

# ANALYSIS OF CAPTURE TRAJECTORIES TO THE VICINITY OF LIBRATION POINTS

M. Nakamiya<sup>\*</sup>, D. J. Scheeres<sup>†</sup>, H. Yamakawa<sup>‡</sup> and M. Yoshikawa<sup>§</sup>

**Spacecraft capture trajectories to the periodic orbits of the L1 and L2 points in the restricted Hill three-body problem are studied. The specific focus is on transfer to these vicinities from interplanetary trajectories. This application is motivated by future proposals to place “Deep Space ports” at the Earth and Mars L1 or L2 points. These spaceports are considered as candidate gateways for interplanetary transfers in the future. We utilize stable manifolds for capture trajectories to periodic orbits around the libration points. As a result, the cost of capture into a periodic orbit is also reduced relative to direct capture into a parabolic orbit. The way of linking between interplanetary transfer trajectories and the stable manifold is also discussed.**

## INTRODUCTION

There has been great interest in the libration points of the circular restricted 3-body problem (CR3BP), which are located where the gravity of the first and second massive bodies and centrifugal force are balanced. In particular, the position of L1 and L2, which lie on the line connecting the two masses, can be considered equivalent to the boundary of the gravity field of the smaller mass of the two. Since an object around these points can maintain the same orientation with respect to the two masses, transfers between the primary body and the libration point have been investigated extensively in the past<sup>1-6</sup>. In fact, several astronomical satellites have already utilized such Halo/Lissajous orbits around the L1 and L2 points of the Sun-Earth system<sup>7</sup>. Moreover, a transfer to the interior and exterior region of the Earth is relatively simple by addition of energy to a spacecraft in the vicinity of the L1 and L2 points. Therefore, these points are also considered as candidate gateways for interplanetary transfers in the future<sup>8-13</sup>. Recently, the analysis and design of transfer orbits using invariant manifolds associated to periodic orbits around the libration points have been a topic of study<sup>14-20</sup>.

Furthermore, if a spaceport is also built around the L1 or L2 points of a target celestial body, the fuel required for landing or for entry orbit at the target celestial body can be supplied there to the spacecraft arriving from interplanetary space, or we can transfer cargo there to another spacecraft which is used exclusively for landing or for an entry orbit. In this way, constructing a spaceport in the vicinity of L1 or L2 of the Sun-Earth and Sun-Target celestial body system can separate the transportation system into three regions: transfer inside the gravity field of the Earth, transfer inside the gravity field of a target celestial body and the interplanetary transfer phase (see Fig. 1). Moreover, this system facilitates round-trip exploration using spaceports as relay points and leads to a reusable transportation system<sup>8, 11</sup>. In the past, the escape trajectories from the libration points of Sun-Earth system also have been examined<sup>21-23</sup>. However,

---

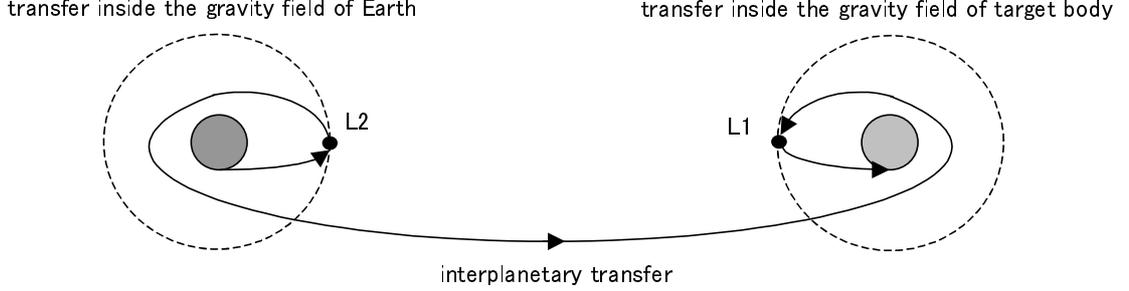
<sup>\*</sup>Ph. D. Candidate, Department of Space and Astronautical Science, The Graduate University for Advanced Studies, 3-1-1 Yoshinodai, Sagami-hara, Kanagawa 229-8510, Japan. E-mail: nakamiya@isas.jaxa.jp

<sup>†</sup>Associate Professor, Department of Aerospace Engineering, University of Michigan, Ann Arbor, Michigan 48109, USA. E-mail: scheeres@umich.edu

<sup>‡</sup>Professor, Kyoto University, Gokasho, Uji, Kyoto 611-0011, Japan. E-mail: yamakawa@rish.kyoto-u.ac.jp

<sup>§</sup> Associate Professor, Japan Aerospace Exploration Agency, 3-1-1 Sagami-hara, Kanagawa 229-8510, Japan. E-mail: makoto@isas.jaxa.jp

capture trajectories from an interplanetary trajectory to the vicinity of the L1 and L2 points of a target celestial body are not fully understood. Thus, spacecraft capture trajectories to the vicinity of the L1 and L2 points in the planar restricted Hill three-body problem are analyzed in this paper.



