Solving fuel-optimal orbital homing problem with continuous thrust using direct methods

Extended abstract to be submitted to 21^{st} ISSFD 2009

Christophe Louembet[1,2] ; Mounir Kara-Zaïtri[1,2] ; Denis Arzelier[1,2] ; Alain Théron[1,2].

[1] *CNRS ; LAAS ; 7 avenue du colonel Roche, F-31077 Toulouse, France*

[2] *Université de Toulouse ; UPS, INSA, INP, ISAE ; LAAS ; F-31077 Toulouse, France*

{louembet ; kara-zaitri ; arzelier ; atheron} @laas.fr

Keywords : satellite, spacecraft, optimal control, direct methods, orbital rendezvous, low and continuous thrust, relative motion, safety constraints

1 Introduction

Spacecraft rendezvous is an enabling technology for present and future space missions. New challenges arise from the International Space Station (ISS) and the numerous formation flight projects (PRISMA, SYMBOL-X).

In this paper, the possibility to achieve a rendezvous with electric propulsion is investigated. In fact, Ion thrusters have increased significantly their power and specific impulse in the recent years. This is the main reason why they have been used and proposed for many space applications including orbit transfers, attitude control, drag compensation for low earth orbits but also for interplanetary and deep space missions. In this paper, the homing phase of a rendezvous will be achieved by means of a continuous signal thrust.

The rendezvous problem, as one of the most fundamental problems in control of spacecraft trajectories, has been extensively studied as an optimal control problem. Both impulsive thrust and the continuous thrust assumptions have been exploited through the Pontryagin's maximum principle respectively in [11] and [9] and the references therein. Unfortunately, considering possible visibility or safety constraints in the frame developed in these works, increase drastically the complexity and may lead to an untractable problem.

On the opposite to the so-called Pontryagin's maximum principle methods, direct methods may easily handle linear and non linear path constraints. These methods, based on the discretization of the original optimal control problem have been successful in solving impulsive thrust rendezvous problem under path constraints as demonstrated in [1] and the references therein. Nevertheless, some computed trajectories may be observed violating the safety/visibility constraints between grid points as in [1]. In fact, there is no guarantee between discretization points.

We propose, in this paper, a new methodology based on differential flatness that is able to ensure constraints satisfaction all along the path on the contrary to classical direct methods.

Flat systems, first introduced in [6], are characterized by the fact that there exists a minimal set of particular outputs (the so-called flat outputs) that characterize all the state space motions and the corresponding input history. Indeed, general optimal control problems could be solved by means of geometric techniques using the flat trajectory parametrization avoiding integration of the dynamics. In other words, the above optimal control problem boils down to find the best flat output motion that lies into the subspace and passes through a given set of points.

In the differential flatness context, a classical and tractable methodology relies on B-splines based collocation [17, 5, 16]. However, as in classical direct methods, this technique involves time sampling: No guarantee on constraints satisfaction between collocation points may be ensured. This may lead to critical issues that need to be detected by an appropriate post-analysis. Thus, both B-splines collocation and direct shooting techniques require an interactive procedure between the trajectory synthesis and a specific post-analysis.
In this paper, our goal is to design flat system trajectories, using the convenient B-splines parametrization, that guarantee continuous constraints satisfaction in time and without the need for a post-analysis. Henrion and Lasserre tackled this problem in the case of linear systems in [10]. They proved that motion planning under constraints can be recast as the inclusion of a univariate polynomial in a linear semi-algebraic subset. By using results on positive polynomials [18] (based on conic duality), [19] (based on sums of squares decomposition), the last problem turns out to be a Linear Matrix Inequalities (LMI) optimization problem. Using the results from [14] on positive piecewise polynomials, the main result of this paper is to provide a new motion planning methodology that allows to design a trajectory fulfilling the constraints continuously in time.
In section 2, we briefly present the relative motion model and the concept of differential flatness. Then, the optimal path planning problem for flat systems is described. Our contribution is detailed in section 3 in two steps. First, in subsection 3.1, the results on positive polynomials from [14] is presented. Subsequently, in subsection 3.2, the constrained B-splines optimization is formulated as a convex optimization problem over linear matrix inequalities (LMI) for which efficient programming (SDP) solvers are available. In section 4, an example of the resolution of an orbital homing problem illustrates the methodology.

