ on the system Oxyz and $f_j(\ell_i, L_i)$, $j = 1,\ldots,6$, $i=1,2,3$, are functions related with the gravity gradient torque and C_{jx}, C_{jy} and C_{jz}, $j = 1,\ldots,6$, can be found in Zanardi[3].

The performance index **J**, associated with the fuel consumption, is introduced by:

$$\mathbf{J} = \frac{1}{2}\int Q^2 dt \quad , \qquad (4)$$

where Q is the magnitude of propulsive torque $\mathbf{Q}$.

The optimization problem consists in determining the optimal control $\mathbf{Q}^*$, which transfers the space vehicle from the initial state (ℓ_{i0}, L_{i0}) at t_0 to the final state (ℓ_{if}, L_{if}) at t_f, such that the consumption is a minimum.

The Mayer problem associated to the minimization of fuel consumption during the attitude maneuvers will be defined as follows. The state vector $\mathbf{x}$ is defined by the Andoyer's variables and the performance index J,

$$\mathbf{x}=\left[\ell_1 \quad \ell_2 \quad \ell_3 \quad L_1 \quad L_2 \quad L_3 \quad \mathbf{J}\right]^T. \qquad (5)$$

The dynamic system is described by equations (3) and by the differential equation associated with J:

$$\frac{d\,\mathbf{J}}{d\,t} = \frac{1}{2}\left[Q_x^2 + Q_y^2 + Q_z^2\right] . \qquad (6)$$

This system can be represent by:

$$\frac{d\mathbf{x}}{d\,t} = \mathbf{F}(\mathbf{x}, \mathbf{Q}) . \qquad (7)$$

The performance index is now given by:

$$J_f = J(t_f) , \qquad (8)$$

and the boundary conditions are determined by the initial state $\mathbf{x}_0 = (\ell_{i\,0}, L_{i\,0}, 0)$ and the final state $\mathbf{x}_f = (\ell_{i\,f}, L_{i\,f}, J_f)$, $i = 1,2,3$.

Following the Pontryagin Maximum Principle[4], the adjoint vector $\boldsymbol{p}_\mathbf{x}$ is introduced and the Hamiltonian function is formed using eqs. (3) and (6):

$$H(\mathbf{x}, \boldsymbol{p}_\mathbf{x}, \mathbf{Q}) = \left[\frac{1}{C} - \frac{1}{A}\right] L_1 p_1 + \frac{1}{A} L_2 p_2 +$$
$$+ p_1 f_1(\ell_i, L_i) + p_2 f_2(\ell_i, L_i) + p_3 f_3(\ell_i, L_i) +$$
$$+ P_2 f_5(\ell_i, L_i) + P_3 f_6(\ell_i, L_i) + p_1\left[C_{1x}Q_x + \right.$$
$$\left. + C_{1y}Q_y\right] + p_2\left[C_{2x}Q_x + C_{2y}Q_y + C_{2z}Q_z\right] +$$
$$+ p_3\left[C_{2x}Q_x + C_{2y}Q_y + C_{3z}Q_z\right] + P_1 Q_z +$$
$$+ P_2\left[C_{5x}Q_x + C_{5y}Q_y + C_{5z}Q_z\right] + P_3\left[C_{6x}Q_x + \right.$$
$$\left. + C_{6y}Q_y + C_{6z}Q_z\right] + \frac{1}{2} P_J\left[Q_x^2 + Q_y^2 + Q_z^2\right] . \qquad (9)$$

The optimal torques Q* must be selected so that the Hamiltonian function reaches its maximum value:

$$\mathbf{Q}^* = \arg\max H(\mathbf{x}, \boldsymbol{p}_\mathbf{x}, \mathbf{Q}) .$$

The equations of control (stationary conditions) are given by:

$$p_1C_{1x} + p_2C_{2x} + p_3C_{3x} + P_2C_{5x} + P_J Q_x = 0$$
$$p_1C_{1y} + p_2C_{2y} + p_3C_{3y} + P_2C_{5y} + P_3C_{6y} + P_J Q_y = 0$$
$$p_2C_{2z} + p_3C_{3z} + P_1 + P_2C_{5z} + P_3C_{6z} + P_J Q_z = 0 . \qquad (10)$$

Solving the system (10), the control Q*, is given by:

$$Q_x^* = -\frac{p_1C_{1x} + p_2C_{2x} + p_3C_{3x} + P_2C_{5x} + P_3C_{6x}}{P_J}$$
$$Q_y^* = -\frac{p_1C_{1y} + p_2C_{2y} + p_3C_{3y} + P_2C_{5y} + P_3C_{6y}}{P_J}$$
$$Q_z^* = -\frac{p_2C_{2z} + p_3C_{3z} + P_1 + P_2C_{5z} + P_3C_{6z}}{P_J} , \qquad (11)$$

and the maximum Hamiltonian function H^*, computed by (9) and (10), can be expressed by:

$$H^* = H^*(\mathbf{x}, \boldsymbol{p}_\mathbf{x}) . \qquad (12)$$

The adjoint variable P_J is a first integral of the canonical system defined by H^* and its value, obtained from the transversality conditions, is equal -1. Consequently, the order of the dynamic system which describes the problem is reduced and the Hamiltonian assumes the form:

$$H^* = H_0^* + H_G^* + H_Q^* , \qquad (13)$$

where: $H_0^* = \left[\frac{1}{C} - \frac{1}{A}\right] L_1 p_1 + \frac{1}{A} L_2 p_2$, (14)

$$H_G^* = \sum_{i=1}^{3}\left[p_i f_i + P_i f_{i+3}\right] , \qquad (15)$$

$$H_Q^* = \frac{p_1^2}{2L_2^2 \sin^2 J} + \frac{p_2^2}{2L_2^2}\left[\cot^2 I + \cot^2 J + \right.$$
$$+ \frac{1}{2} P_2^2 + \frac{1}{2} P_3^2 - \frac{p_1 p_2}{L_2^2 \sin J}\left[\cot J + \cot I \cos\ell_2\right] +$$
$$+ \frac{p_1 p_3 \cos\ell_2}{L_2^2 \sin I \sin J} + \frac{p_1 P_3}{2L_2 \sin J}\sum_{\varepsilon} \varepsilon\cos(I + \varepsilon\ell_2) -$$
$$- \frac{p_2 p_3}{L_2^2 \sin I}\left[\cot I + \cot J \cos\ell_2\right] +$$
$$+ \frac{p_2 P_1}{2L_2} \cot I \sum_{\varepsilon} \varepsilon \cos(J + \varepsilon\ell_2) +$$
$$+ \frac{p_2 P_3}{2L_2}\left\{\cot J \sum_{\varepsilon} \varepsilon\cos(I + \varepsilon\ell_2) - \right.$$
$$- \frac{1}{2}\cot I\left[\sum_{\varepsilon} \varepsilon\cos(I + 2\varepsilon\ell_2) + \sum_{\varepsilon,\delta} \varepsilon\delta\cos(I + 2\delta + \right.$$
$$\left.\left. + \varepsilon\ell_2) - \frac{1}{2}\sum_{\varepsilon,\delta} \delta\cos(I + 2\varepsilon J + 2\delta\ell_2)\right]\right\} +$$