**Figure 1: Vision of an Interplanetary Transfer in the Future**

## HILL THREE-BODY MODEL

The physical system considered is the restricted Hill three-body model. This model is a limiting case of the circular restricted three-body problem (CR3BP) and describes the dynamics of a massless particle attracted by two point masses revolving around each other in a circular orbit (see Fig. 2). In fact, the Hill model can be obtained from the CR3BP by setting the center of the coordinate system to be at the secondary body and then assuming that the distance of the satellite from the center is small compared to the distance between the spacecraft and the sun. The resulting equations of motion provide a good description for the motion of a spacecraft in the vicinity of the L1 and L2 libration points of the smaller body<sup>24</sup>.

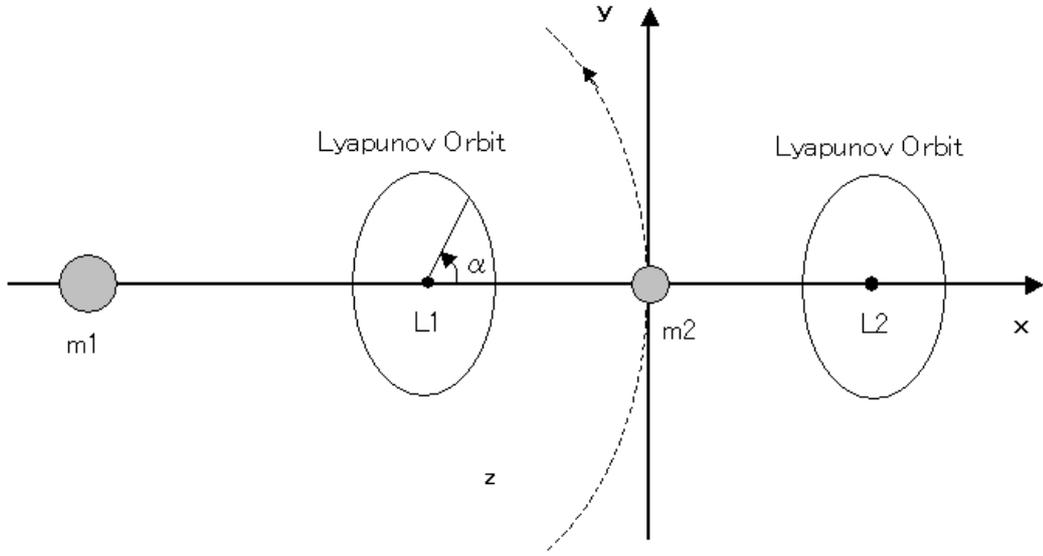
### Equations of Motion

The equations of motion for spacecraft in this planar Hill model are given by<sup>25,26</sup>,

$$\ddot{x} - 2\omega\dot{y} - 3\omega^2 x = -\frac{\mu}{r^3} x \quad (1)$$

$$\ddot{y} + 2\omega\dot{x} = -\frac{\mu}{r^3} y \quad (2)$$

where  $r = \sqrt{x^2 + y^2}$  is the distance from the center of second smaller body to a spacecraft,  $\omega$  is the angular velocity of the secondary body ( $m_2$ ) about the primary body ( $m_1$ ), and  $\mu$  is the gravitational parameter of the secondary body.



**Figure 2: Geometry of the planar Restricted Hill Three-Body Model**

### Libration Points

In the circular restricted Three-body problem model there are five points where the gravity of the first and second massive bodies and centrifugal force acting on S/C are balanced, which are called libration points. In the restricted Hill Three-body problem only the L1 and L2 libration points exist, but they are symmetric about the origin with coordinates  $x = \pm(\mu/3\omega^2)^{1/3}$ ,  $y = 0$ .

### Jacobi Integral

Equations (1-2) have an integral of motion similar to the CR3BP. The following equation denotes the *Jacobi integral*, which is a conservative quantity determined from the initial conditions,

$$J = \frac{1}{2}v^2 - \frac{\mu}{r} + \frac{3x^2}{2}\omega \quad (3)$$

where  $v = \sqrt{\dot{x}^2 + \dot{y}^2}$  is the velocity of the particle in the rotating frame. This constant has a deep influence on the dynamics of motion. The condition  $v^2 \geq 0$  in Eq. 3 impose a restriction on the allowable position for the motion at any given value of J. Setting  $v = 0$  defines the zero-velocity surface, which sets a physical boundary of the allowable motion (Fig. 2). In particular, the critical value of J at L1 and L2 defines the energy at which the zero-velocity surfaces open at L1 and L2, and is expressed as:

$$J_{L1,2} = -\frac{1}{2}(9\mu\omega)^{2/3} \quad (4)$$

## Normalization

Next, we normalize the above equations setting the unit length and the unit time as follows:

$$l = \left(\frac{\mu}{\omega}\right)^{1/3} \text{ and } \tau = \frac{1}{\omega} \quad (5)$$

The normalized equations of motion are then

$$\ddot{x} - 2\dot{y} - 3x = -\frac{x}{r^3} \quad (6)$$

$$\ddot{y} + 2\dot{x} = -\frac{y}{r^3} \quad (7)$$

This normalization allows us to eliminate all free parameters from the equations, thus, computations performed for them can be scaled to any physical system by multiplying by the unit length and time, which only depend on the properties of the primary and secondary bodies. Moreover, this normalization is equivalent to  $\omega = 1$  and  $\mu = 1$ , thus, the normalized x coordinate of libration points and the normalized value of J at L1 and L2 are equal to

$$x = \pm(1/3)^{1/3} = \pm 0.693... \quad (8)$$

$$J_{L1,2} = \frac{1}{2}9^{2/3} = 2.16337... \quad (9)$$

Table 1 gives the normalized radius for most planets of the solar system.

