OPTIMAL CONTROL AND SMOOTHING TECHNIQUES FOR COMPUTING MINIMUM FUEL ORBITAL TRANSFERS AND RENDEZVOUS

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ABSTRACT

We investigate in this paper the computation of minimum fuel orbital transfers and rendezvous. Each problem is seen as an optimal control problem and is solved by means of shooting methods [1]. This approach corresponds to the use of Pontryagin's Maximum Principle (PMP) [2-4] and leads to the solution of a Two Point Boundary Value Problem (TPBVP). It is well known that this last one is very difficult to solve when the performance index is fuel consumption because in this case the optimal control law has a particular discontinuous structure called "bang-bang".

We will show how to modify the performance index by a term depending on a small parameter in order to yield regular controls. Then, a continuation method on this parameter will lead us to the solution of the original problem. Convergence theorems will be given.

Finally, numerical examples will illustrate the interest of our method. We will consider two particular problems: The GTO (Geostationary Transfer Orbit) to GEO (Geostationary Equatorial Orbit) transfer and the LEO (Low Earth Orbit) rendezvous.

1. INTRODUCTION

The computation of optimal "bang-bang" controls is of particular interest because of the difficulty in obtaining the optimal solution when using the PMP. In this case, the main difficulty comes from the fact that the shooting function [1] may not be defined everywhere, may be discontinuous or non differentiable at some points and may have a singular Jacobian matrix on a large domain. This induces a very small convergence radius for the solving methods as Newton's or Powell's [5].

One way to avoid this difficulty is to add a specific term to the objective function depending on a small parameter e. For each e, the optimal control associated with the perturbed performance index is regular and e has to be updated through a continuation or homotopy [6] method. This general approach, called "continuation-smoothing method", has been developed by the authors in [7]. A statistical point of view in terms of density functions throws light on the method (see [7]) and provides a general framework for deriving new perturbed performance indexes and regular controls.

2. PROBLEM STATEMENT

2.1 Formulation

Let us denote by x the vector of equinoxial orbital parameters:

$$x = \begin{pmatrix} a \\ e_x = e\cos(w+W) \\ e_y = e\sin(w+W) \\ h_x = \tan(i/2)\cos(W) \\ h_y = \tan(i/2)\sin(W) \\ L = w + W + v \end{pmatrix} \quad (1)$$

where (a,e,i,w,W,v) denotes the set of classical Keplerian parameters. We will use Gauss equations associated with parameters (1) to describe the motion of the satellite and the perturbative accelerations will be expressed in the (T,N,W) local orbital frame where vector T in collinear to the satellite velocity vector.

Thus, our problem falls under the form of the following optimal control problem:

$$\begin{cases} Min \quad J(u) = -m(t_f) \\ \dot{x}(t) = f(x(t)) + F_{max}\, g(x(t)).\dfrac{u(t)}{m(t)} \\ \dot{m}(t) = -F_{max}\dfrac{\|u(t)\|}{g_0 Isp} \\ \|u(t)\| \le 1 \quad t \in [t_0, t_f] \\ x(t_0) = x_0 \qquad y(x(t_f)) = 0 \\ m(t_0) = m_0 \end{cases} \quad (2)$$

where m denotes the mass of the satellite, F_{max} the maximum thrust modulus of the engine and Isp its specific impulse. The constant g_0, u, t_0 and t_f denote respectively the Earth's sea-level gravitational acceleration, the control variable i.e. the scaled thrust vector and the initial and final dates **which are both fixed**.