$$+\frac{p_3 P_1}{2L_2 \sin I}\sum_{\varepsilon}\varepsilon\cos(J+\varepsilon\ell_2)+\frac{p_3 P_3}{L_2^2}\Bigg[\ \frac{1}{2}\sin 2\ell_2 +$$

$$+\frac{1}{4}\sum_{\varepsilon}\varepsilon\sin 2(J+\varepsilon\ell_2)+\frac{1}{\sin I}\sum_{\varepsilon,\delta}\varepsilon\delta\cos(I+2\delta J+$$

$$+\varepsilon\ell_2)\ \Bigg]+P_1\,P_2\cos J-\frac{1}{2}P_1 P_3\Bigg[\ \cos(I+J)\ +$$

$$+\cos(I-J)\ +\ \frac{1}{2}\sum_{\varepsilon,\delta}\cos(I+\delta J+\varepsilon\ell_2)\ \Bigg]\ -$$

$$-\frac{1}{2}P_2\,P_3\Bigg[\ \cos(I+2J)+\cos\ (I-2J)+$$

$$+\frac{1}{2}\sum_{\varepsilon,\delta}\varepsilon\cos(I+2\,\varepsilon J+\delta\ell_2)\ \Bigg]\ , \qquad (16)$$

where $\sum_{\varepsilon}$ and $\sum_{\varepsilon,\delta}$ means that ε and δ assume the values: +1 and -1; H_0^* is the unperturbed Hamiltonian associated with the torque-free rotational motion, H_G^* and H_Q^* are the perturbed functions. H_G^* is associated with the gravity gradient torque, and H_Q^* is related with optimal control. The two first parcels of the eq. (13) is associated with the conservative system : $(H_0^* + H_G^*)$ and the other parcel (H_Q^*) is associated with the non-conservative system.

ATTITUDE OPTIMAL CORRECTIONS

The problem of optimal attitude corrections, with fixed duration, given by the linearized Hamiltonian around the reference attitude is considered.

The reference attitude is given by the solution of the torque-free rotational motion[1,2] :

$$\bar{\ell}_1 = \bar{\ell}_{10} + \bar{\omega}_1\,(t - t_0)$$
$$\bar{\ell}_2 = \bar{\ell}_{20} + \bar{\omega}_2\,(t - t_0)$$
$$\bar{\ell}_3 = \bar{\ell}_{30} \qquad \bar{L}_i = \bar{L}_{i\,0}$$
$$i = 1,2,3\ ; \qquad (17)$$

where the bar means reference attitude and, $\bar{L}_1 = \bar{L}_2\cos\bar{J}$, $\bar{L}_3 = \bar{L}_2\cos\bar{I}$, $\bar{\omega}_1 = \left[\frac{1}{C}-\frac{1}{A}\right]\bar{L}_1$ and $\bar{\omega}_2 = \frac{1}{A}\bar{L}_2$.

The analytical solution for the attitude maneuvers, considering only the propulsive torque, is described by following Hamiltonian:

$$H^* = H_0^* + H_Q^*\ , \qquad (18)$$

with H_0^* and H_Q^* defined by eqs (13) and (15), respectively, and linearized around the reference attitude defined by (17).

A first order analytical solution for the problem of attitude corrections can be gotten by simple quadratures. This solution will be expressed by:

$$\Delta\mathbf{x} = E\,\boldsymbol{p}_{\mathbf{x}_0} + D\ , \qquad (19)$$

where : $\Delta\mathbf{x}$ represents the variations determined over the Andoyer's variables; $\boldsymbol{p}_{\mathbf{x}_0}$ means the initial values of the adjoint variables; E is a 6x6 matrix, related with the propulsive system and D is a 6x1 vector, related with the solution of non-perturbed problem. The adjoints variables are first integral of the canonical system, associated by the maximum Hamiltonian H^*, linearized around the reference attitude.

The solution (19) represents a complete solution for the attitude optimal corrections and contains secular terms and short periodic terms. The six integration constants must be determined in order to satisfy the two-point boundary value problem, which consists in starting in initial condition A_0 at initial time and reaching the final attitude A_1 at fixed final time .

The non-null elements of the matrices D and E, are given by:

$$d_1 = \bar{\omega}_1\ \Delta t \qquad d_2 = \bar{\omega}_2\ \Delta t$$

$$e_{11} = \frac{\Delta t}{\bar{L}_{20}^2\sin^2\bar{J}} \qquad e_{14} = \frac{1}{2}\left[\frac{1}{C}-\frac{1}{A}\right]\Delta t^2$$

$$e_{15} = \frac{1}{2}\left[\frac{1}{C}-\frac{1}{A}\right]\cos\bar{J}\,\Delta t^2 \qquad e_{25} = \frac{1}{2A}\ \Delta t^2$$

$$e_{31} = \frac{\sin\bar{\ell}_2 - \sin\bar{\ell}_{20}}{\bar{L}_{20}^2\,\bar{\omega}_2\,\sin\bar{I}\,\sin\bar{J}} \qquad e_{33} = \frac{\Delta t}{\bar{L}_{20}^2\sin^2\bar{I}}$$

$$e_{44} = e_{55} = e_{66} = \Delta t \qquad e_{45} = e_{54} = \cos\bar{J}\,\Delta t$$

$$e_{12} = \frac{1}{\bar{L}_{20}^2}\left[\frac{\cos\bar{J}\Delta t}{\sin^2\bar{J}} + \frac{\cot\bar{I}}{\bar{\omega}_2\sin\bar{J}}\left(\sin\bar{\ell}_2 - \sin\bar{\ell}_{20}\right)\right] -$$
$$-\frac{\cot.\bar{I}}{2\bar{L}_{20}\bar{\omega}_2^2}\left[\frac{1}{C}-\frac{1}{A}\right]\sum_{\varepsilon}\left[\cos(\bar{J}+\varepsilon\bar{\ell}_2) - \cos(\bar{J}+\varepsilon\bar{\ell}_{20})\right]$$