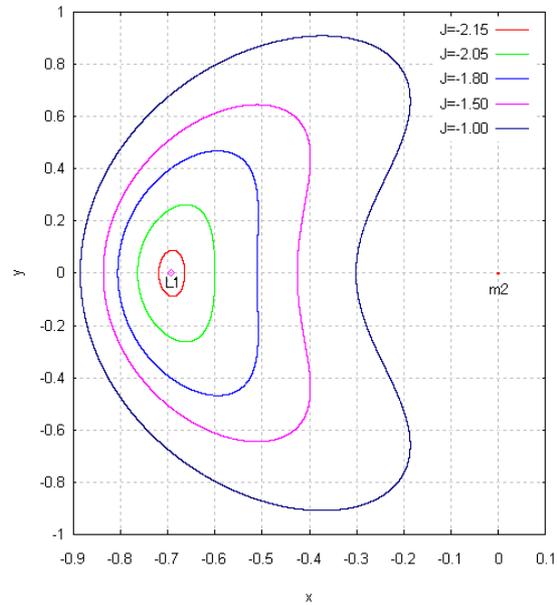
**Table 1**

### NORMALIZED RADIUS OF THE SOLAR SYSTEM PLANET

Planet	Mass ( $\times 10^{23}$ kg)	Gravitational parameter ( $\times 10^5$ km <sup>3</sup> /s <sup>2</sup> )	Mean motion (rad/s)	Normalized radius
Mercury	0.3302	0.220329	$8.27 \times 10^{-7}$	0.007663
Venus	4.869	3.248889	$3.24 \times 10^{-7}$	0.00415
Earth	5.9742	3.986345	$1.99 \times 10^{-7}$	0.002955
Mars	0.64191	0.428321	$1.06 \times 10^{-7}$	0.002173
Jupiter	1899	1267.127	$1.68 \times 10^{-8}$	0.000933
Saturn	568.8	379.5375	$6.76 \times 10^{-9}$	0.000641
Uranus	86.86	57.9582	$2.67 \times 10^{-9}$	0.000253
Neptune	102.4	68.32742	$1.21 \times 10^{-9}$	0.000148

## Lyapunov Orbit

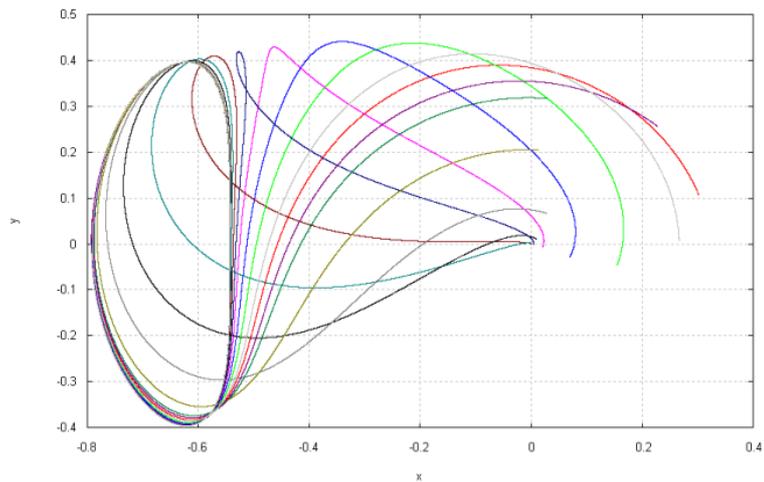
There exist periodic orbits near the libration points in two-dimensional space<sup>27-31</sup> called Lyapunov orbits, respectively, whose sizes depend on the value of the Jacobi constant (see Fig. 3). In this paper, we first analyze capture trajectories to a planar Lyapunov orbit to outline our procedure and then will consider the three-dimensional periodic orbit transfer case for future work.



**Figure 3: Lyapunov Orbits around L1**

## Invariant Manifolds

Above-mentioned periodic orbits, Lyapunov orbits, are not stable completely. There exist invariant structures associated with Lyapunov orbits, called unstable and stable manifolds (see Fig. 4). These are very sensitive and are affected by initial conditions. We exploit this stable manifold for capture trajectories to periodic orbits around the libration points.



