KALMAN FILTER ESTIMATION OF SPINNING SPACECRAFT ATTITUDE USING MARKLEY VARIABLES

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ABSTRACT

There are several different ways to represent spacecraft attitude and its time rate of change. For spinning or momentum-biased spacecraft, one particular representation has been put forward as a superior parameterization for numerical integration. Markley has demonstrated that these new variables have fewer rapidly varying elements for spinning spacecraft than other commonly used representations and provide advantages when integrating the equations of motion. The current work demonstrates how a Kalman filter can be devised to estimate the attitude using these new variables.

The seven Markley variables are subject to one constraint condition, making the error covariance matrix singular. The filter design presented here explicitly accounts for this constraint by using a six-component error state in the filter update step. The reduced dimension error state is unconstrained and its covariance matrix is nonsingular.

1. INTRODUCTION

Attitude estimation for a torque-free spinning spacecraft reduces to a comparatively simple problem if the angular momentum vector lies along the major principal axis. In this case, the spin axis remains constant in both the body frame and the inertial frame, and mean values for the spin direction, period, and phase can be determined using a batch method.

It is more difficult to estimate a time-dependent attitude, such as occurs during maneuvers or when wire booms or other appendages are vibrating. Even in the absence of torques, the angular momentum will nutate in the body frame if it is not parallel to a principal axis. This effect can be important in case of a failure in the nutation damper.

For these reasons, a new attitude determination utility for spinning spacecraft is being developed by the Flight Dynamics Analysis Branch at the National Aeronautics and Space Administration (NASA) Goddard Space Flight Center. This paper describes the Kalman filter design proposed for this utility.

Attitude estimation methods for non-spinning, three-axis stabilized spacecraft often make use of the Euler symmetric parameters (commonly called the quaternion) to represent the body attitude with respect to an inertial reference frame [1]. The quaternion is a four-component, globally nonsingular attitude representation. The state vector typically has seven components consisting of the quaternion plus three elements that provide bias corrections to the measured rates. However, spinning spacecraft usually do not carry rate-sensing gyros, so the rotation rate vector itself must be estimated rather than just the biases. Rates are needed for time propagation since the attitude dynamics satisfy a 2nd-order differential equation.

One disadvantage of using the quaternion for spinning spacecraft is that all four components generally are rapidly varying. It is shown in [2] that alternative descriptions can be devised that have fewer rapidly varying elements, and thus, are easier to integrate numerically. These elements will be referred to here as Markley variables. The main goal of this paper is to present a design for an extended Kalman filter to estimate the attitude using these new variables. This filter is expected to have superior convergence properties compared to one using the quaternion and rates to represent attitude dynamics. The Markley variables used in this paper form a set of seven parameters, defined in Section 2. Section 3 presents the Kalman filter design.

One major complication is that the Markley variables are subject to a constraint condition. This constraint makes the state error covariance singular. However, only six parameters are required to represent the attitude dynamics (e.g., roll, pitch, yaw, and their time-derivatives). It is always possible to define a six-element error state that is unconstrained. Section 4 presents a new formulation of the filter based on a reduced, unconstrained, six-component representation for the state update and error covariance. Section 5 gives a summary and plans for future work.

2. MARKLEY VARIABLES

The seven variables of the attitude dynamics representation described in [2] are the angular momentum in an inertial attitude reference frame (typically the geocentric inertial frame), $\boldsymbol{L}_I$, the angular momentum in the body frame, $\boldsymbol{L}_B$, and a rotation angle, ζ, defined below. (Vectors will be denoted with bold characters.) These are subject to the constraint that the magnitude of the angular momentum vector is the same in the inertial and body frames.

The angle ζ can be defined in a number of different ways. For this paper, Markley's second representation will be used, as follows. An intermediate frame is defined by the rotation matrix

$$A_{EI} \equiv L^{-2}\Big[\boldsymbol{L}_B \cdot \boldsymbol{L}_I I + \frac{(\boldsymbol{L}_B \times \boldsymbol{L}_I)(\boldsymbol{L}_B \times \boldsymbol{L}_I)^T}{L^2 + \boldsymbol{L}_B \cdot \boldsymbol{L}_I} - \boldsymbol{L}_I \boldsymbol{L}_B^T + \boldsymbol{L}_B \boldsymbol{L}_I^T\Big] \quad (1)$$

where I is the 3×3 identity, $\boldsymbol{L}$ is the angular momentum vector, L is its magnitude, and the subscripts I, B, and E refer to the inertial, body, and intermediate frames, respectively. This definition is nonsingular as long as the angular momentum is nonzero and the body frame is not rotated exactly 180 degrees from the inertial frame. (In practice, this geometric limitation can be circumvented by introducing extra rotations to redefine the inertial reference frame as needed.)