$$e_{13} = -\frac{\cot\bar{I}}{2\bar{L}_{20}\bar{\omega}_2^2\sin\bar{I}}\left[\frac{1}{C}-\frac{1}{A}\right]\sum_{\varepsilon}\varepsilon\left[\cos(\bar{J}+\varepsilon\bar{\ell}_2\ ) -\right.$$

$$- \cos(\bar{J} + \varepsilon\bar{\ell}_{20})\Big] - \frac{\sin\bar{\ell}_2 - \sin\bar{\ell}_{20}}{\bar{L}_{20}^2\bar{\omega}_2 \sin\bar{I}\sin\bar{J}}$$

$$e_{16} = -\frac{1}{4}\left[\frac{1}{C} - \frac{1}{A}\right]\left\{ \left[\cos(\bar{I} + \bar{J}) + \cos(\bar{I} - \bar{J})\right]\Delta t^2 - \right.$$

$$\left. -\frac{1}{\bar{\omega}_2^2}\sum_{\varepsilon,\delta}\left[\cos(\bar{I} + \delta\bar{J} + \varepsilon\bar{\ell}_2) - \cos(\bar{I} + \delta\bar{J} + \varepsilon\bar{\ell}_{20})\right]\right\} +$$

$$+ \frac{1}{2\bar{L}_{20}\bar{\omega}_2\sin\bar{I}}\sum_{\varepsilon}\left[\sin(\bar{I} + \varepsilon\bar{\ell}_2) - \sin(\bar{I} + \varepsilon\bar{\ell}_{20})\right]$$

$$e_{21} = -\frac{1}{\bar{L}_{20}^2\sin\bar{J}}\left[\frac{\cos\bar{J}\Delta t}{\sin\bar{J}} + \frac{\cot\bar{I}}{\bar{\omega}_2}\left(\sin\bar{\ell}_2 - \sin\bar{\ell}_{20}\right)\right]$$

$$e_{22} = \frac{2}{\bar{L}_{20}^2\bar{\omega}_2}\cot\bar{I}\cot\bar{J}\left(\mathrm{sen}\,\bar{\ell}_2 - \mathrm{sen}\,\ell_{20}\right) +$$

$$+ \frac{1}{\bar{L}_{20}^2}[\cot^2\bar{I} + \cot^2\bar{J}]\Delta t$$

$$e_{23} = e_{32} = -\frac{1}{\bar{L}_{20}^2\sin\bar{J}}\cot\bar{I}\ \Delta t +$$

$$+ \frac{1}{\bar{L}_{20}^2\bar{\omega}_2\sin\bar{J}}\cot\bar{J}\left(\sin\bar{\ell}_2 - \sin\ell_{20}\right)$$

$$e_{24} = \frac{\cot\bar{I}}{2\bar{L}_{20}\bar{\omega}_2}\sum_{\varepsilon}\left[\sin(\bar{J} + \varepsilon\bar{\ell}_2) - \sin(\bar{J} + \varepsilon\bar{\ell}_{20})\right] +$$

$$+ \frac{1}{2A}\cos\bar{J}\ \Delta t^2$$

$$e_{26} = -\frac{1}{4A}\left[\frac{1}{\bar{\omega}_2^2}\sum_{\varepsilon,\delta}\varepsilon\left[\sin(\bar{I} + 2\varepsilon\bar{J} + \delta\bar{\ell}_2) - \sin(\bar{I} + \right.\right.$$

$$\left.\left. + 2\varepsilon\bar{J} + \delta\bar{\ell}_{20})\right] - \left[\cos(\bar{I} + 2\bar{J}) + \cos(\bar{I} - 2\bar{J})\right]\Delta t^2\right] +$$

$$+ \frac{\cot\bar{J}}{2\bar{L}_{20}\bar{\omega}_2}\sum_{\varepsilon}\left[\sin(\bar{I} + \varepsilon\bar{\ell}_2) - \sin(\bar{I} + \varepsilon\bar{\ell}_{20})\right] -$$

$$- \frac{\cot\bar{I}}{4\bar{L}_{20}\bar{\omega}_2}\left[\frac{1}{2}\sum_{\varepsilon}\left[\sin(\bar{I} + 2\varepsilon\bar{\ell}_2) - \sin(\bar{I} + 2\varepsilon\bar{\ell}_{20})\right] + \right.$$

$$+ \sum_{\varepsilon,\delta}\delta\left[\sin(\bar{I} - 2\delta\bar{J} + \varepsilon\bar{\ell}_2) - \sin(\bar{I} - 2\delta\bar{J} + \varepsilon\bar{\ell}_{20})\right] +$$

$$\left. + \frac{1}{4}\sum_{\varepsilon,\delta}\left[\sin(\bar{I} + 2\varepsilon\bar{J} + 2\delta\bar{\ell}_2) - \sin(\bar{I} + 2\varepsilon\bar{J} + 2\delta\bar{\ell}_{20})\right]\right]$$

$$e_{34} = e_{43} = \sum_{\varepsilon}\frac{\left[\sin(\bar{J} + \varepsilon\bar{\ell}_2) - \sin(\bar{J} + \varepsilon\bar{\ell}_{20})\right]}{2\bar{L}_{20}\bar{\omega}_2\sin\bar{I}}$$

$$e_{36} = e_{63} = \frac{1}{\bar{L}_{20}\bar{\omega}_2}\left[-\frac{1}{8}\sum_{\varepsilon}\left[\cos 2(\bar{J} + \varepsilon\bar{\ell}_2) - \right.\right.$$

$$\left. -\cos 2(\bar{J} + \varepsilon\bar{\ell}_{20})\right] - \frac{1}{\sin\bar{I}}\sum_{\varepsilon,\delta}\delta\left[\sin(\bar{I} + 2\delta\bar{J} + \varepsilon\bar{\ell}_2) - \right.$$

$$\left.\left. - \sin(\bar{I} + 2\delta\bar{J} + \varepsilon\bar{\ell}_{20})\right] - \frac{1}{4}\left(\cos 2\bar{\ell}_2 - \cos 2\bar{\ell}_{20}\right)\right]$$

$$e_{42} = \frac{\cot\bar{I}}{\bar{L}_{20}\bar{\omega}_2}\sum_{\varepsilon}\left[\sin(\bar{J} + \varepsilon\bar{\ell}_2) - \sin(\bar{J} + \varepsilon\bar{\ell}_{20})\right]$$

$$e_{46} = e_{64} = -\frac{1}{2}\left[\cos(\bar{I} + \bar{J}) + \cos(\bar{I} - \bar{J})\right]\Delta t +$$

$$+ \frac{1}{2\bar{\omega}_2}\sum_{\varepsilon,\delta}\varepsilon\left[\sin(\bar{I} + \delta\bar{J} + \varepsilon\bar{\ell}_2) - \sin(\bar{I} + \delta\bar{J} + \varepsilon\bar{\ell}_{20})\right]$$