**Figure 4: Stable Manifold of Lyapunov orbit**

## TRANSFERS BETWEEN SECONDARY BODIES AND LIBRATION POINT ORBITS

### Assumption of Capture Trajectories

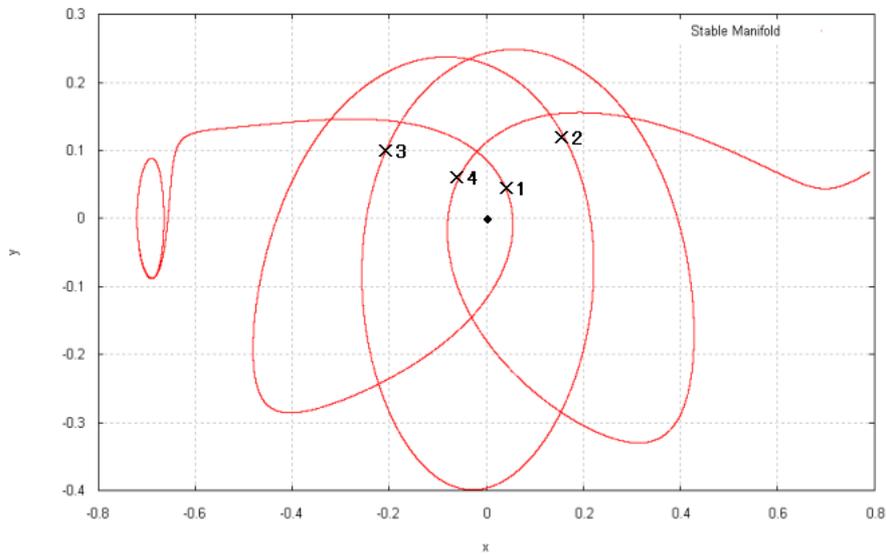
In this paper, we assume that capture trajectories are trajectories that enter the sphere of influence of a target body from interplanetary space and have a close flyby to the target body. At closest approach an impulsive maneuver is performed to put into the spacecraft on the stable manifold that leads to capture to a periodic orbit in the vicinity of L1 or L2 of the target body. The reason why an impulsive maneuver is performed near the surface of the target body (periapsis) is because it is the energetically most efficient place to reduce the approach energy, and may also be reduced by using an aero assist with the planetary atmosphere. After orbiting around the target body a few times, S/C is placed into the Lyapunov orbit. At this time, an infinitesimal impulsive maneuver is necessary to keep the Lyapunov orbits, but it is negligible. In this way, the stable manifolds are used for capture trajectories to Lyapunov orbits.

### Definition of Periapsis Points

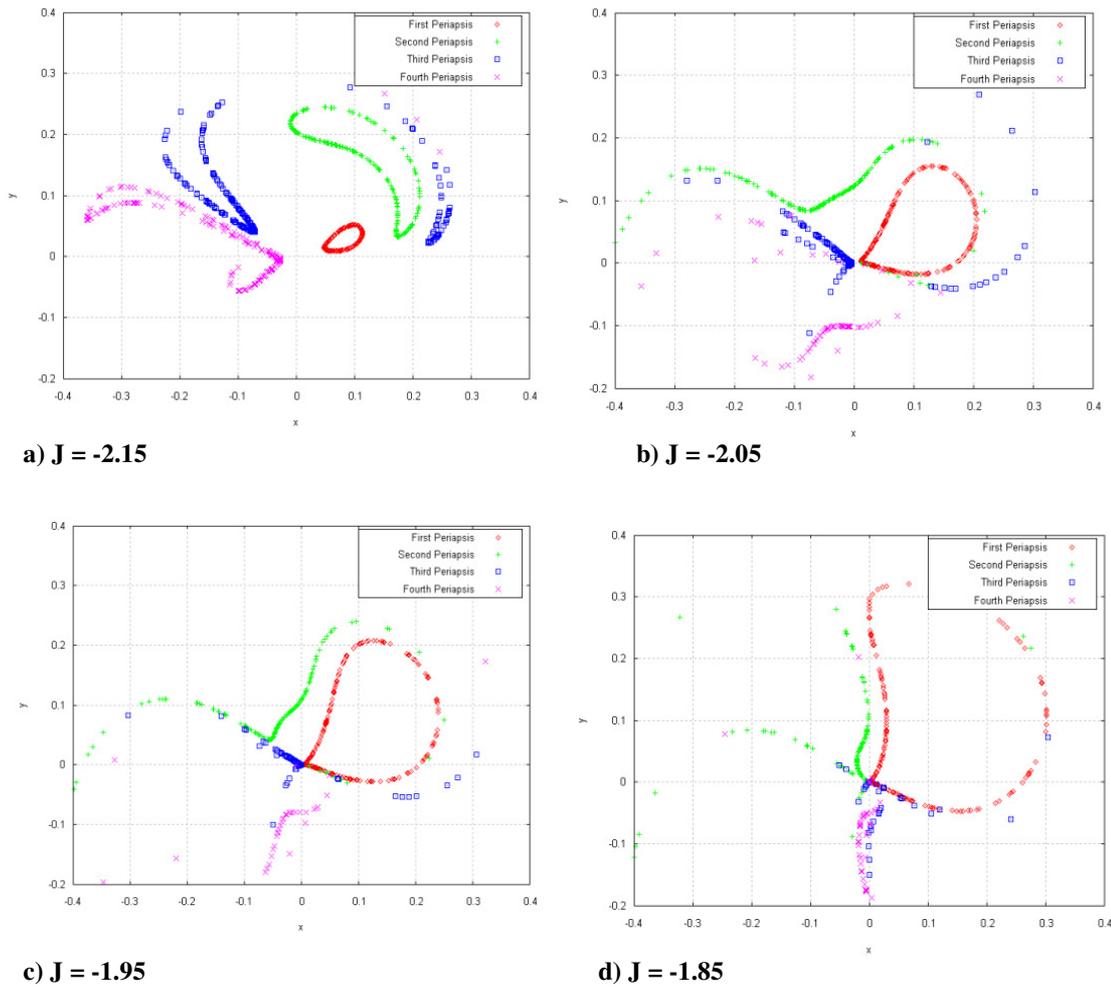
Periapsis points are the closest points of stable manifolds to the target body. In this study, we investigate the first four periapsis passage points of stable manifolds propagated backwards in time. Fig. 5 shows the first four periapsis passage points of one example trajectory of the stable manifold ( $J = -2.15$ ).