The definition of A_{EI} uses the angular momentum expressed in both inertial and body frames. Caution is needed because of this mixing of frames; *in the context of Eqn.* 1, the $\boldsymbol{L}_B$ and $\boldsymbol{L}_I$ cannot be thought of as representations of an abstract vector.

The matrix A_{EI} has the property that

$$A_{EI} \boldsymbol{L}_I = \boldsymbol{L}_B \quad (2)$$

The attitude matrix, A_{BI}, is the transformation from the inertial to the body frame; thus,

$$A_{BI} \boldsymbol{L}_I = \boldsymbol{L}_B \quad (3)$$

The A_{EI} and A_{BI} matrices both transform $\boldsymbol{L}_I$ to $\boldsymbol{L}_B$, but differ by a rotation about $\boldsymbol{L}_B$. This distinction is seen as follows. Define matrix A_{BE} in terms of A_{EI} and A_{BI},

$$A_{BE} \equiv A_{BI} A_{EI}^T \quad (4)$$

then multiply $\boldsymbol{L}_I$ by A_{BI} and use Eqn. 4,

$$A_{BE} A_{EI} \boldsymbol{L}_I = A_{BI} \boldsymbol{L}_I \quad (5)$$

With Eqns. 2 and 3, this becomes

$$A_{BE} \boldsymbol{L}_B = \boldsymbol{L}_B \quad (6)$$

Thus, A_{BE} represents a rotation about $\boldsymbol{L}_B$.

Let the A_{BE} rotation define the angle ζ, so A_{BE} can be written [1]

$$A_{BE} = \cos\zeta\, I + \frac{1-\cos\zeta}{L^2} \boldsymbol{L}_B \boldsymbol{L}_B^T - \frac{\sin\zeta}{L}[\boldsymbol{L}_B \times] \quad (7)$$

where the cross-product matrix is defined as

$$[\boldsymbol{v}\times] \equiv \begin{bmatrix} 0 & -\boldsymbol{v}_z & \boldsymbol{v}_y \\ \boldsymbol{v}_z & 0 & -\boldsymbol{v}_x \\ -\boldsymbol{v}_y & \boldsymbol{v}_x & 0 \end{bmatrix} \quad (8)$$

for any vector $\boldsymbol{v}$.

2.1 Equations of Motion

Expressions for the time dependence of the Markley variables are needed to propagate the attitude between sensor updates in the sequential filter. The angular momentum in the inertial frame satisfies the equation of motion

$$\frac{d\boldsymbol{L}_I}{dt} = \boldsymbol{N}_I \quad (9)$$

where $\boldsymbol{N}_I$ is the total external torque. The corresponding equation of motion expressed in the body frame is

$$\frac{d\boldsymbol{L}_B}{dt} = \boldsymbol{N}_B - \boldsymbol{\omega}_{BI} \times \boldsymbol{L}_B \quad (10)$$

where

$$\boldsymbol{\omega}_{BI} = J^{-1}(\boldsymbol{L}_B - \boldsymbol{h}_B) \quad (11)$$

is the body rotation rate vector, J is the spacecraft moment of inertia tensor, and $\boldsymbol{h}_B$ is the angular momentum relative to the body frame of any internal moving components. The external torque expressed in the body frame is

$$\boldsymbol{N}_B = A_{BI} \boldsymbol{N}_I \quad (12)$$

Finally, [2] shows that ζ satisfies the equation

$$\frac{d\zeta}{dt} = \frac{L\big[(\boldsymbol{L}_B + \boldsymbol{L}_I)\cdot \boldsymbol{\omega}_{BI} + L^{-2}(\boldsymbol{L}_B \times \boldsymbol{L}_I)\cdot(\boldsymbol{N}_B + \boldsymbol{N}_I)\big]}{L^2 + \boldsymbol{L}_B \cdot \boldsymbol{L}_I} \quad (13)$$

3. KALMAN FILTER

There are two main parts that need to be specified for a Kalman filter design: the propagation step and the update step for both the state vector and the state covariance matrix. These steps are discussed in Sections 3.1 and 3.2. Other details about Kalman filtering for spacecraft applications can be found in [3], for example.