$$e_{56} = e_{65} = -\frac{1}{2}\left[\cos(\bar{I} + 2\bar{J}) + \cos(\bar{I} - 2\bar{J})\right]\Delta t +$$

$$+ \frac{1}{4\bar{\omega}_2}\sum_{\varepsilon,\delta}\varepsilon\left[\sin(\bar{I} + \delta\bar{J} + \varepsilon\bar{\ell}_2) - \sin(\bar{I} + \delta\bar{J} + \varepsilon\bar{\ell}_{20})\right]$$

$$e_{61} = \frac{1}{2\bar{L}_{20}\bar{\omega}_2\sin\bar{J}}\sum_{\varepsilon}\left[\sin(\bar{I} + \varepsilon\bar{\ell}_2) - \sin(\bar{I} + \varepsilon\bar{\ell}_{20})\right]$$

$$e_{62} = \frac{\cot\bar{J}}{2\bar{L}_{20}\bar{\omega}_2}\sum_{\varepsilon}\left[\sin(\bar{I} + \varepsilon\bar{\ell}_2) - \sin(\bar{I} + \varepsilon\bar{\ell}_{20})\right] -$$

$$- \frac{\cot\bar{I}}{4\bar{L}_{20}\bar{\omega}_2}\left[\frac{1}{2}\sum_{\varepsilon}\left[\sin(\bar{I} + 2\varepsilon\bar{\ell}_2) - \sin(\bar{I} + 2\varepsilon\bar{\ell}_{20})\right] - \right.$$

$$\left. - \frac{1}{4}\sum_{\varepsilon,\delta}\delta\left[\sin(\bar{I} - 2\varepsilon\bar{J} + \delta\bar{\ell}_2) - \sin(\bar{I} - 2\varepsilon\bar{J} + \delta\bar{\ell}_{20})\right]\right] -$$

$$- \sum_{\varepsilon,\delta}\delta\left[\sin(\bar{I} - 2\delta\bar{J} + \varepsilon\bar{\ell}_2) - \sin(\bar{I} - 2\delta\bar{J} + \varepsilon\bar{\ell}_{20})\right], \qquad (20)$$

where Δt is the duration of the maneuver.

For artificial satellite with spherical symmetry, A = C, the analytical solutions given by (18) to (20) simplifies, since H_0^* is computed by:

$$H_0^* = \frac{\bar{L}_{20}}{A}\,p_2, \qquad (21)$$

and $d_1 = 0$, $e_{12} = e_{21}$, $e_{13} = e_{31}$, $e_{16} = e_{61}$, $e_{14} = e_{15} = 0$.

The attitude's corrections maneuvers problem can be simplified if we consider only the secular terms. The Hamiltonian function associated to long duration maneuvers is described by eqs. (14) and (18), with the perturbed Hamiltonian given by:

$$H_Q^* = \frac{1}{2\bar{L}_{20}^2}\left\{\frac{p_{10}^2}{\sin^2\bar{J}} + p_{20}^2\left[\cot^2\bar{I} + \cot^2\bar{J}\right] + \right.$$

$$+\frac{p_{30}^2}{\sin^2\bar{I}}\Bigg\}+\frac{1}{2}\left[P_{10}^2+P_{20}^2+P_{30}^2\right]-\frac{p_{20}}{\bar{L}_{20}^2}\left[\frac{p_{10}\cot\bar{J}}{\sin\bar{J}}+\right.$$

$$\left.+\frac{p_{30}\cot\bar{I}}{\sin\bar{I}}\right]+P_{10}P_{20}\cos\bar{J}-\frac{1}{2}P_{30}\Big\{P_{10}\left[\cos(\bar{I}+\bar{J})+\right.$$

$$\left.+\cos(\bar{I}-\bar{J})\right]-P_{20}\left[\cos(\bar{I}+2\bar{J})+\cos(\bar{I}-2\bar{J})\right]\Big\}. \qquad (22)$$

The differential equations for the maneuvers are:

$$\frac{d\ell_1}{dt}=\left[\frac{1}{C}-\frac{1}{A}\right]\left[\bar{L}_{10}+\frac{d}{dt}\left(t\,\Delta\bar{L}_1\right)-t\frac{d\bar{L}_1}{dt}\right]+$$

$$+\frac{p_{10}}{\bar{L}_{20}^2\sin^2\bar{J}}-\frac{p_{20}\cot\bar{J}}{\bar{L}_{20}^2\sin\bar{J}}$$

$$\frac{d\ell_2}{dt}=\frac{1}{A}\left[\bar{L}_{20}+\frac{d}{dt}\left(t\,\Delta\bar{L}_2\right)-t\frac{dL_2}{dt}\right]-\frac{p_{10}\cot\bar{J}}{\bar{L}_{20}^2\sin\bar{J}}+$$

$$+\frac{p_{20}}{\bar{L}_{20}^2}\left[\cot^2\bar{I}+\cot^2\bar{J}\right]-\frac{p_{30}\cot\bar{I}}{\bar{L}_{20}^2\sin\bar{I}}$$

$$\frac{d\ell_3}{dt}=-\frac{p_{20}\cot\bar{I}}{\bar{L}_{20}^2\sin\bar{I}}+\frac{p_{30}}{\bar{L}_{20}^2\sin^2\bar{I}}$$

$$\frac{dL_1}{dt}=P_{10}+P_{20}\cos\bar{J}-\frac{1}{2}P_{30}\left[\cos(\bar{I}+\bar{J})+\cos(\bar{I}-\bar{J})\right]$$

$$\frac{dL_2}{dt}=P_{10}\cos\bar{J}+P_{20}-$$

$$-\frac{1}{2}P_{30}\left[\cos(\bar{I}+2\bar{J})+\cos(\bar{I}-2\bar{J})\right]$$

$$\frac{dL_3}{d\,t}=-\frac{1}{2}P_{10}\left[\cos(\bar{I}+\bar{J})+\cos(\bar{I}-\bar{J})\right]-$$

$$-\frac{1}{2}P_{20}\left[\cos(\bar{I}+2\bar{J})+\cos(\bar{I}-2\bar{J})\right]+P_{30}. \qquad (23)$$

The simplified analytical solutions for eqs. (23) are given in the matrix form as:

$$\Delta\,\mathbf{x}=E_S\,\boldsymbol{p}_{\mathbf{x}_0}+D_S, \qquad (24)$$

where the subscript S means the secular part, with non-null elements of the matrices D_S and E_S are presented in Santos et al[5] .