Based on this result, Fig. 6 plots the first four periapsis point's locations of the stable manifold for  $J = -2.15$ . We can see that each periapsis points region spread out and these periapsis location depends on the size of Jacobi constant.

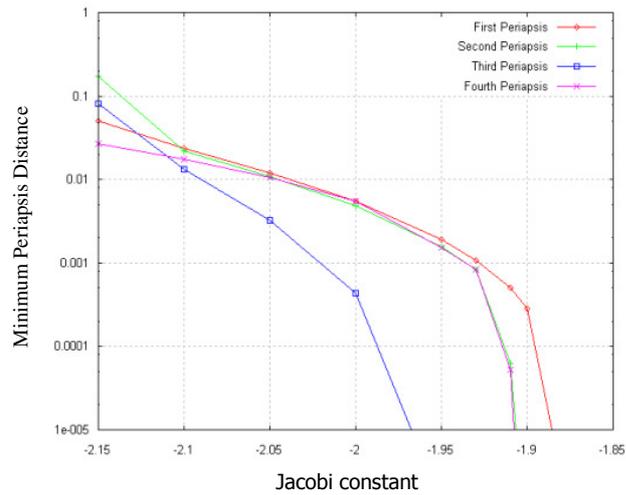
Fig. 7 shows the relation between the minimum periapsis distance and the value of  $J$ . The minimum periapsis distance means the distance from the origin to the periapsis point, which is closest from the origin in each periapsis points. It is found that the minimum periapsis distance decreases as the value of  $J$  increases. Moreover, each minimum periapsis distance becomes smaller than 0.007663 (which is the largest normalized planetary radius, Mercury) when the value of  $J$  is large enough. Thus the stable manifold of first four periapsis passage point can intersect the surface of any of the planets in the solar system.



**Figure 5: First Four Periapsis Passage Points of one example trajectory of the stable manifold propagated backward for  $J = -2.15$ .**



**Figure 6: First Four Periapsis Location.**

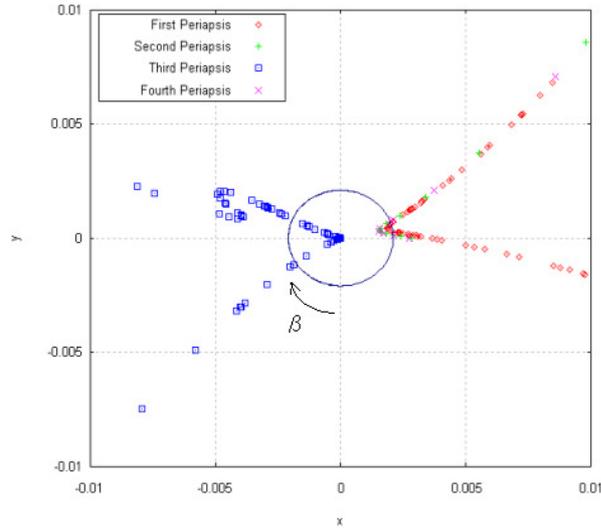


**Figure 7: Minimum Distance as a Function of Size of Lyapunov Orbit (Value of Jacobi Constant)**

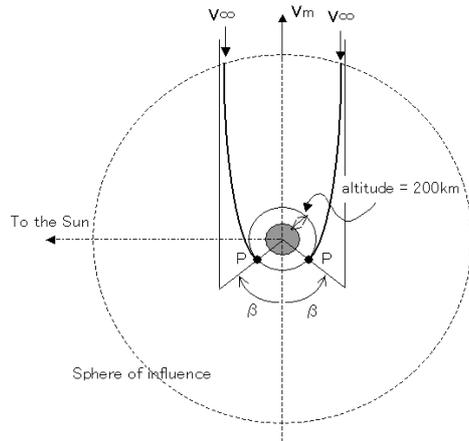
## APPLICATION TO EARTH-MARS TRANSFER

In this section, a patched conic approximation for the interplanetary transfer is applied to our study of capture trajectories to Lyapunov orbits planar dimensional cases. We focus our attention on a transfer from the Earth to Mars. However, these results can be applied to other planets of the solar system as well. We compare the cost of transfer into the Lyapunov orbit with the cost of parabolic capture to Mars, assuming the 2-body problem. At this time, a trajectory correction maneuver is assumed at the periapsis near the surface of the Mars ( $h \cong 200$  km) after an interplanetary transfer.