3.1 Time Propagation

Section 2.1 presented the equations of motion for the Markley variables. These equations can be integrated numerically to propagate the full seven-component state forward in time between sensor observations. In the test version of the filter currently being developed, the time integration is implemented using a simple 2nd-order accurate Euler method (time-centered), with the total torque expressed as a sum of the command torques and environmental torques from gravity gradient and residual magnetic dipole perturbations. The constraint on the norm of $\boldsymbol{L}$ can be approximately maintained by using the average of the norms of $\boldsymbol{L}_B$ and $\boldsymbol{L}_I$ at each step, or $\boldsymbol{L}_I$ can be forced to have norm exactly equal to that of $\boldsymbol{L}_B$, as seen below.

The state error covariance is defined as the expectation value of the outer product of the state errors, $P = E[(X - \overline{X})(X - \overline{X})^T]$, where $\overline{X}$ is the mean state vector. This is a 7×7 matrix. However, since the norms of $\boldsymbol{L}_B$ and $\boldsymbol{L}_I$ are constrained to be equal, P is of rank 6 for an unbiased estimator. Anticipating this problem and the resolution that will be given below in Section 4, the full state vector now is defined to be

$$X = \begin{bmatrix} \hat{\boldsymbol{L}}_I \\ \boldsymbol{L}_B \\ \zeta \end{bmatrix} \tag{14}$$

where $\hat{\boldsymbol{L}}_I$ is the unit vector in the direction of the inertial frame angular momentum. This state vector still has seven-components, but the constraint now is that the first three elements have unit norm.

The 7×7 error covariance, P, satisfies the equation

$$\frac{dP}{dt} = FP + PF^T + Q(t) \tag{15}$$

where $Q(t)$ is the process noise, and F is obtained from the state dynamics equation,

$$\frac{dX}{dt} = f(X,t) \tag{16}$$

with

$$F \equiv \frac{\partial f(X,t)}{\partial X} \tag{17}$$

The function $f(X,t)$ is given by Eqns. 9, 10, and 13, except that Eqn. 9 now becomes

$$\frac{d\hat{\boldsymbol{L}}_I}{dt} = \left(I - \hat{\boldsymbol{L}}_I\hat{\boldsymbol{L}}_I^T\right)\frac{\boldsymbol{N}_I}{L} \tag{18}$$

With this change, the expression for F is

$$F = \begin{bmatrix} -\left(\hat{\boldsymbol{L}}_I \cdot \boldsymbol{N}_I I + \hat{\boldsymbol{L}}_I \boldsymbol{N}_I^T\right)/L & -\left(I - \hat{\boldsymbol{L}}_I\hat{\boldsymbol{L}}_I^T\right)\boldsymbol{N}_I \boldsymbol{L}_B^T / L^3 & 0_{3\times1} \\ 0_{3\times3} & [\boldsymbol{L}_B \times]J^{-1} - \left[J^{-1}\boldsymbol{L}_B \times\right] & 0_{3\times1} \\ d_I L/(L^2 + \boldsymbol{L}_I \cdot \boldsymbol{L}_B) & d_B L/(L^2 + \boldsymbol{L}_I \cdot \boldsymbol{L}_B) & 0_{1\times1} \end{bmatrix} \tag{19}$$

with

$$d_I = -\boldsymbol{L}_B^T \frac{d\zeta}{dt} + L\left(J^{-1}\boldsymbol{L}_B\right)^T + (\boldsymbol{N}_B + \boldsymbol{N}_I)^T\left[\hat{\boldsymbol{L}}_B \times\right] \tag{20}$$

and

$$\begin{aligned} d_B = &-\left(\hat{\boldsymbol{L}}_B + \hat{\boldsymbol{L}}_I\right)^T \frac{d\zeta}{dt} + \left(J^{-1}\boldsymbol{L}_B\right)^T - (\boldsymbol{N}_B + \boldsymbol{N}_I)^T\left[\hat{\boldsymbol{L}}_B \times\right]/L \\ &+ (\boldsymbol{L}_B + \boldsymbol{L}_I)^T J^{-1} + \left(\hat{\boldsymbol{L}}_I \cdot J^{-1}\hat{\boldsymbol{L}}_B\right)\boldsymbol{L}_B^T \\ &- \left(\hat{\boldsymbol{L}}_B \times \hat{\boldsymbol{L}}_I\right) \cdot (\boldsymbol{N}_B + \boldsymbol{N}_I)\hat{\boldsymbol{L}}_B^T / L \end{aligned} \tag{21}$$