The analytical solutions for the optimal corrections of the artificial satellite with spherical symmetry , whose Hamiltonian is described in the eqs (18), (21) and (22), are also given by the matrix form (24), but $D_S=[0\ \ d_2\ \ 0\ \ 0\ \ 0\ \ 0]^T$, with $d_2=\bar{\varpi}_2\ \Delta t$ and $e_{14}=e_{15}=e_{16}=0$.

If we consider the satellite with the cylindrical symmetry and the influence of the gravity gradient torque, the Hamiltonian function will be given by (13), (14) e (22). The Hamiltonian function $H_G{}^*$ associated with the gravity gradient torque , taking in account terms up to the inverse of the cubic of the distance between the Earth's center of mass and the satellite's center of mass and assuming circular orbit, the orbital inclination and the longitude of ascending node equal zero degree, is given by:

$$H_G{}^*=\frac{3}{4\bar{L}_{20}}\frac{\mu}{r^3}[C-A]\Big\{\ p_{10}\left[1-3\cos^2\bar{I}\right]\cos\bar{J}-$$

$$-p_{20}\left[\left[1-3\cos^2\bar{I}\right]\cos^2\bar{J}+\left[1-3\cos^2\bar{J}\right]\cos^2\bar{I}\right]+$$

$$+\ p_{30}\left[1-3\cos^2\bar{J}\right]\ \cos\bar{I}\ \Big\}, \qquad (25)$$

where μ is the Gaussian constant and r is the radial distance.

If the satellite's orbit is circular and the gravity gradient torque is taken in account during the long duration maneuvers, the analytical solutions are also given by eq. (24), but the elements of $D_S=[d_1\ \ d_2\ \ d_3\ \ 0\ \ 0\ \ 0]^T$ are expressed by:

$$d_1=\left[\frac{1}{C}-\frac{1}{A}\right]\bar{L}_{10}+\frac{3\mu}{4r^3}\left(\frac{C-A}{\bar{L}_{20}}\right)\left[1-3\cos^2\bar{I}\right]\cos\bar{J}\Delta t$$

$$d_2=\frac{1}{A}\bar{L}_{20}-\frac{3\mu}{4r^3}\left(\frac{C-A}{\bar{L}_{20}}\right)\left[\left[1-3\cos^2\bar{I}\right]\cos^2\bar{J}+\right.$$

$$\left.+\left[1-3\cos^2\bar{J}\right]\ \cos^2\bar{I}\right]\Delta t$$

$$d_3=\frac{3\mu}{4r^3}\left(\frac{C-A}{\bar{L}_{20}}\right)\ \left[1-3\cos^2\bar{J}\right]\cos\bar{I}\,\Delta t. \qquad (26)$$

For elliptic orbit, the Hamiltonian $H_G{}^*$, taking in account terms up to second order in eccentricity and assuming the orbital inclination and the longitude of ascending node equal zero degree, has the following form:

$$H_G{}^*=\frac{3\mu}{a^3}\left(\frac{C-A}{4\bar{L}_{20}}\right)\left[1+\frac{3}{2}e^2\right]\Big\{p_{10}\left[1-3\cos^2\bar{I}\right]\cos\bar{J}-$$

$$- p_{20}\left[\left[1 - 3\cos^2 \bar{I}\right]\cos^2 \bar{J} + \left[1 - 3\cos^2 \bar{J}\right]\cos^2 \bar{I}\right] +$$

$$+ \; p_{30}\left[1 - 3\cos^2 \bar{J}\right] \cos \bar{I} \quad \Big\} . \qquad (27)$$

where a is the semi-major axis and e is the eccentricity . In this case , the analytical solution has also the matrix form (24), with $D_S = [d_1 \;\; d_2 \;\; d_3 \;\; 0 \;\; 0 \;\; 0]^T$ and

$$d_1 = \frac{3\mu}{4a^3}\left(\frac{C - A}{\bar{L}_{20}}\right)\left[1 + \tfrac{3}{2}e^2\right]\left[1 - 3\cos^2 \bar{I}\right]\cos\bar{J}\Delta t +$$

$$+ \left[\tfrac{1}{C} - \tfrac{1}{A}\right] \bar{L}_{10}$$

$$d_2 = \frac{3\mu}{4a^3}\left(\frac{C - A}{\bar{L}_{20}}\right)\left[1 + \tfrac{3}{2}e^2\right]\left[\left[1 - 3\cos^2 \bar{I}\right]\cos^2 \bar{J} +\right.$$

$$\left. + \left[1 - 3\cos^2 \bar{J}\right] \cos^2 \bar{I}\right] \Delta t + \tfrac{1}{A} \bar{L}_{20}$$

$$d_3 = \frac{3\mu}{4a^3}\left(\frac{C\text{-}A}{\bar{L}_{20}}\right)\left[1 + \tfrac{3}{2}e^2\right]\left[1 - 3\cos^2 \bar{J}\right]\cos\bar{I}\Delta t \, . \qquad (28)$$

THE OPTIMAL CONSUMPTION

The optimal consumption J that is necessary for the satellite's attitude correction is obtained by quadrature of the equation:

$$\dot{J} = \tfrac{1}{2} \; Q^{*2} , \qquad (29)$$

where the components of the optimal torques are expressed in terms of the Andoyer's variables and their adjoints by:

$$Q_x^* = \frac{p_1 \cos \ell_1}{\bar{L}_2 \sin \bar{J}} - \frac{p_2}{\bar{L}_2}\left[\cos\ell_1 \cot \bar{J} + \cot \bar{I}(\cos\ell_1 \cos\ell_2 - \right.$$

$$\left. - \sin\ell_1 \sin\ell_2 \cos\bar{J})\right] - \frac{p_3}{\bar{L}_2 \sin \bar{I}} \left(\cos\ell_1 \cos\ell_2 - \right.$$

$$\left. - \sin\ell_1 \sin\ell_2 \cos\bar{J}\right) + P_2 \sin\bar{J} \sin\ell_1 +$$

$$+ P_3\left[\sin\bar{I}\cos\ell_1\sin\ell_2 + \sin\ell_1(\cos\bar{I}\sin\bar{J} + \sin\bar{I}\cos\bar{J}\cos\ell_2)\right]$$

$$Q_y^* = \frac{p_1 \sin \ell_1}{\bar{L}_2 \sin \bar{J}} + \frac{p_2}{\bar{L}_2}\left[\sin\ell_1 \cot \bar{J} + \cot \bar{I}(\sin\ell_1 \cos\ell_2 + \right.$$

$$\left. + \cos\ell_1 \sin\ell_2 \cos\bar{J})\right] - \frac{p_3}{\bar{L}_2 \sin \bar{I}} \left(\sin\ell_1 \cos\ell_2 + \right.$$

$$\left. + \cos\ell_1 \sin\ell_2 \cos\bar{J}\right) + P_2 \sin\bar{J} \cos\ell_1 +$$

$$+ P_3\left[-\sin\bar{I} \sin\ell_1 \sin\ell_2 + \cos\ell_1(\cos\bar{I} \sin\bar{J} + \sin\bar{I}\cos\bar{J}\cos\ell_2)\right]$$

$$Q_z^* = \frac{p_2}{\bar{L}_2}\sin\ell_2 \sin \bar{J} \cot \bar{I} + \frac{p_3}{\bar{L}_2 \sin \bar{I}}\sin\ell_2 \text{sen}\bar{J} +$$

$$+ P_1 + P_2\cos\bar{J} + P_3(\cos\ell_2 \sin\bar{I} \sin\bar{J} - \cos\bar{I} \cos\bar{J}) \, . \qquad (30)$$