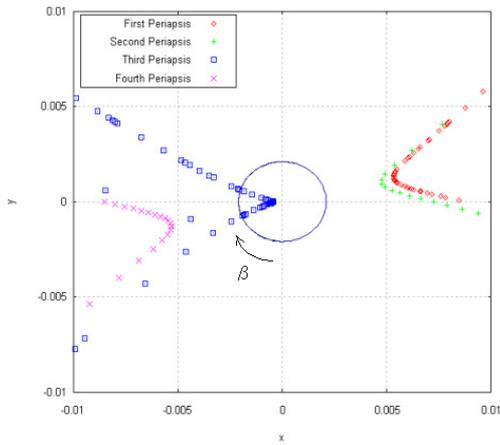
From Fig. 7 and Table 1, the stable manifold intersects the surface of Mars for first periapsis when Jacobi constant is equal to  $-1.961$ . Fig. 8 shows that the minimum first periapsis point of stable manifold intersects the surface of Mars ( $J = -1.961$ ). However, the location of periapsis after interplanetary space is determined from interplanetary transfer trajectory (see Fig. 9). In case of 200 km periapsis altitude, angle  $\beta$  is equal to 51.1 deg. Thus, it is impossible to link the interplanetary trajectory and the stable manifold at the first periapsis point. On the other hand, the third periapsis point of stable manifold are more likely to connect to the periapsis point after interplanetary space. Fig. 10 shows the relation between angle  $\beta$  of the third periapsis of stable manifold and Jacobi constant, and Fig. 11 plot the relation between them. We can see that angle  $\beta$  of the third periapsis becomes 51.1 deg when  $J$  is  $-1.898$ . Fig. 12 shows the stable manifold starting from the third periapsis (in backward integration) in case of  $J = -1.898$ .



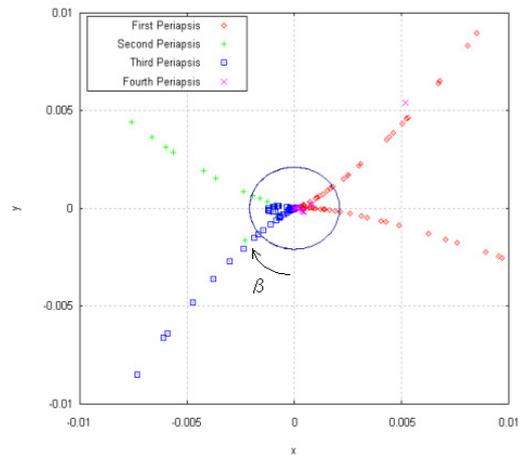
**Figure 8: Periapsis Location for  $J = -1.961$ :  $\bigcirc$  is the Surface of Mars.**



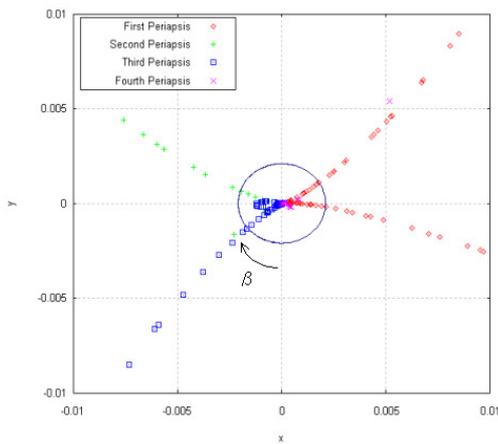
**Figure 9: Periapsis Location (angle  $\beta$ ) after Interplanetary Space for Hohmann Transfer.**