3.2 Sensitivity Matrix

The sensitivity, H, is the matrix of partials of the sensor observation with respect to the state X. Only vector observations are considered here. For example, the output from V-slit Sun and star sensors and magnetometers are vectors. For these sensors, H is a 3×7 matrix.

Body frame vector observations can be modeled as

$$\begin{aligned} \boldsymbol{v}_B^{obs} &= A_{BI}\boldsymbol{v}_I + \boldsymbol{n} \\ &= \exp([\boldsymbol{\alpha}\times])A_{est}\boldsymbol{v}_I + \boldsymbol{n} \\ &\approx (I + [\boldsymbol{\alpha}\times])\boldsymbol{v}_B \\ &= \boldsymbol{v}_B - [\boldsymbol{v}_B \times]\boldsymbol{\alpha} \end{aligned} \tag{22}$$

where $\boldsymbol{v}_I$ and $\boldsymbol{v}_B$ are the reference vector expressed in the inertial frame and body frame, respectively, A_{est} is the current estimate of the attitude matrix, $\boldsymbol{\alpha}$ is the negative of the unknown attitude error vector, and $\boldsymbol{n}$ represents random noise with covariance R. Thus, the partial of $\boldsymbol{v}_B^{obs}$ with respect to $\boldsymbol{\alpha}$ is $-[\boldsymbol{v}_B\times]$. Using $\boldsymbol{\alpha}$ in the error state rather than $-\boldsymbol{\alpha}$ simplifies signs elsewhere when constructing a quaternion filter.

Similarly, the negative attitude error, $\boldsymbol{\alpha}$, can be related to the attitude quaternion, so that

$$\frac{\partial \boldsymbol{q}}{\partial \boldsymbol{\alpha}} = -\frac{q_4}{2}I - \frac{1}{2}[\boldsymbol{q}\times] \tag{23}$$

where $\boldsymbol{q}$ is the vector part of the quaternion and q_4 is its fourth component. This is evaluated approximately using the current attitude estimate on the right-hand side.

Next, the attitude quaternion can be expressed as a function of the Markley variables, and the partials of $\boldsymbol{q}$ with respect to X can be determined. This yields

$$\frac{\partial \boldsymbol{q}}{\partial \hat{\boldsymbol{L}}_I} = \frac{C_1 \cos(\zeta/2)}{L\sqrt{2\left(L^2 + \boldsymbol{L}_I \cdot \boldsymbol{L}_B\right)}} + \frac{C_2 \sin(\zeta/2)}{\sqrt{2\left(L^2 + \boldsymbol{L}_I \cdot \boldsymbol{L}_B\right)}} \tag{24}$$

$$\frac{\partial \boldsymbol{q}}{\partial \boldsymbol{L}_B} = \frac{C_3 \cos(\zeta/2)}{L\sqrt{2\left(L^2 + \boldsymbol{L}_I \cdot \boldsymbol{L}_B\right)}} + \frac{C_4 \sin(\zeta/2)}{\sqrt{2\left(L^2 + \boldsymbol{L}_I \cdot \boldsymbol{L}_B\right)}} \tag{25}$$

and

$$\frac{\partial \boldsymbol{q}}{\partial \zeta} = -\frac{(\boldsymbol{L}_B \times \boldsymbol{L}_I)\sin(\zeta/2)}{2L\sqrt{2\left(L^2 + \boldsymbol{L}_I \cdot \boldsymbol{L}_B\right)}} + \frac{(\boldsymbol{L}_B + \boldsymbol{L}_I)\cos(\zeta/2)}{2\sqrt{2\left(L^2 + \boldsymbol{L}_I \cdot \boldsymbol{L}_B\right)}} \tag{26}$$