In terms of the elements of the matrix E and the initial values of the adjoint variables, the optimal consumption is given by:

$$\Delta J = \tfrac{1}{2} e_{11} p_{10}^2 + \tfrac{1}{2} e_{22} p_{20}^2 + \tfrac{1}{2} e_{33} p_{30}^2 + \tfrac{1}{2} e_{44} P_{10}^2 +$$

$$+ \tfrac{1}{2} e_{55} P_{20}^2 + \tfrac{1}{2} e_{66} P_{30}^2 + e_{21}\, p_{10} p_{20} + e_{31}\, p_{10} p_{30} +$$

$$+ e_{61} p_{10} P_{30} + e_{23} p_{20} p_{30} + e_{34} p_{30} P_{10} + e_{36} p_{30} P_{30} +$$

$$+ e_{42} p_{20} P_{10} + e_{45} P_{10} P_{20} + e_{46} P_{10} P_{30} +$$

$$+ e_{56} P_{20} P_{30} + e_{62}\, p_{20} P_{30} \, . \qquad (31)$$

In the case where only the secular terms are considered, the eq. (31) is simplified and given in terms of the elements of the matrix E_S:

$$\Delta J_S = \tfrac{1}{2} e_{11} p_{10}^2 + \tfrac{1}{2} e_{22} p_{20}^2 + \tfrac{1}{2} e_{33} p_{30}^2 + \tfrac{1}{2} e_{44} P_{10}^2 +$$

$$+ \tfrac{1}{2} e_{55} P_{20}^2 + \tfrac{1}{2} e_{66} P_{30}^2 + e_{12}\, p_{10} p_{20} + e_{23}\, p_{20} p_{30} +$$

$$+ e_{45} P_{10} P_{20} + e_{46} P_{10} P_{30} + e_{56} P_{20} P_{30} \, . \qquad (32)$$

NUMERICAL SIMULATION

In this section, numerical results are presented for a long-time attitude maneuver of a cylindrical satellite. The effects of gravity gradient torque are not included in this simulation. The minimal consumption, the magnitudes of the optimal torques and the evolution of the analytical solution are computed by using eqs (32), (11) and (24), respectively. The physical characteristic of the satellite, the initial state and the final state are presented in the following tables. The initial values for the adjoint variables are gotten numerically solving the algebraic system (24).

The temporal evolutions of the Euler angles (ϕ, θ, ψ) and their rate variations ($\dot{\phi}$, $\dot{\theta}$, $\dot{\psi}$) are shown in the figures 3 and 4, respectively, during the attitude maneuver . The propagations of the Andoyer variables $(\ell_1, \ell_2, \ell_3, L_1, L_2, L_3)$ are shown in the figures 5 and 6. The magnitudes of the optimal torques and the magnitude of the performance index are presented in the figures 7 and 8, respectively.

Table 1 – Physical characteristic
Initial mass = $1.0\text{x}\ 10^3$ kg
Principal Moments of inertia
A = B = $3.95\text{x}10^2$ kg m ^{2} C = $1.05\text{x}10^2$ kg m ^{2}

Table 2 - Initial and Final values for the Euler Angles and their Rate Variations	
Initial Attitude t_0 = 0.0 s	*Final Attitude* t_f = $5.0\text{x}10^4$ s
$\phi_0 = 1.0\text{x}10^1$ deg	$\phi_f = 1.3\text{x}10^1$ deg
$\theta_0 = 1.5\text{x}10^1$ deg	$\theta_f = 1.7\text{x}10^1$ deg
$\psi_0 = 1.0\text{x}10^1$ deg	$\psi_f = 1.1\text{x}10^1$ deg
$\dot{\phi}_0 = 2.0\text{x}10^{-3}$ deg /s	$\dot{\phi}_f = 1.7\text{x}10^{-3}$ deg /s
$\dot{\theta}_0 = 2.0\text{x}10^{-3}$ deg /s	$\dot{\theta}_f = 2.0\text{x}10^{-3}$ deg /s
$\dot{\psi}_0 = .2.0\text{x}10^{-3}$ deg /s	$\dot{\psi}_f = 1.5\text{x}10^{-3}$ deg /s

Table 3 - Initial Adjoint Variables
p_{10} = - $3.858\text{x}10^{-1}$
p_{20} = - $6.671\text{x}10^{-1}$
p_{30} = $1.0\text{x}10^{-2}$
P_{10} = - $1.91\text{x}10^{-2}$
P_{20} = - $1.91\text{x}10^{-2}$
P_{30} = - $1.78\text{x}10^{-2}$
P_{J0} = - 1.0

Table 4 : Terminal Values for the Andoyer's Variables	
Initial Attitude t_0 = 0 s	Final Attitude t_f= $5\text{x}10^4$ s
ℓ_{10} = $8.55\text{x}10^1$ deg	ℓ_{1f} = $8.70\ \text{x}10^1$ deg
ℓ_{20} =$1.97\ \text{x}10^2$ deg	ℓ_{2f} =$1.98\ \text{x}10^2$ deg
ℓ_{30} = $1.07\ \text{x}10^2$ deg	ℓ_{3f} =$1.1\ \text{x}10^2$ deg
L_{10} = $2.35\ \text{x}10^{-1}$ kg m^2 / s	L_{1f} = $1.87\ \text{x}10^{-1}$ kg m^2 / s
L_{20} = $5.21\ \text{x}10^{-1}$ kg m^2 / s	L_{2f} = $5.00\ \text{x}10^{-1}$ kg m^2 / s
L_{30} = $2.57\ \text{x}10^{-1}$ kg m^2 / s	L_{3f} = $2.12\ \text{x}10^{-1}$ kg m^2 / s