2 Problem statement

2.1 Relative motion model

In this paper, the Rendezvous mission consists of two spacecraft: one chaser satellite with full 3-axis capability and one passive target spacecraft on an arbitrary elliptic orbit. The relative motion between the two satellites in close space was firstly described in the space context in [22] and [21]. In the following, we describe briefly the developement of the equations of motions in order to obtain a state model of the relative dynamics. Let $\vec{\rho}$ be the relative vector between spacecrafts. Under the keplerian assumption, the dynamic equation of ρ is:

$$\left(\frac{d^2\vec{\rho}}{dt^2}\right)_{B_{in}} = \Delta\vec{g} + \frac{\vec{F}_{chaser}}{m_{chaser}} \tag{1}$$

where B_{in} is the inertial earth frame, $\Delta\vec{g} = \vec{g}(M_{chaser}) - \vec{g}(M_{target})$ is the differential gravity force with F_{chaser} and m_{chaser} are respectively the propulsion force and the mass of the chaser spacecraft. Therefore, the equations will be expressed in the Gauss-Hill frame $(O_S, \vec{R}, \vec{S}, \vec{W})$ where O_S is the center of mass of the target satellite (see figure 1).

In this framework, the relative position between spacecraft is such that:

$$\vec{\rho} = x\vec{R} + y\vec{S} + z\vec{W} \tag{2}$$

After the expansion of $\frac{d^2\vec{\rho}}{dt^2}$ and the linearization of the differential gravity term $\Delta\vec{g}$, equation (1) leads to the so-called Tshauner-Hempel equations:

$$\begin{cases} \ddot{x} = 2n\frac{(1+e\cos\nu)^2}{(1-e^2)^{3/2}}\dot{y} - 2n^2 e\sin\nu\left(\frac{1+e\cos\nu}{1-e^2}\right)^3 y + n^2\left(\frac{1+e\cos\nu}{1-e^2}\right)^3(3+e\cos\nu)x + n^2u_R \\ \ddot{y} = -2n\frac{(1+e\cos\nu)^2}{(1-e^2)^{3/2}}\dot{x} + 2n^2 e\sin\nu\left(\frac{1+e\cos\nu}{1-e^2}\right)^3 x + n^2\left(\frac{1+e\cos\nu}{1-e^2}\right)^3(e\cos\nu)y + n^2u_S \\ \ddot{z} = -n^2\left(\frac{1+e\cos\nu}{1-e^2}\right)^3 z + n^2u_W \end{cases} \tag{3}$$

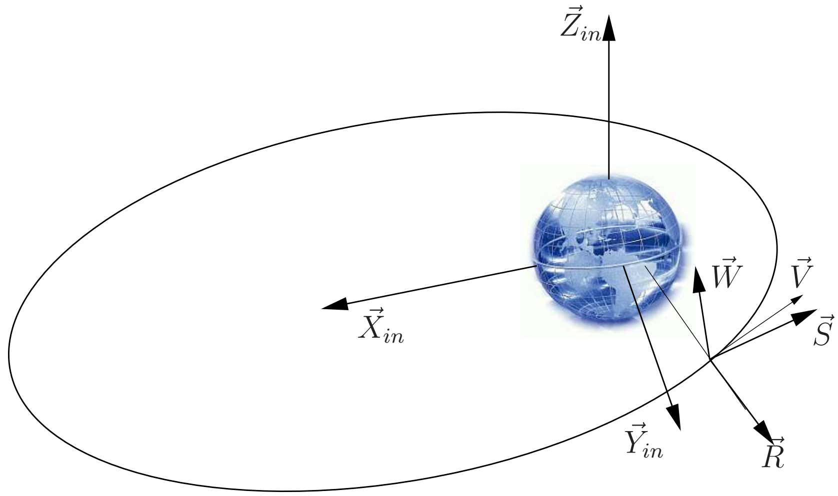


Figure 1: Hill frame

where ν is the true anomaly, n the mean motion and the input vector $u(t)$ is defined by:

$$\begin{cases} u_R = \frac{F_{chaser,R}}{m_{chaser} n^2} \\ u_S = \frac{F_{chaser,S}}{m_{chaser} n^2} \\ u_W = \frac{F_{chaser,W}}{m_{chaser} n^2} \end{cases} \tag{4}$$