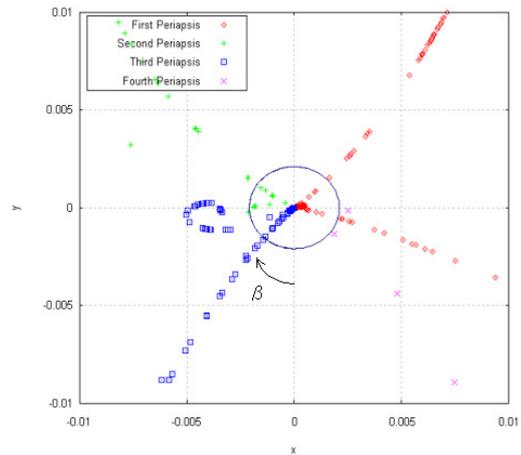
**a)  $J = -2.00$**



**b)  $J = -1.95$**

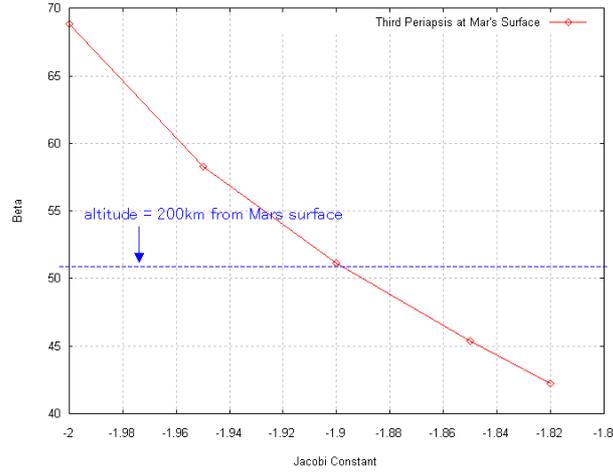


**c)  $J = -1.90$**

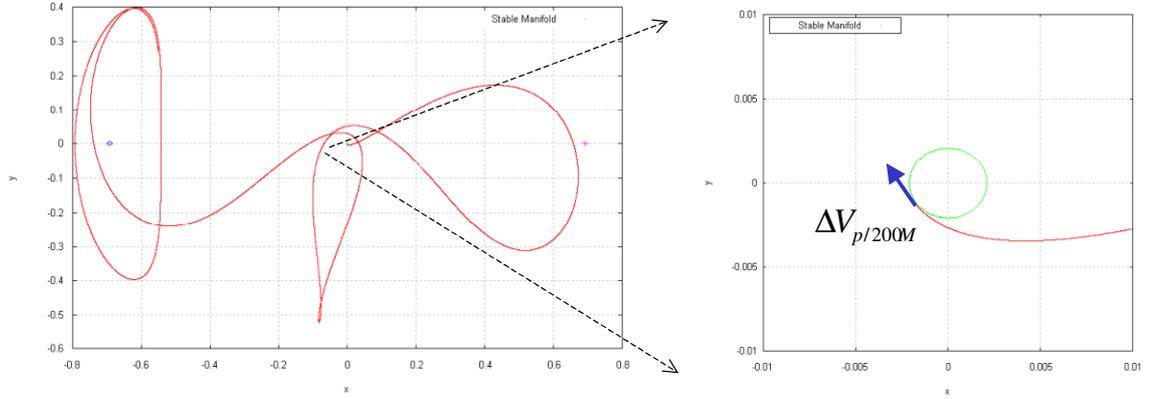


**d)  $J = -1.85$**

**Figure 10: Relation between Angle  $\beta$  of Third Periapsis of Stable Manifold and Jacobi Constant.**



**Figure 11: Angle  $\beta$  of Third Periapsis of Stable Manifold plotted against Jacobi Constant.**



**Figure 12: Stable Manifold until Third Periapsis in case of  $J = -1.898$ .**

The hyperbolic arrival velocity of a spacecraft with respect to Mars is  $v_{\infty/M} = 2.648$  km/s if we assume a simple Hohmann transfer from Earth to Mars, and use the patched conic approximation for arrival at Mars. Subsequently, the velocity near the surface of Mars,  $v_{p/M}$ , is expressed as:

$$v_{p/M} = \sqrt{v_{\infty/M}^2 + (2\mu_M / (r_M + 200))} \quad (12)$$

where  $\mu_M$  is the gravitational parameter of Mars and  $r_M$  is the radius of Mars, leading to a periapsis velocity  $v_{p/M} = 5.551$  km/s. On the other hand, the velocity at the third periapsis passage point of stable manifold from Lyapunov orbit ( $J = -1.898$ ) is 5.009 km/s. Therefore, the required  $\Delta V_{p/M}$  for entry to the Lyapunov orbit with  $J = -1.898$  is  $5.551 - 5.009 = 0.542$  km/s. It is instructive to compare the  $\Delta V$  for capture to the periodic orbit with simple 2-body capture criterion at Mars:  $\Delta V_{p/M} = v_{p/M} - 2^{1/2} v_{c/M} = 0.673$  km/s. Thus capture to the Lyapunov orbit allows a significant improvement even with respect to capture into a parabolic orbit, or in other words after our capture maneuver is performed the spacecraft is initially on a hyperbolic orbit, but one that transitions to a bound periodic orbit. Therefore, the required  $\Delta V$  for capture to Lyapunov orbit is reduced by over 19% from a simple capture maneuver. If a spacecraft is

located at Sun-Mars Lyapunov orbit, the interplanetary cargo ship does not need to carry the fuel after departing from spaceport, that is, fuel for injection into Mars circular orbit ( $\Delta V_{C/M} = 1.56$  km/s).

## CONCLUSIONS

This paper investigates capture trajectories to Lyapunov/orbits in the planar Hill three-body problem. We concentrate on transfer into these vicinities from interplanetary trajectories. The characteristics of the relation between the minimum periapsis distance and the size of Lyapunov orbits are obtained using the manifold trajectories from Lyapunov orbits by backward integration. Numerical results show that the manifold trajectories of the first four periapsis passage point from Lyapunov orbits can intersect the surface of any of the planets in the solar system if the value of Jacobi integral is large enough. Thus this technique can be used to reduce the cost of capture at any of the planets in the solar system.

As an example we consider transfers from an interplanetary transfer into a Lyapunov orbit near the Mars surface. It was found that the required velocity increment for could be reduced by approximately 19 % compared with the capture into a parabolic orbit although connection point between the periapsis point after interplanetary space and stable manifold trajectories.

Extension to three-dimension (capture to Halo orbit) and analysis of aero assist using planetary atmosphere are future work.

## REFERENCES

1. D' Amario, L. A., and Edelbaum, T. N., "Minimum Impulse Three-Body Trajectories," *AIAA Journal*, Vol. 12, No. 4, 1974, pp. 455-462.
2. Broucke, R., "Travelling Between the Lagrange Points and the Moon," *Journal of Guidance and Control*, Vol. 2, No. 4, 1979, pp. 257-263.
3. Gomez, G., Jorba, A., Masdemont, J. and Simo, C., "Study of the Transfer from the Earth to a Halo Orbit Around the Equilibrium Point L1," *Celestial Mechanics and Dynamical Astronomy*, Vol. 56, No. 4, 1993, pp. 541-562.
4. Prado, A. F. B. A., "Travelling Between the Lagrange Points and the Earth," *Acta Astronautica*, Vol. 39, No. 7, 1996, pp. 483-486.
5. J.D. Strizzi, J. Kutrieb, P. Damphousse, J. Carrico, "Sun-Mars Libration Points and Mars Mission. Simulations," AAS 01-159, AAS/AIAA Spaceflight Mechanics Meeting, Santa Barbara, California, USA., February 2001
6. Baoyin, H., McInnes, R. C., "Trajectories to and from the Lagrange points and the primary body surfaces," *Journal of Guidance, Control, and Dynamics*, Vol. 29, No. 4., 2006, pp 998-1003.
7. Canalias, E., Gomez, G., Marcote, M., and Masdemont, J. J., "Assessment of Mission Design Including Utilization of Libration Points and Weak Stability Boundaries," *ESA Advanced Concept Team*, URL: <http://www.esa.int/act>.
8. Farquhar, R. W., "Future Missions for Libration-Point Satellites," *Astronautics and Aeronautics*, Vol. 7, No. 5, May 1969, pp. 52-56.
9. Lo, M., and Ross, S., "The Lunar L1 Gateway: Portal to the Stars and Beyond," AIAA Paper 2001-4768, August 2001.
10. Folta, D. C., et al., "Servicing and Deployment of National Resources in Sun-Earth. Libration-Point Orbits," IAC Paper 02-Q.6.08, October 2002.