In the case of an orbital transfer, all the components of x are fixed at time t_f except the true longitude L. So, the final conditions take the form:

$$y\,(x(t_f)) = \begin{bmatrix} a(t_f) - a_f \\ e_x(t_f) - e_{x,f} \\ e_y(t_f) - e_{y,f} \\ h_x(t_f) - h_{x,f} \\ h_y(t_f) - h_{y,f} \end{bmatrix} \quad (3)$$

In the case of a rendezvous, all the components of x are fixed at time t_f for a given number of revolutions. Thus, the final conditions can be written under the following compact form:

$$y\,(x(t_f)) = x(t_f) - x_f \quad (4)$$

In order to apply the PMP, we have to build first the Hamiltonian of problem (2) given by:

$$H(x(t),m(t),u(t),p(t)) = -F_{max}\frac{\|u(t)\|}{g_0 Isp}p_m(t) + p_x(t)^T\left(f(x(t)) + F_{max}\,g(x(t)).\frac{u(t)}{m(t)}\right) \quad (5)$$

where $p(t) = \begin{bmatrix} p_x(t) \\ p_m(t) \end{bmatrix}$ and ^{T} denote the costate vector and the transpose operator respectively.

Then, the optimal control $\overline{u}(t)$ is obtained by minimizing the Hamiltonian (5):

$$\overline{u}(t) = arg \min_{\|w\|\le 1} -\frac{p_m(t)}{g_0 Isp}\|w\| + p_x(t)^T g(x(t)).\frac{w}{m(t)} \quad (6)$$

and this yields **a "bang-bang" control law**.

In fact, let the switching function $r(t)$ be given by:

$$r(t) = \frac{\|g(x(t))^T.p_x(t)\|}{m(t)} + \frac{p_m(t)}{g_0 Isp} \quad (7)$$

and let us define $b(t)$ by:

$$b(t) = \begin{cases} 0 & if\ r(t) < 0 \\ 1 & if\ r(t) > 0 \\ z \in [0,1] & if\ r(t) = 0 \end{cases} \quad (8)$$

Then, if $g(x(t))^T.p_x(t) = 0$, $\overline{u}(t)$ has only to satisfy:

$$\|\overline{u}(t)\| = b(t) \quad (9)$$

and if $g(x(t))^T.p_x(t) \neq 0$, $\overline{u}(t)$ is given by:

$$\overline{u}(t) = -b(t)\frac{g(x(t))^T.p_x(t)}{\|g(x(t))^T.p_x(t)\|} \quad (10)$$

2.2 Solution method

The PMP gives necessary optimality conditions for problem (2). In addition to (6), it gives the costates differential equations:

$$\begin{cases} \dot{p}_x(t) = -\frac{\partial H}{\partial x}(x(t),m(t),\overline{u}(t),p(t)) \\ \dot{p}_m(t) = -\frac{\partial H}{\partial m}(x(t),m(t),\overline{u}(t),p(t)) \end{cases} \quad (11)$$

and the following transversality conditions:

$$p_m(t_f) = -1 \quad (12)$$

and, only in the case of a transfer:

$$p_L(t_f) = 0 \quad (13)$$

where p_L denotes the last component of vector p_x.

So, the PMP yields a TPBVP defined by the states differential equations with initial and final conditions (see (2), (3) and (4)) and by (7), (8), (9), (10), (11), (12), and (13) in the case of a transfer. Then, the numerical solution of problem (2) reduces to the computation of the zero of a function, called the shooting function, whose unknown is vector $p(t_0)$.

2.3 Numerical issues

When solving the shooting equations associated with a "bang-bang" optimal control problem, numerical difficulties arise. First, for computing the shooting function itself it is necessary to integrate a system of Ordinary Differential Equations (ODEs) whose right-hand side is a discontinuous function of time. So, common algorithms like Runge-Kutta's one with adaptive step encounter problems to achieve the required precision especially when the number of switching dates is large [8-9] and the shooting function is then evaluated with a poor precision. On the other hand, the convergence theorems for Newton's method require the equations to be twice continuously differentiable and require their Jacobian matrix to be non singular in the vicinity of the solution. So, Newton's or Powell's methods often fail to converge in the case of non smooth equations. This is why smoothing techniques are so useful in this framework.