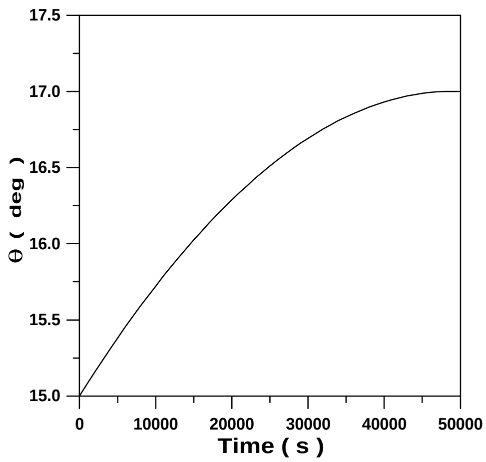

Figure 3a – Evolution of Euler Angle θ

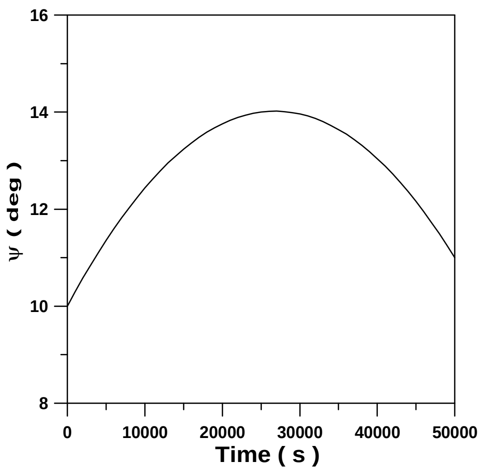

Figure 3b – Evolution of Euler Angle ψ

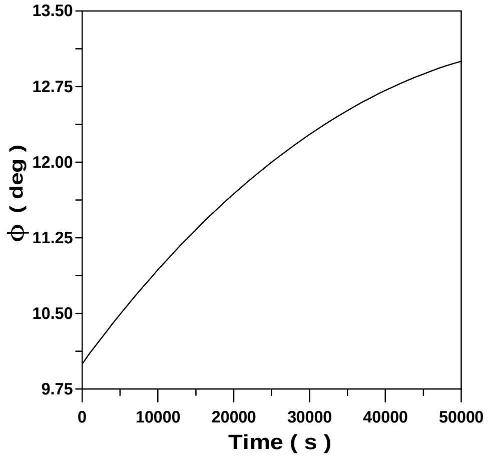

Figure 3c– Evolution of Euler Angle φ

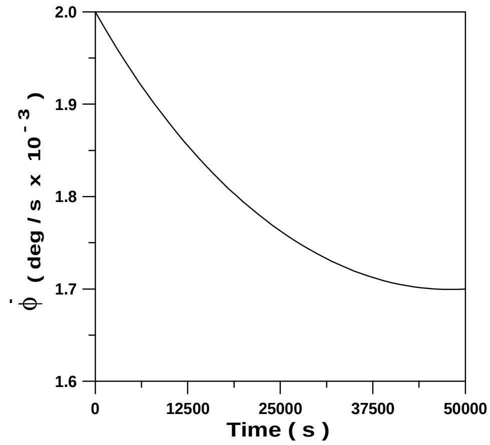

Figure 4c – Evolution of rate variation $\dot{\phi}$

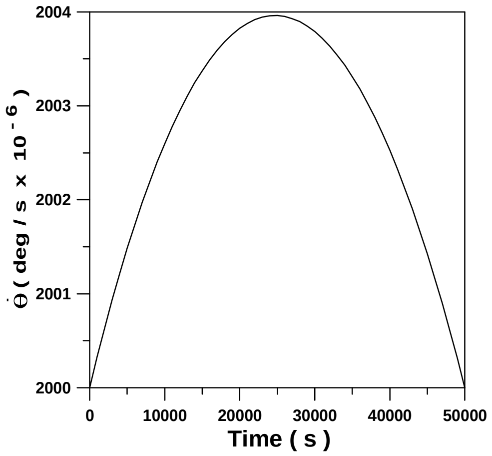

Figure 4a – Evolution of Rate variation $\dot{\theta}$

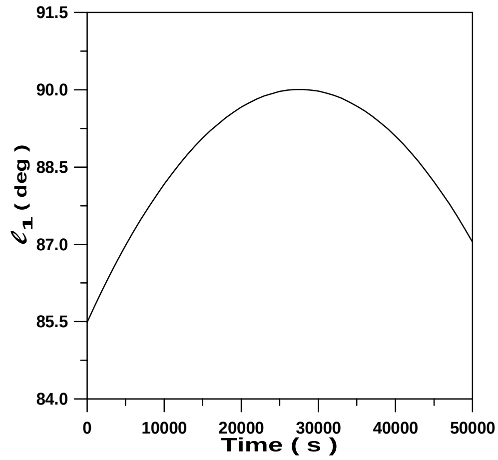

Figure 5a – Evolution of the Andoyer variable ℓ_1

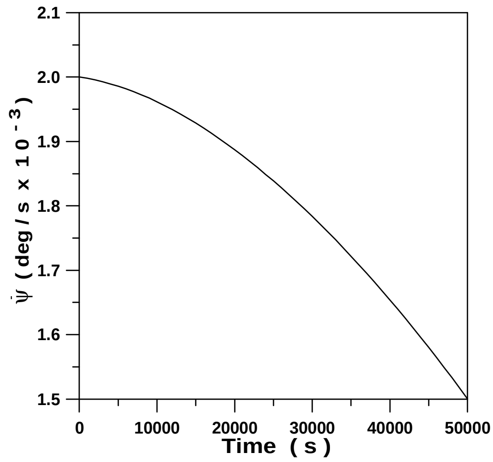

Figure 4b – Evolution of rate variation $\dot{\psi}$

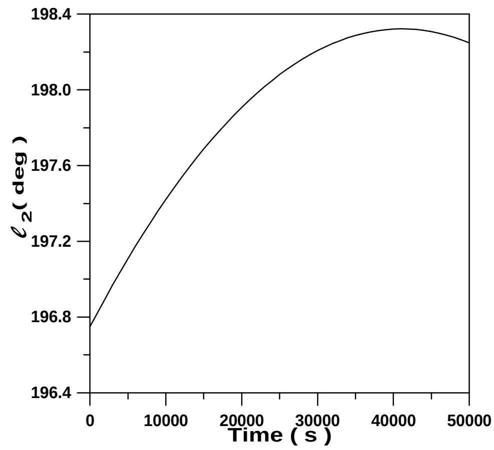

Figure 5b – Evolution of the Andoyer variable ℓ_2

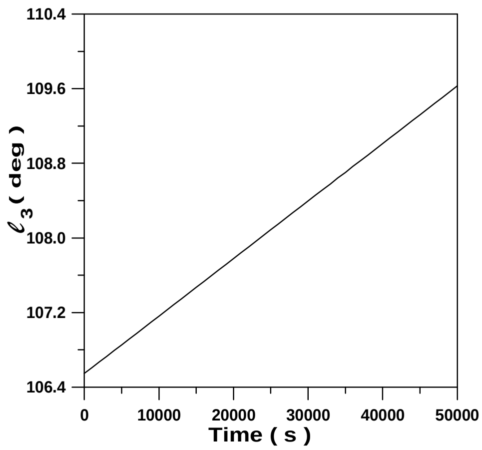

Figure 5c – Evolution of the Andoyer variable ℓ_3

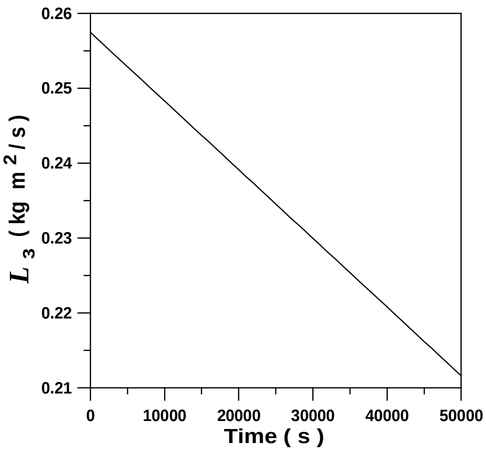

Figure 6c – Evolution of the Andoyer variable L_3

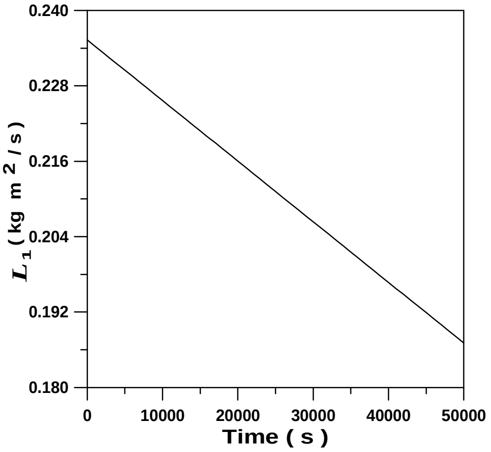

Figure 6a – Evolution of the Andoyer variable L_1