where

$$C_1 = -\frac{(\boldsymbol{L}_B \times \boldsymbol{L}_I)\, L\boldsymbol{L}_B^T/2}{L^2 + \boldsymbol{L}_I \cdot \boldsymbol{L}_B} + L[\boldsymbol{L}_B \times] \tag{27}$$

$$C_2 = -\frac{(\boldsymbol{L}_B + \boldsymbol{L}_I)\, L\boldsymbol{L}_B^T/2}{L^2 + \boldsymbol{L}_I \cdot \boldsymbol{L}_B} + L\, I \tag{28}$$

$$C_3 = -(\boldsymbol{L}_B \times \boldsymbol{L}_I)\left(\frac{\boldsymbol{L}_B^T}{L^2} + \frac{(\boldsymbol{L}_B + \boldsymbol{L}_I/2)^T}{L^2 + \boldsymbol{L}_I \cdot \boldsymbol{L}_B}\right) - [\boldsymbol{L}_I \times] \tag{29}$$

and

$$C_4 = I - \frac{(\boldsymbol{L}_B + \boldsymbol{L}_I)(\boldsymbol{L}_B + \boldsymbol{L}_I/2)^T}{L^2 + \boldsymbol{L}_I \cdot \boldsymbol{L}_B} \tag{30}$$

The 3×7 sensitivity matrix is obtained by combining all these pieces using the chain rule,

$$H = \frac{\partial \boldsymbol{v}_B^{obs}}{\partial X} = -[\boldsymbol{v}_B \times]\left(\frac{\partial \boldsymbol{q}}{\partial \boldsymbol{\alpha}}\right)^{-1}\left[\frac{\partial \boldsymbol{q}}{\partial \hat{\boldsymbol{L}}_I}\ ,\ \frac{\partial \boldsymbol{q}}{\partial \boldsymbol{L}_B}\ ,\ \frac{\partial \boldsymbol{q}}{\partial \zeta}\right] \tag{31}$$

4. REDUCED REPRESENTATION

The state error covariance is a 7×7 matrix of rank 6. It can be difficult numerically to maintain P as a rank 6 matrix during the filter propagation and update steps. To avoid this problem entirely, it is preferable to cast the filter in terms of a six-component error state, reducing the dimensionality of the covariance matrix and the filter update.

Sections 4.1 and 4.2 derive several relationships that are needed in Section 4.3 where the reduced form of the filter is presented. In particular, the reduced state vector is presented in Section 4.2.

4.1 Unit Vector Covariance

The 3×3 block of P corresponding to the uncertainty in the unit vector angular momentum can be related to the covariance corresponding to the 3-vector $\boldsymbol{L}_I$ as follows. If the estimated angular momentum and its error are written

$$\boldsymbol{L}' = \boldsymbol{L} + \boldsymbol{\delta L} \tag{32}$$

where $\boldsymbol{\delta L}$ has zero mean, then the corresponding unit vector and its error are

$$\begin{aligned}\hat{\boldsymbol{L}}' &= (\boldsymbol{L} + \boldsymbol{\delta L})\left(L^2 + 2\boldsymbol{\delta L}\cdot\boldsymbol{L} + \delta L^2\right)^{-1/2} \\ &\approx \hat{\boldsymbol{L}} + \left(I - \hat{\boldsymbol{L}}\hat{\boldsymbol{L}}^T\right)\frac{\boldsymbol{\delta L}}{L}\end{aligned} \tag{33}$$

where the subscript I has been dropped here for simplicity. If the angular momentum covariance is

$$P_L = E\left[\boldsymbol{\delta L}\boldsymbol{\delta L}^T\right] \tag{34}$$

then the covariance of its unit vector is

$$P_{\hat{L}} = L^{-2}\left(I - \hat{\boldsymbol{L}}\hat{\boldsymbol{L}}^T\right)P_L\left(I - \hat{\boldsymbol{L}}\hat{\boldsymbol{L}}^T\right) \tag{35}$$

4.2 Reduction to 6-Component State

Next, construct the matrix M that rotates the Z-axis of the inertial frame to be parallel to $\boldsymbol{L}_I$. With this transformation, it is possible to reduce the dimensionality of the state simply by discarding the third component of $M^T\hat{\boldsymbol{L}}_I$, which is manifestly equal to unity. Only the X- and Y-components are kept since they can deviate from zero in the 1st order at the update step. The state update is shown explicitly in Section 4.3.