11. Farquhar, R. W., Dunham D. W., Guo, Yan-ping and Madams, V. J., "Utilization of libration points for human exploration in the sun-earth-moon system and beyond," *Acta Astronautica*, Vol. 55, pp. 687-700, 2004.
12. J.E. Marsden & S.D. Ross, "New Methods in Celestial Mechanics and Mission Design," *Bulletin of the American Mathematical Society*, Vol. 43, No. 1, pp. 43-73, 2006
13. Website: "JAXA Vision - JAXA 2005 -," URL: [http://www.jaxa.jp/2025/index\\_e.html](http://www.jaxa.jp/2025/index_e.html).
14. Conley, C.C., "Low energy transit in the restricted three body problem," *SIAM Journal on Applied Mathematics*, Vol. 16, No. 4, pp. 732-746, 1968.
15. Koon, W. S., Lo, M. W., Marsden, J. E. & Ross, S. D. "Heteroclinic Connections Between Periodic Orbits and Resonance Transitions in Celestial Mechanics," *Chaos*, Vol. 10, No. 2, 2000, pp. 427-469.
16. Koon, W. S., Lo, M. W., Marsden, J. E. & Ross, S. D., "Constructing a Low Energy Transfer Between Jovian Moons Contemp," *Contemporary Mathematics*, Vol., 292, 2002, pp. 129-145.
17. Lo, M. "The Interplanetary Superhighway and the Origins Program", IEEE Space 2002 Conference, Big Sky, MT, March. 2002.
18. Yamato, H. & Spencer, D. B., "Transit-Orbit Search for Planar Restricted Three-Body Problems with Perturbations.," *Journal of Guidance, Control, and Dynamics*, Vol. 27, No. 6, pp. 1035-1045, 2004.
19. Topputo F., Vasile M., Bernelli-Zazzera F., "Low Energy Interplanetary Transfers Exploiting Invariant Manifolds of the Restricted Three Body Problem," *Journal of the Astronautical Sciences*, 2005, Vol. 53, Number 4, pp 353-372.
20. Howell, K., and Kakoi, M., "Transfers between the Earth-Moon and Sun-Earth systems using manifolds and transit orbits," *Acta Astronautica*, Vol. 59, pp. 367-380, 2006.
21. Matsumoto, M., and Kawaguchi, J., "Deep Space Quay and Solar Voyage Era," *The Sixth IAA International Conference on Low-Cost Planetary Missions*, Kyoto, Japan, 11-13 October, 2005, pp. 401-408.
22. Nakamiya, M., and Yamakawa, H., "Analysis of Earth Escape Trajectories Starting from Sun-Earth L2 Point," *16th Workshop on Astrodynamics and Flight Mechanics*, ISAS, 1-2 August 2006.
23. Nakamiya, M., and Yamakawa, H., "Earth Escape Trajectories Starting from L2 Point," *AIAA/AAS Astrodynamics Specialist Conference and Exhibit*, Keystone, Colorado, Aug. 21-24, 2006.
24. Scheeres, D., Guman, M., and Villac, B., "Stability Analysis of Planetary Satellite Orbiters:. Application to the Europa Orbiter," *Journal of Guidance, Control and Dynamics*, Vol. 24, No. 4., 2001, pp. 778-787.
25. Szebehely, V., *Theory of Orbits: The Restricted Problem of Three Bodies*, Academic Press, New York, 1967.
26. Villac, B. F. & Scheeres, D. J., "Escaping Trajectories in the Hill Three-Body Problem and Applications," *Journal of Guidance, Control, and Dynamics*, Vol. 26, No. 2., 2003, pp. 224-232.
27. Henon, M., "Numerical exploration of the restricted problem. V," *Astronomy and Astrophysics 1*, pp. 223-238, 1969.
28. Breakwell, J. V., and Brown, J. V., "The 'Halo' Family of 3-Dimensional Periodic Orbits in the Earth-Moon' Restricted 3-Body Problem," *Celestial Mechanics*, Vol. 20, Nov. 1979, pp. 389-404.
29. Farquhar, R. W., "The Control and Use of Libration-Point Satellites," NASA Technical Report, R-346, 1970
30. Richardson, D. L., "Analytic Construction of Periodic Orbits About the Collinear Points," *Celestial Mechanics*, Vol. 22, Oct. 1980, pp. 241-253.
31. Howell, K. C., "Families of Orbits in the Vicinity of the Collinear Libration Points," *Journal of the Astronautical Sciences*, Vol. 49, No. 1, Jan.-March 2001, pp. 107-125.