3. THE CONTINUATION-SMOOTHING METHOD

3.1 The method

The basic idea of the method developed in [7] is to deduce the solution of problem (2) from the successive solutions of an auxiliary problem. This one takes the following form:

$$\begin{cases} Min \quad J_e(u) = -m(t_f) - e\,F_{max}\int_{t_0}^{t_f} F\left(\|u(t)\|\right)dt \\ \dot{x}(t) = f(x(t)) + F_{max}\, g(x(t)).\dfrac{u(t)}{m(t)} \\ \dot{m}(t) = -F_{max}\dfrac{\|u(t)\|}{g_0 Isp} \\ \|u(t)\| \le 1 \quad t \in \left[t_0, t_f\right] \\ x(t_0) = x_0 \qquad y\,(x(t_f)) = 0 \\ m(t_0) = m_0 \end{cases} \qquad (14)$$

where F is a continuous function and where e is assumed to be in the interval $]0,1]$.

Depending on the properties of F and according to the vocabulary used in mathematical programming, this approach may be called a penalty or a barrier one. In fact, if $|F(w)| \to +\infty$ as w approaches one or zero, F is called a barrier function otherwise it is a penalty one.

The "continuation-smoothing method" consists first in solving problem (14) for $e = 1$. Then, after defining a decreasing sequence of e values $e_1 = 1 > e_2 > \dots e_n > 0$, the current TPBVP for problem (14), associated with $e = e_k, k = 2 \dots n$ is solved with the solution of the previous one as a starting point. This iterative process terminates for example when a certain precision on the performance index is reached:

$$\left| J_{e_{k+1}}(\bar{u}_{e_{k+1}}) - J_{e_k}(\bar{u}_{e_k}) \right| \le h.\left(e_k - e_{k+1}\right) \quad h > 0 \quad (15)$$

where $\bar{u}_{e_k}$ denotes the optimal control solution of problem (14) for $e = e_k$.

Now, in the case F satisfies one of the two following properties:

$$F(w) \ge 0 \quad \forall w \in \left[0,1\right] \qquad (16)$$

$$F(w) \le 0 \quad \forall w \in \left[0,1\right] \qquad (17)$$

some interesting propositions can be derived (see [7] for the proofs). If F satisfies (16), we obtain:

<u>*Proposition 1:*</u>

$$J_{e_1}(\bar{u}_{e_1}) \le J_{e_2}(\bar{u}_{e_2}) \le \dots \le J_{e_n}(\bar{u}_{e_n}) \le J(\bar{u}) \le J(\bar{u}_{e_k})$$
$$k = 1 \dots n$$

A similar proposition can be derived in the case F satisfies (17) with the inequalities in reverse order.

Moreover, in both cases we have the following result:

<u>*Proposition 2:*</u>

$$\begin{cases} \lim\limits_{e \to 0} J_e(\bar{u}_e) = J(\bar{u}) \\ \lim\limits_{e \to 0} J(\bar{u}_e) = J(\bar{u}) \end{cases}$$

3.2 Choice of function *F*

The useful quadratic penalty approach corresponds to the following function F:

$$F(w) = w.(1-w) \quad \forall w \in \left[0,1\right] \qquad (18)$$

which satisfies property (16).

The logarithmic barrier approach corresponds to the following function F:

$$F(w) = log(w) + log(1-w) \quad \forall w \in \left]0,1\right[\qquad (19)$$

which satisfies property (17) and yields the following optimal control for problem (14):

Let $r(t)$ be given by (7) and let us define $b_e(t)$ by:

$$b_e(t) = \frac{2e}{2e - r(t) + \sqrt{r(t)^2 + 4e^2}} \tag{20}$$

Then, if $g(x(t))^T . p_x(t) = 0$, $\bar{u}_e(t)$ has only to satisfy:

$$\left\| \bar{u}_e(t) \right\| = b_e(t) \tag{21}$$

and if $g(x(t))^T . p_x(t) \neq 0$, $\bar{u}_e(t)$ is given by:

$$\bar{u}_e(t) = -b_e(t) \frac{g(x(t))^T . p_x(t)}{\left\| g(x(t))^T . p_x(t) \right\|} \tag{22}$$

As the switching function $r(t)$ is continuous with respect to time *t*, it is clear from (20), (21) and (22) that $\bar{u}_e(t)$ satisfies the same property. Let us notice that function *F* given by (19) is a barrier function.