To this end, define

$$M \equiv I\cos\vartheta + (1 - \cos\vartheta)\hat{\boldsymbol{n}}\hat{\boldsymbol{n}}^T + \sin\vartheta[\hat{\boldsymbol{n}}\times] \tag{36}$$

where ϑ is the angle between the Z-axis and $\boldsymbol{L}_I$, and

$$\hat{\boldsymbol{n}} = \frac{\hat{\boldsymbol{Z}} \times \hat{\boldsymbol{L}}_I}{\left|\hat{\boldsymbol{Z}} \times \hat{\boldsymbol{L}}_I\right|} \tag{37}$$

With this definition, one has

$$M\hat{\boldsymbol{Z}} = \hat{\boldsymbol{L}}_I \tag{38}$$

by construction. Also, define

$$m_{23} \equiv \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \quad , \quad m_{32} \equiv \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \tag{39}$$

One can show that

$$M\, m_{32}\, m_{23}\, M^T = I - \hat{\boldsymbol{L}}_I \hat{\boldsymbol{L}}_I^T \tag{40}$$

Now, define the six-component state to be

$$x = \begin{bmatrix} m_{23} M^T \hat{\boldsymbol{L}}_I \\ \boldsymbol{L}_B \\ \zeta \end{bmatrix} \equiv S^T X \tag{41}$$

where S^T is the 6×7 matrix

$$S^T = \begin{bmatrix} m_{23} M^T & 0_{2\times 3} & 0_{2\times 1} \\ 0_{3\times 3} & I & 0_{3\times 1} \\ 0_{1\times 3} & 0_{1\times 3} & 1 \end{bmatrix} \tag{42}$$

The m_{23} matrix in Eqn. 41 explicitly removes the Z-component of the rotated unit angular momentum vector. Eqn. 41 also shows that

$$\frac{\partial x}{\partial X} = S^T \tag{43}$$

One can show that $S^T S = I_{6\times 6}$ and

$$I_{7\times 7} = S\, S^T + Z_L \tag{44}$$

using Eqn. 40, and where $I_{6\times 6}$ and $I_{7\times 7}$ are the 6×6 and 7×7 identity matrices, and with Z_L defined as

$$Z_L \equiv \begin{bmatrix} \hat{\boldsymbol{L}}_I \hat{\boldsymbol{L}}_I^T & 0_{3\times 3} & 0_{3\times 1} \\ 0_{3\times 3} & 0_{3\times 3} & 0_{3\times 1} \\ 0_{1\times 3} & 0_{1\times 3} & 0_{1\times 1} \end{bmatrix} \tag{45}$$

4.3 Reduced Form for the Filter

The 6×6 covariance matrix for the reduced state now can be defined as

$$\overline{P} \equiv S^T P\, S \tag{46}$$

with time derivative

$$\begin{aligned} \frac{d\overline{P}}{dt} &= S^T \frac{dP}{dt} S \\ &= S^T F\left(SS^T + Z_L\right)PS \\ &\quad + S^T P\left(SS^T + Z_L\right)F^T S + S^T Q(t) S \end{aligned} \tag{47}$$

where the identity $I_{7\times 7}$ has been introduced in the form given in Eqn. 44. The time-dependence of S has been neglected. In the absence of external torques, S is exactly constant, and in practice, the time-dependence is small over each short numerical integration interval. Recomputing S once for each time step should be sufficiently accurate for propagation of the covariance.

Note that

$$P Z_L = Z_L P = 0 \tag{48}$$

since $\boldsymbol{L}_I$ is annihilated by the unit vector covariance, as shown by Eqn. 35. Thus, one can define

$$\overline{F} \equiv S^T F\, S \tag{49}$$

and

$$\overline{Q} \equiv S^T Q\, S \tag{50}$$

to obtain a reduced covariance propagation equation,

$$\frac{d\overline{P}}{dt} = \overline{F}\,\overline{P} + \overline{P}\,\overline{F}^T + \overline{Q}(t) \tag{51}$$

which is of the same standard form as Eqn. 15 and carries the same information content, but is not subject to any constraint condition.