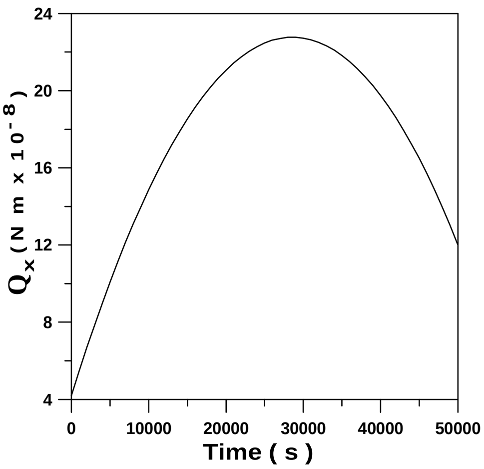

Figure 7a – Evolution of control variable Q_x

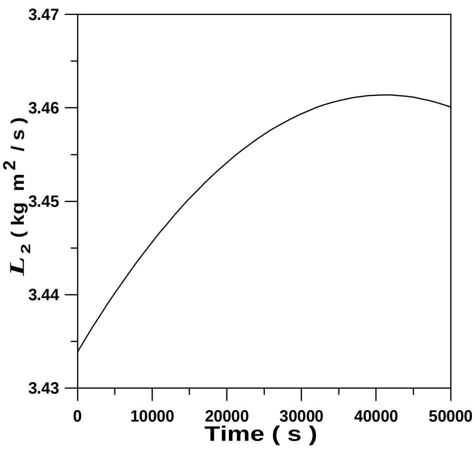

Figure 6b – Evolution of the Andoyer variable L_2

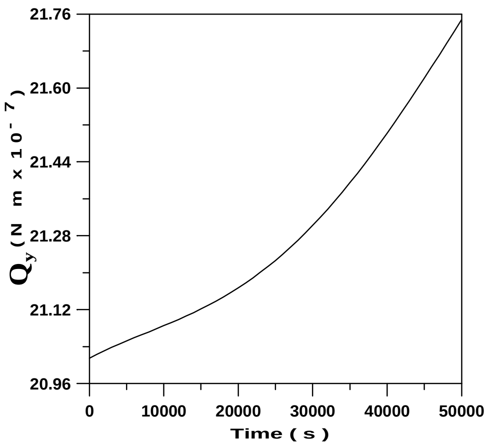

Figure 7b – Evolution of control variable Q_y

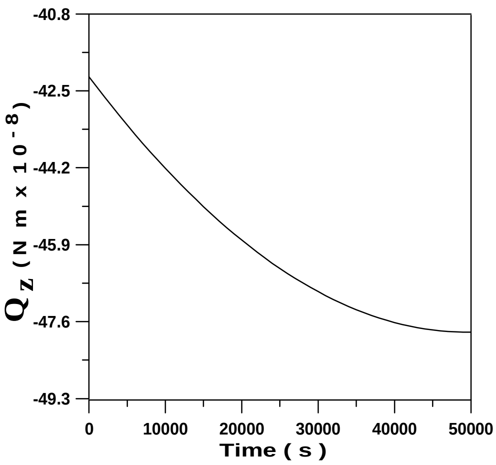

Figure 7c – Evolution of control variable Q_z

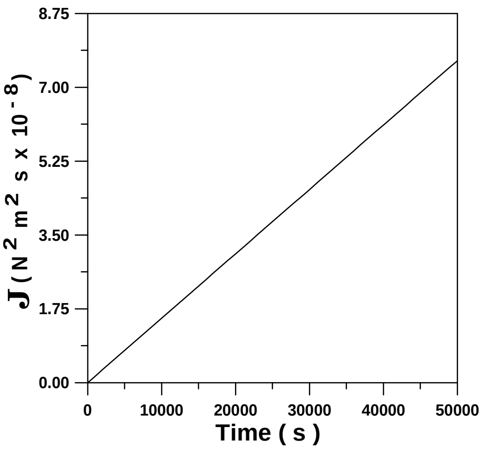

Figure 8 – Evolution of the performance index J

CONCLUSION

The first order solution presented here for the optimal maneuvers for artificial satellite's attitude corrections shows that the perturbations due to the gravity gradient torque and to the propulsive system are uncoupled and that is possible to establish an optimal control law for artificial satellite's attitude. The presented results are similar to the solution determined for orbit corrections maneuvers made with identical propulsion system[6].

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