Simplified Tschauner-Hempel equations can be obtained by replacing time as the independent variable by the true anomaly: ν

$$\frac{d(\cdot)}{dt} = \frac{d(\cdot)}{d\nu}\frac{d\nu}{dt} = (\cdot)'\dot{\nu}$$

$$\frac{d^2(\cdot)}{dt^2} = \frac{d^2(\cdot)}{d\nu^2}\dot{\nu}^2 + \ddot{\nu}\frac{d(\cdot)}{d\nu} \tag{5}$$

$$= \dot{\nu}^2(\cdot)'' + \ddot{\nu}(\cdot)'$$

and by using the classical change of variables:

$$\begin{aligned} \begin{bmatrix} \tilde{x} \\ \tilde{y} \\ \tilde{z} \end{bmatrix} &= (1 + e\cos\nu) \begin{bmatrix} x \\ y \\ z \end{bmatrix} \\ \begin{bmatrix} \tilde{x}' \\ \tilde{y}' \\ \tilde{z}' \end{bmatrix} &= (1 + e\cos\nu) \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} - e\sin\nu \begin{bmatrix} x \\ y \\ z \end{bmatrix} \\ \begin{bmatrix} \tilde{u}_R \\ \tilde{u}_S \\ \tilde{u}_W \end{bmatrix} &= \left(\frac{1-e^2}{1+e\cos\nu}\right)^3 \begin{bmatrix} u_R \\ u_S \\ u_W \end{bmatrix} \end{aligned} \tag{6}$$

Tschauner-Hempel equations become:

$$\begin{cases} \tilde{x}'' = 2\tilde{y}' + \frac{3}{1+e\cos\nu}\tilde{x} + \tilde{u}_R \\ \tilde{y}'' = -2\tilde{x}' + \tilde{u}_S \\ \tilde{z}'' = -\tilde{z} + \tilde{u}_W \end{cases} \tag{7}$$

We can now define the state vector $\tilde{X}(\nu) = \begin{bmatrix}\tilde{x} & \tilde{y} & \tilde{z} & \tilde{x}' & \tilde{y}' & \tilde{z}'\end{bmatrix}$ and the associated input vector $\tilde{u} = \begin{bmatrix}\tilde{u}_R & \tilde{u}_S & \tilde{u}_W\end{bmatrix}$ and deduce the linear time-periodic state space model:

$$\frac{d\tilde{X}(\nu)}{d\nu} = \tilde{A}_{TH}\tilde{X}(\nu) + \tilde{B}_{TH}\tilde{u}(\nu) \tag{8}$$

with:

$$\tilde{A}_{TH} = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ \frac{3}{1+e\cos\nu} & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & -2 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \end{bmatrix} \tilde{B}_{TH} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{9}$$

2.2 Differential flatness

Differential flatness, or flatness in short, has been introduced by Fliess et al. [7] in 1992. Consider a nonlinear system:

$$\dot{X} = f(X, u), \tag{10}$$

where X is the n-component state vector and u the m-component control assuming that $m \leq n$.

Definition 1 *The nonlinear system* (10) *is differentially flat if there exists an m-dimensional vector χ, whose elements are differentially independent, such that:*

$$\chi(t) = \Phi\left(X(\nu), u(\nu), \dot{u}(\nu), \ldots, u^{(\alpha)}(\nu)\right), \tag{11}$$

and:

$$\begin{cases} X(\nu) = & \Psi_X\left(\chi(\nu), \dot{\chi}(\nu), \ldots, \chi^{(\beta-1)}(\nu)\right), \\ u(\nu) = & \Psi_u\left(\chi(\nu), \dot{\chi}(\nu), \ldots, \chi^{(\beta)}(\nu)\right), \end{cases} \tag{12}$$

where $\Psi_X$and Ψ_u are smooth functions, $\chi_i^{(k)}(\nu)$ denoting the k^{th} order time derivative of the i^{th} component of $\chi(\nu)$, and the multi-index $\beta = (\beta_1, \ldots, \beta_m)$ contains the characteristic numbers associated to the flat outputs and is defined by::

$$\beta_i = \min\left\{k \in \mathbb{N}^* : \partial\left(\frac{d^k\chi_i}{dt^k}\right)/\partial u_j \neq 0,\, j \in \{1, \ldots, m\}\right\}, \tag{13}$$

with $i = 1, \ldots, m$. The elements of $\chi \in \mathbb{R}^m$ are called flat outputs.