Similarly, the 7×3 Kalman gain

$$K = PH^T\left(HPH^T + R\right)^{-1} \tag{52}$$

can be written in a reduced 6×3 form by introducing $I_{7\times 7}$ from Eqn. 44. Thus,

$$\begin{aligned} S^T K &= S^T P\left(SS^T + Z_L\right)H^T \\ &\qquad \left[H\left(SS^T + Z_L\right)P\left(SS^T + Z_L\right)H^T + R\right]^{-1} \\ &= \overline{P}\,\overline{H}^T\left[\overline{H}\,\overline{P}\,\overline{H}^T + R\right]^{-1} \\ &\equiv \overline{K} \end{aligned} \tag{53}$$

where R is the covariance of the sensor noise, $\boldsymbol{n}$, in Eqn. 22, and where the reduced 3×6 sensitivity matrix is defined as

$$\overline{H} \equiv H\, S \tag{54}$$

Finally, the state update is

$$\delta x = \overline{K}\left(\boldsymbol{v}_B^{obs} - \boldsymbol{v}_B^{ref}\right) \tag{55}$$

The 3rd, 4th, and 5th elements of the update vector δx are added to the a priori estimate of $\boldsymbol{L}_B$. The 6th element of δx is added to the a priori estimate of ζ. The 1st and 2nd elements of δx are corrections to the unitized angular momentum vector in the rotated frame; the corresponding a priori elements of x are zero by construction. Thus, the updated inertial frame unitized angular momentum vector is

$$\hat{\boldsymbol{L}}_I = M \begin{bmatrix} \delta x_1 \\ \delta x_2 \\ 1 \end{bmatrix} \left(1 + \delta x_1^2 + \delta x_2^2\right)^{-1/2} \qquad (56)$$

where M is given by Eqn. 36. The full vector $\boldsymbol{L}_I$ can be recovered at any time from $\hat{\boldsymbol{L}}_I$ and the norm of $\boldsymbol{L}_B$.

5. CONCLUSIONS

An extended Kalman filter design has been presented for spinning spacecraft that makes use of the Markley variables. The Markley variables have much less variability than other attitude representations such as the quaternion and rotation rate for spinning spacecraft, even in the presence of nutation or attitude maneuvers. This behavior makes them more suitable for use in an attitude determination algorithm.

The Markley variables are a seven-parameter set subject to one constraint condition. The constraint leads to a singular error covariance matrix. A method was given for reducing the error state in the filter update step to six independent components, resulting in a nonsingular 6×6 error covariance matrix.

The convergence properties of a Kalman filter for spinning spacecraft using the Markley variables are expected to be superior to those of a filter using the quaternion attitude representation. A prototype system is being developed to study the filter performance. Much additional work is needed before this filter can be migrated to the operational environment. In particular, the filter must be able to accept the different measurement types used on spinning spacecraft. Only vector sensors were treated in Section 3.2. The form of the H matrix will be somewhat different for each sensor type; although, the steps leading to Eqn. 31 will be similar for all sensors.

Another feature that must be included in an operational version of the filter is the ability to handle the 180 degree restriction in the Markley variables. The denominator in Eqn. 1 goes to zero if $\boldsymbol{L}_B \cdot \boldsymbol{L}_I = -L^2$, that is, if the body frame is rotated 180 degrees from the inertial frame. It is possible to redefine the inertial reference frame with an extra rotation whenever the body-to-inertial frame angle approaches 180 degrees. Then, to reconstruct the usual attitude, it is only a matter of bookkeeping to keep track of the times these extra rotations were applied. A similar problem occurs in Eqn. 37 if $\boldsymbol{L}_I$ and $\boldsymbol{Z}$ are 0 or 180 degrees apart. The matrix M given by Eqn. 36 then is undefined. In these cases, M is either the identity or an arbitrary rotation by 180 degrees, and the component of $\boldsymbol{L}_I$ to be discarded is already known to be parallel to $\boldsymbol{Z}$.

Another major enhancement would be to estimate sensor biases along with the attitude dynamics state. The ability to estimate these biases is already an important part of the current batch-method spinning spacecraft attitude determination system, where the spin vector is assumed to be constant for the entire batch. However, sensor biases may not be sufficiently observable in the time-dependent case. More work will be needed to study this possibility. Nonetheless, even the current filter design for estimating only the attitude dynamics will add a significant new capability to the attitude ground support system used for Flight Dynamics support for a number of spinning spacecraft missions at the NASA Goddard Space Flight Center.

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