4. NUMERICAL RESULTS

4.1 The GTO-GEO transfer

In the case of a high thrust GTO-GEO transfer, our approach based on optimal control leads to the same strategies than the methods based on mathematical programming: The maneuvers are mainly located at the apogees and almost collinear to vector *T*. The consumptions are very closed due to a low variation of the thrust direction. Mathematical programming methods indeed assume that the thrust direction is constant in an inertial frame and try to compute the thrust dates and directions for a given number of thrusts.

Let us consider now a medium thrust transfer to a final orbit with a lower semimajor axis than the GEO orbit. This backup case of a GTO-GEO transfer could be completed for example by an electric TOP-UP. The data are as follows:

$$\begin{cases} a_0 = 24371.138\ km & w_0 = 180.0\ deg \\ e_0 = 0.73 & W_0 = 41.0\ deg \\ i_0 = 7\ deg & v_0 = 0.0\ deg \\ m_0 = 8000.0\ kg & \end{cases} \tag{23}$$

$$\begin{cases} a_f = 37226.138\ km & w_f = 180.0\ deg \\ e_f = 0.1691822 & W_f = 41.0\ deg \\ i_f = 0.0\ deg & v_f\ free \end{cases} \tag{24}$$

$$\begin{cases} F_{max} = 35.0\ N & Isp = 270.0\ s \\ t_0 = 0.0\ h & t_f = 200.0\ h = 8.33\ days \end{cases} \tag{25}$$

We consider in this case a Keplerian motion without any gravitational perturbation. The optimal consumption obtained by solving problem (2) with the "continuation-smoothing method" is equal here to:

$$m_0 - m(t_f) = 3085.1\ kg \tag{26}$$

For the same number of maneuvers, the consumption obtained with mathematical programming is equal to *3140.4 kg*. The difference (*55 kg*) **comes from the optimization of the thrust direction and induces an important increase of the satellite lifetime (about *36 weeks*)**. In fact, the variation of thrust direction is more important here than in a classical GTO-GEO transfer (see Fig. 2). This underlines the interest of optimal control in this case. The first maneuver is located at the first perigee and the other ones are located at the following apogees (see Fig. 1). The optimal trajectory, the thrust history and the evolution of the orbital parameters are given below:

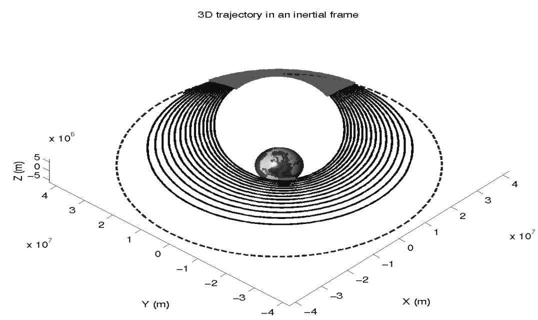

Fig. 1. Trajectory with GEO orbit in dotted line

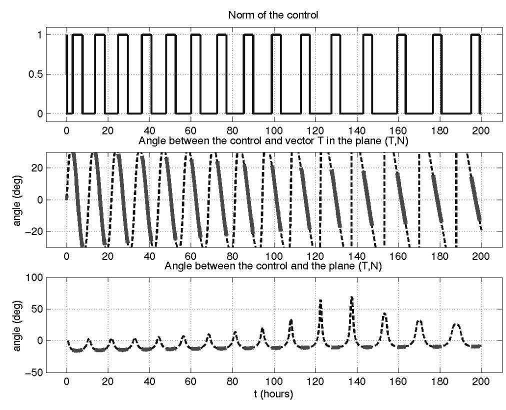

Fig. 2. Control history – 16 maneuvers

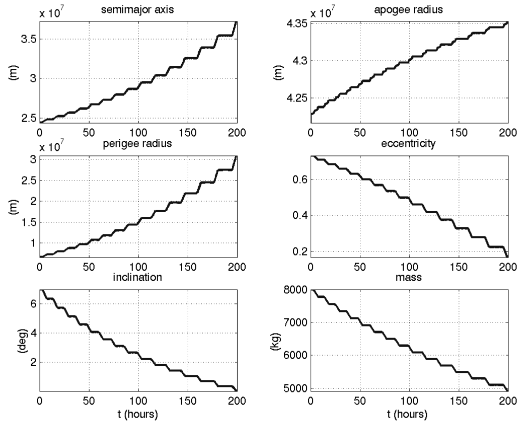

Fig. 3. Orbital parameters and mass of the satellite

4.2 The LEO rendezvous

In the following, we take into account Earth's oblateness in Gauss equations.

In the case of classical rendezvous between almost circular orbits, optimal control leads to the same strategies than a software based on mathematical programming and the associated consumptions are very closed. The maneuvers are located on the first and the last orbits and the two optimization methods make use of the natural drift of the angle *W*.

Remark: The final longitude for the rendezvous is fixed to $L_f = w_f + W_f + v_f + 360 * N_R \ (deg)$. The optimal number of revolutions N_R is determined by trying different values around N_T, the number of revolutions for the transfer problem which has to be solved first.

Let us consider now a rendezvous between slightly elliptical orbits associated with the following data:

$$\begin{cases} a_0 = 7178.139\ km & w_0 = 3.96585\ deg \\ e_0 = 0.083587 & W_0 = 160.01859\ deg \\ i_0 = 45.9934\ deg & v_0 = 180.0\ deg \\ m_0 = 120.0\ kg & t_0 = 0.0\ h \end{cases} \quad (27)$$

$$\begin{cases} a_f = 6977.523\ km & w_f = 255.72884\ deg \\ e_f = 0.01298109 & W_f = 102.528445\ deg \\ i_f = 45.99644\ deg & v_f = 114.26059\ deg \end{cases} \quad (28)$$

$$\begin{cases} Isp = 210.0\ s \\ t_f = 13.2974\ days \end{cases} \quad (29)$$

The optimal consumption obtained by solving problem (2) with the "continuation-smoothing method" is equal here to:

$$\begin{cases} m_0 - m(t_f) = 19.484\ kg & if\ F_{max} = 4\ N \\ m_0 - m(t_f) = 19.394\ kg & if\ F_{max} = 40\ N \end{cases} \quad (30)$$

The first maneuvers **are not located on the first orbits** (see Fig. 5 and Fig. 7 below) and our tool based on **mathematical programming is not able to guess such a strategy alone**. More precisely, if we tell the tool at which orbit begins the first maneuver, it obtains in three impulsive maneuvers:

$$m_0 - m(t_f) = 19.407\ kg \quad with\ F_{max} = +\infty \quad (31)$$

Let us notice that the consumption obtained in (30) with $F_{max} = 40\ N$ is lower than the one obtained in (31). This is due to the fact that the mathematical programming tool encounters difficulties to locate with precision the longitude of the first maneuver.

Now, if we locate as an initial guess the first maneuver at the first orbit (like in the case of a rendezvous between circular orbits) our mathematical programming software remains stuck to this solution which is a **local minimum** for the problem. Four maneuvers are needed and the associated consumptions are equal to:

$$\begin{cases} m_0 - m(t_f) = 19.634\ kg & with\ F_{max} = +\infty \\ m_0 - m(t_f) = 19.909\ kg & with\ F_{max} = 40\ N \end{cases} \quad (32)$$

When $F_{max} \neq +\infty$ the mathematical programming tool assumes again that the thrust direction is constant in an inertial frame. As the number of maneuvers is fixed here to four, the thrust durations are rather long hence the difference between the two consumptions in (32). In this case the maneuvers should be split in order to reduce this gap. Let us see now the solution obtained with optimal control:

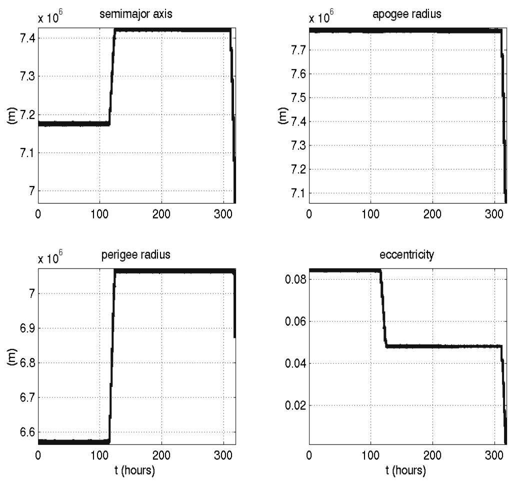

Fig. 4. $F_{\max} = 40\ N$ - Part 1/2

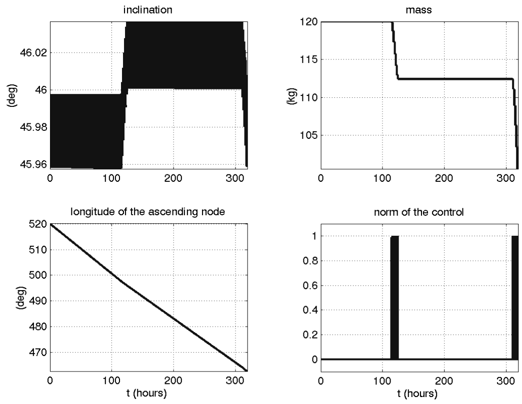

Fig. 5. - $F_{max} = 40\ N$ - Part 2/2 - 12 maneuvers (7 + 5)

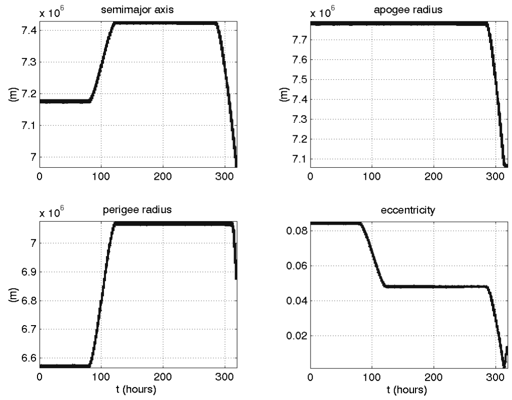

Fig. 6. $F_{max} = 4\ N$ - Part 1/2

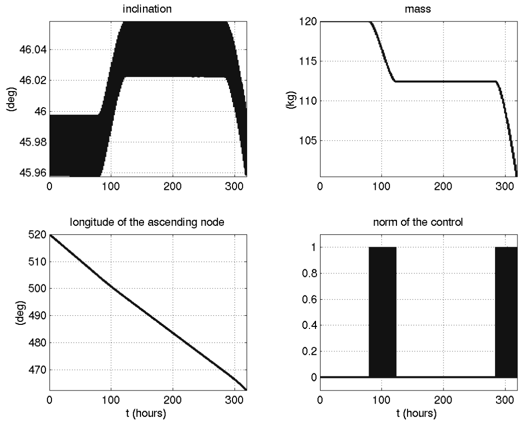

Fig. 7. $F_{max} = 4\ N$ - Part 2/2 - 45 maneuvers (25 + 20)

5. CONCLUSION

We have shown in this paper that in case of medium thrust orbital transfers, optimal control allows to save fuel compared to mathematical programming methods. The gain comes from the optimization of the thrust direction, considered as a function of time.

Moreover, in case of non classical rendezvous problems between elliptical orbits, optimal control computes the optimal strategy that can be used as an initial guess for a mathematical programming software. Without this initial guess mathematical programming tools remain stuck in general to a local minimum of the problem.

Finally, in the framework of shooting methods, we have demonstrated that smoothing techniques are very efficient for solving such "bang-bang" optimal control problems (transfers or rendezvous) even in the case of a large number of maneuvers. Moreover, let us notice that automatic differentiation tools make now the computation of the right-hand side of the costates differential equations straightforward.

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