Since we focus attention on linear systems, we recall from [8] the following result:

Proposition 1 *A linear sytem is flat if and only if it is controllable*

In our case, system (8) is fully controllable since the chaser is fully actuated. One can easily check that the controllability matrix of the pair (A_{TH}, B_{TH}) is full-row rank for $e < 1$. Consequently, system (8) is differentially flat. Moreover, an elligible flat output is the position vector $\chi = [\tilde{x}\,\tilde{y}\,\tilde{z}]^T$ (see [12, footnote 2] for complementary explanations). Indeed, by inversing system (7), we can express the input vector $\tilde{u}$ in terms of $\overline{\chi}$:

$$\begin{cases} \tilde{u}_R = \chi_1'' - \frac{3}{1+e\cos\nu}\chi_1 - 2\chi_2' \\ \tilde{u}_S = \chi_2'' + 2\chi_1' \\ \tilde{u}_W = \chi_3'' + \chi_3 \end{cases} \tag{14}$$

The real interest of flatness for optimal control problem is that it also defined a Lie-Bäcklund equivalence between a nonlinear system and a trivial system [8]. As χ represent the state of the trivial system,

the m-components of χ are differentially independent. Indeed, a $\overline{\chi}$-space of dimension n_χ can be considered with the coordinates $\overline{\chi} = \{\chi, \chi', \chi'', \dots, \chi^{(p)}\}$ with $p \in \mathbb{N}$ where any curve of this space is equivalent to the system trajectories.
As it will be described in the next paragraph, the solution to the optimal control problem can be described as a particular curve of the $\overline{\chi}$-space.

2.3 Optimal path planning for flat systems

The generation of constrained trajectory consists in determining a finite-time trajectory $t \mapsto (\tilde{X}(\nu), \tilde{u}(\nu))$ with $t \in [t_i\,; t_f]$, satisfying the set of constraints related to the dynamics of the underlying system, boundary conditions, path and actuators constraints. The problem can be formulated as follows:

$$\begin{array}{c}\min_{\tilde{u}} J(\tilde{X}, \tilde{u}) \\ \text{subject to:}\end{array} \quad \begin{cases} \frac{d\tilde{X}(\nu)}{d\nu} = \tilde{A}_{TH}\tilde{X}(\nu) + \tilde{B}_{TH}\tilde{u}(\nu), \\ \tilde{X}(\nu_i) = \tilde{X}_0, \quad \tilde{u}(\nu_i) = \tilde{u}_0, \\ \tilde{X}(\nu_f) = \tilde{X}_f, \quad \tilde{u}(\nu_f) = \tilde{u}_f, \\ \gamma(\tilde{X}(\nu), \tilde{u}(\nu)) \geq 0, \end{cases} \tag{15}$$

where $J(\tilde{X}, \tilde{u})$ represents a particular objective function and $\gamma(\tilde{X}(\nu), \tilde{u}(\nu))$ the path and actuators constraints. The path and actuators constraints are such that:

$$H_{vis}\tilde{X} \leq K_{vis} \tag{16}$$
$$-\tilde{u}_{max} \leq \tilde{u}_i \leq \tilde{u}_{max} \tag{17}$$

Equation (16) represents the visibility constraint: The chaser's trajectory must lie in a visibility cone defined by a polytope characterized by its cartesian coordinates (H_{vis}, K_{vis}). Equation (17) gives saturation bounds on the actuators.

Using the specific flatness properties, the optimal control problem 15 is transformed into the following problem:

Problem 1 *Considering the flat system* (10)*, the optimal path planning problem can be formulated as the following optimization problem:*

$$\begin{array}{c}\min_{\overline{\chi}} J(\overline{\chi}(\nu)) \\ \textit{subject to:}\end{array} \quad \begin{cases} \overline{\chi}(t_i) = \overline{\chi}_i, \\ \overline{\chi}(t_f) = \overline{\chi}_f, \\ \overline{\chi}(\nu) \in S_{\overline{\chi}}, \end{cases} \tag{18}$$

where $\overline{\chi}$ are flat space coordinates and the subset $S_{\overline{\chi}}$, the so-called feasible region, is such that:

$$S_{\overline{\chi}} = \{\overline{\chi} |\, \gamma_\chi(\overline{\chi}(\nu)) \geq 0\}, \tag{19}$$

with $\gamma_\chi(\overline{\chi}(\nu))$ describing the path and actuators constraints in terms of of $\overline{\chi}$.

Since the considered relative motion and constraints functional γ are assumed to be linear, $S_{\overline{\chi}}$ is a polytopic subset of $O_{\overline{\chi}}$. $J(\overline{\chi})$ is assumed to be convex in terms of $\overline{\chi}$.

Then, by virtue of (18), it turns out that the optimal control problem for a flat system consists in determining a finite time trajectory $\nu \mapsto \chi(\nu)$ that connects two points of the $\overline{\chi}$-space and belongs to the subset $S_{\overline{\chi}}$. Since all curves of $\overline{\chi}$-space verify the nonlinear system dynamics, problem (15) is equivalent to the geometric and integration-free problem (18). One of the advantages of problem 1 is that it can be solved by all algorithms able to determine curves belonging to a well-determined subspace. Among eligible algorithms, B-splines collocation methods have been popular and largely investigated [17, 15, 16]. However, the drawback of this method is that it does not guarantee the constraints satisfaction on the time continuum [16]. Nevertheless, the B-splines formalism offers a convenient framework to define piecewise polynomial trajectories offering high flexibility with a low number of parameters. Indeed, in this paper, the trajectories of the flat output χ components and their derivatives are represented with a B-splines basis:

$$\begin{array}{lll} \chi_i(\nu) = & \sum_{j=1}^{n_B} C_{i,j}.B_{j,k}(\nu), & i=1,\dots,m \\ \chi_i^{(r)}(\nu) = & \sum_{j=1}^{n_B} C_{i,j}.B_{j,k}^{(r)}(\nu), & i=1,\dots,m. \end{array} \tag{20}$$

Here $\{B_{j,k}\}$ is a k^{th} order B-splines basis built on a given knot sequence T (see appendix A for definitions and [3, chap. VIII] for more details). The control points $C_{i,j}$ are the coordinates of the piecewise polynomials $\chi_i(\nu)$ in the B-splines basis.
Let $C = (C_{1,1},\dots,C_{1,n_B},C_{2,1},\dots,C_{2,n_B},C_{3,1},\dots,C_{3,n_B})$ be the vector of the control points defining the trajectories $\chi(\nu)$. Using a B-splines parameterization of the flat output, the control points C become the decision variables of the flat optimal control problem (18).

Problem 2 *Consider flat system* (10)*, the optimal path planning problem using B-splines parametrization can be formulated as follows:*

$$\begin{array}{c} \min_C J(\overline{\chi}(C)) \\ \textit{subject to:} \end{array} \quad \left\{ \begin{array}{c} \overline{\chi}(t_i, C) = \overline{\chi}_i, \\ \overline{\chi}(t_f, C) = \overline{\chi}_f, \\ \overline{\chi}(C) \in S_{\overline{\chi}}. \end{array} \right. . \tag{21}$$

The constraint $\overline{\chi}(C) \in S_{\overline{\chi}}$ can be seen as an inclusion of $\overline{\chi}(\nu)$ trajectories within the intersection of several half-spaces. In fact, it will be shown that positioning the trajectory $\overline{\chi}(\nu)$ in a half-space is equivalent to evaluate the sign of the piecewise polynomial gap function, $\kappa(\nu)$, between $\overline{z}(\nu)$ and the hyper-plane boundary.Thus, this positioning problem is equivalent to a positivity problem of piecewise polynomials. By using the concept of positive B-splines developed through LMI approach in [14], the problem 2 will be recast as a semidefinite programming problem whose solution will effectively verify the constraint $\overline{\chi}(C) \in S_{\overline{\chi}}$ all along the path.

3 Path planning as a B-splines positivity problem

3.1 Elements of piecewise polynomial positivity

In [14], the sums of squares representation of piecewise polynomials function has been developed. This representation is convenient since its positiveness only depends on the semi definite positiveness of a weighting matrix Y. Through the linear operator Λ^*, the set of the coefficients μ may be described on a B-spline basis $v(t)$ that define a positive piecewise polynomial function. In fact, this set is shown to be a linear image of the cone of the positive semidefinite matrices.

Theorem 1 *Let μ be an element of the closed, pointed and convex cone K defined by:*

$$\mathsf{K} = \{\mu \in \mathbb{R}^{n_v} \, : \, \mu = \Lambda^*(Y), \, Y \succeq 0\}. \tag{22}$$

Each element μ of K describes a positive semidefinite polynomial on the basis $v(t)$ so that

$$P(t) = \sum_{i=1}^{n_v} \mu_i v_i(t) \geq 0. \tag{23}$$

In sake of conciseness, definitions of Λ^* and proof of the theorem 1 are detailed in [14].

3.2 Motion planning as an LMI problem

This result, mainly based on theorem 1, is the description of the piecewise polynomial trajectory inclusion into a polytope as a B-spline positivity problem and consequently as an LMI problem.
Le $O_{\overline{\chi}}$ be the finite dimensional flat output space with the following coordinates:

$$\overline{\chi} = (\chi_1, \dots, \chi_m, \dot{\chi}_1, \dots, \dot{\chi}_m, \chi_1^{(r)}, \dots, \chi_m^{(r)})$$

Recall that the flat trajectories $[\nu_0,\, \nu_f] \to \mathbb{R}^{n_{\overline{\chi}}},\ \nu \mapsto \chi(\nu)$ are parametrized on k-order B-splines basis $\{B_k\}$ (see equation (20)).
Let the feasible region $S_{\overline{\chi}}$ be an intersection of n_c half-spaces of $O_{\overline{\chi}}$ and H_i be the i^{th} half-space described by its Cartesian coordinates:

$$H_i = \{\overline{\chi} \in \mathbb{R}^{n_{\overline{\chi}}} |\, a_i^T \overline{\chi} \leq b_i\}, \tag{24}$$

where $a_i \in \mathbb{R}^{n_{\overline{\chi}}}$ and $b_i \in \mathbb{R}$ with $i = 1, \dots, n_c$. We note that $\overline{\chi}(\nu)$ belong to the half-space H_i if and only if:

$$a_i^T \overline{\chi}(\nu) \leq b_i \tag{25}$$

Theorem 2 *Solving the path planning problem defined by* (21)*, is equivalent to solving the following SDP problem:*

$$\min_C J(\overline{\chi}(C))$$

$$\textit{subject to: } \begin{cases} \alpha_i C - b_i = \Lambda^*(Y_i) \\ Y_i \succeq 0 \\ \Theta C = \theta \end{cases} \quad , \forall i = 1, \dots, n_c. \tag{26}$$

with the objective function assumed to be linear in $\overline{\chi}$ *and in the control points*

$$C = (C_{1,1}, \dots, C_{1,n}, C_{2,1}, \dots, C_{2,n}, C_{3,1}, \dots, C_{3,n})$$

as well.
Λ^* *is the dual operator defined in [14].* $\alpha_i \in \mathbb{R}^{n_v \times N_C}$ *are linear matrix functions of* a_i*, with* a_i *and* b_i *associated to the* i^{th} *half-space* H_i *(cf. equation* (24)*). The equality constraint* $\Theta C = \theta$ *represents the initial and final conditions. The proof is detailed in appendix B.*

4 Orbital homing example

In this section we detailed the rendezvous problem and propose a solution using the methodology presented in section 3. The studied case is inspired by the ATV mission. The Keplerian parameters of the target orbit are given in table 4: The Rendez-vous is given by the initial and final relative state:

Excentricity e	0.0052
Inclinaison i	$52°$
RAAN Ω	0
Perigee argument ω	0
Initial homing anomaly ν_1	0
Final homing anomaly ν_f	$5rad$

Table 1: Keplerian parameters of the target orbit

$$X_1 = [-800,\, 300,\, 0,\, 0,\, 0,\, 0] \tag{27}$$

$$X_f = [-1,\, 0,\, 0,\, 0,\, 0,\, 0] \tag{28}$$

$$\tag{29}$$

The homing problem is formally described by the optimal control problem (15). Using flatness and B-spline parametrization as explained in subsections 2.2 and 2.3, the problem to be solved is now the following:

$$\begin{array}{c} \min_C J(\overline{\chi}(C)) \\ \text{subject to:} \end{array} \quad \left\{ \begin{array}{c} \overline{\chi}(t_i, C) = \overline{\chi}_i, \\ \overline{\chi}(t_f, C) = \overline{\chi}_f, \\ \overline{\chi}(C) \in S_{\overline{\chi}}. \end{array} \right. . \tag{30}$$

The cost to be minimized is the fuel consumption defined by i.e. $\int_{\nu_1}^{\nu_f} |\tilde{u}(\nu)| d\nu$. The feasible region $S_{\overline{\chi}}$ is given by the constraints on system (8).

The first constraint comes from the saturation on the actuators. In the sequel of the paper, saturation bound in each direction $\tilde{U}_{max}$ is 2 N such that:

$$\begin{cases} -\tilde{U}_{max} \leq \tilde{u}_R \leq \tilde{U}_{max} \\ -\tilde{U}_{max} \leq \tilde{u}_S \leq \tilde{U}_{max} \\ -\tilde{U}_{max} \leq \tilde{u}_W \leq \tilde{U}_{max} \end{cases} \tag{31}$$

Indeed, from equation (14), the saturation constraints may be expressed in terms of $\overline{\chi}$:

$$\begin{cases} -\tilde{U}_{max} \leq \chi_1'' - \frac{3}{1+e\cos\nu}\chi_1 - 2\chi_2' \leq \tilde{U}_{max} \\ -\tilde{U}_{max} \leq \chi_2'' + 2\chi_1' \leq \tilde{U}_{max} \\ -\tilde{U}_{max} \leq \chi_3'' + \chi_3 \leq \tilde{U}_{max} \end{cases} \tag{32}$$

Since only the second derivative of χ is involved in system (32), we will consider the $\overline{\chi}-$space $O_{\overline{\chi}}$ such that:

$$O_{\overline{\chi}} = \{\chi, \chi', \chi''\} \tag{33}$$

Note that the dimension of $O_{\overline{\chi}}$ is 9. Alternatively, the saturation contraint can be defined as the membership of the trajectory $\overline{\chi}(t)$ to polytope of $O_{\overline{\chi}}$ described by its cartesian coordinates:

$$H_{sat}\overline{\chi}(\nu) \leq K_{sat}. \tag{34}$$

In order to have a constant matrix H, we replace the variant term $\frac{3}{1+e\cos\nu}$ by its upper and lower bound $\frac{3}{1+e}$ and $\frac{3}{1-e}$ such that:

$$H_{sat} = \begin{bmatrix} \frac{3}{1-e} & 0 & -1 & 0 & 2 & 0 & 0 & 0 & 0 \\ -\frac{3}{1+e} & 0 & 1 & 0 & -2 & 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \end{bmatrix}, \quad K_{sat} = \tilde{U}_{max}. \tag{35}$$

The visibility constraint will be defined as: The chaser must keep its position in an open polytope behind the target satellite (see figure 2). The visibility cone is defined by the angle β such that the polytope is given by its cartesian coordinates H_{vis} and K_{vis} such that:

$$H_{vis} \begin{bmatrix} \chi_1 \\ \chi_2 \\ \chi_3 \end{bmatrix} \leq K_{vis} \tag{36}$$

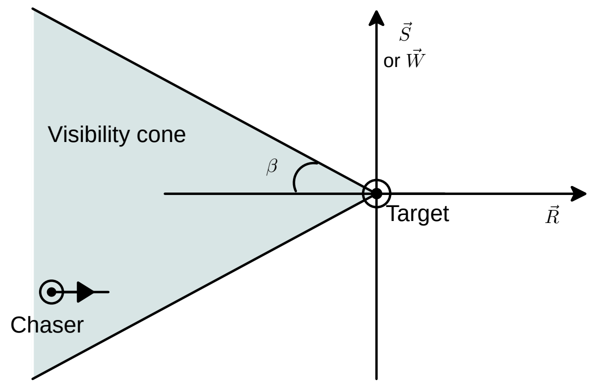


Figure 2: Visibility cone

with:

$$H_{vis} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & \tan(\frac{\pi}{2} - \beta) \\ 1 & 0 & -\tan(\frac{\pi}{2} - \beta) \\ 1 & \tan(\frac{\pi}{2} - \beta) & 0 \\ 1 & -\tan(\frac{\pi}{2} - \beta) & 0 \end{bmatrix}, \quad K_{vis} = 0. \tag{37}$$

Here, $\beta = 20°$. Thus, n_c, the number of linear constraints in $\overline{\chi}$, is 11 (6 for the saturation contraints and 5 for the visibility one).

The trajectory $t \mapsto \chi(\nu)$ is a 5^{th} order piecewise polynomial function defined on the sequence of equidistant knots $\xi = \{\xi_1, \ldots, \xi_{10}\}$. Indeed $\chi(\nu)$ as element of $\mathbb{P}_{k,\xi,\nu}$ admits $B(\nu)$ as B-splines basis. The continuity parameter vector ν are given in table 1. The dimension of the basis $B(t)$ is $n = 14$.
In order to define Λ^*, we need to characterize the gap function $\kappa(\nu)$. Since the higher derivation order involved in $O_{\overline{\chi}}$ is two, $\kappa(\nu)$ belongs to $\mathbb{P}_{k,\xi,\nu\ominus 2}$ (see appendix B) and thus it admits a B-splines basis $v(\nu)$ of dimension $n_v = 32$. Then, the corresponding basis $w(\nu)$ is computed. Its dimension is $n_w = 12$. The operator Λ^* is deduced from $v(\nu)$, $w(\nu)$ and definitions given in [14].
Coefficient matrices α_i of problem (26) are then calculated for each half-space of polytope S_p (see (45)). The problem equivalent to (30) is finally set as:

$$\begin{array}{c} \min_C J(C) \\ \text{subject to:} \end{array} \begin{cases} \alpha_1 C - b_1 = \Lambda^*(Y_1),\ Y_1 \succeq 0, \\ \vdots \\ \alpha_{11} C - b_{11} = \Lambda^*(Y_{11}),\ Y_1 1 \succeq 0, \\ \Theta C = \theta \end{cases} \tag{38}$$

where $C \in \mathbb{R}^{3n}$, $\alpha_i \in \mathbb{R}^{n_v \times 3n}$. The equality constraint $\Theta C = \theta$ represents the initial and final conditions of the rendezvous. The cost need to be linear in C to be handle by the semidefinite programming: $J(C) = \ddot{\chi}(t_f, C)$. Problem (38) is solved using Yalmip [13] and Sedumi 1.02 [20].
The obtained trajectory is given in figure 1. Figure 3 shows that the in-plane trajectory is clearly included in the visibility cone.
For sake of comparison, we solve problem (21) by means of flatness and collocation methods described in [16, 17, 5]. Recall that the constraints are checked in a finite number of time called collocation points. The problem is solved with the quadratic solver MATLAB `quadprog` considering 20 collocation points that are equidistant in anomaly. Although an admissible solution for the collocation problem is quickly

B-splines basis	order	ν_i	
		$i = 1,\ 10$	$i = 2 - 9$
$B(t)$	5	0	4
$v(t)$	5	0	2
$w(t)$	3	0	2

Table 2: B-splines basis parameters

obtained, the trajectory violates the constraints between the collocation points(see figures 3 and 4) on several intervals. Moreover, we can observe on figure 4a that our methodology can produce trajectories very close to the bounds without violating them. On the contrary, when collocation points get closed to the boundaries, the situation could lead to constraints violation (see figures 4). With the above collocation methods, an iterative process is needed to re-distribute the sequence or increase the number the collocation points. This is to be compared to the one-shot method exposed in this paper.

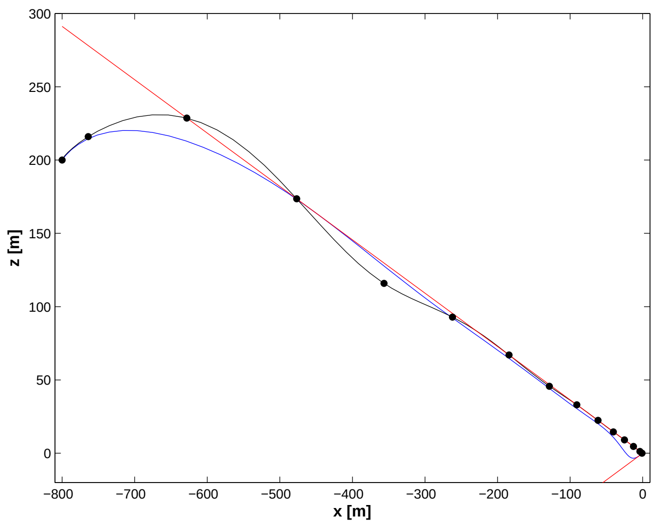


Figure 3: Trajectories $\overline{\chi}(t)$ obtained by SDP (blue) and by collocation (black), the collocation points are the black point, red lines give visibility constraints

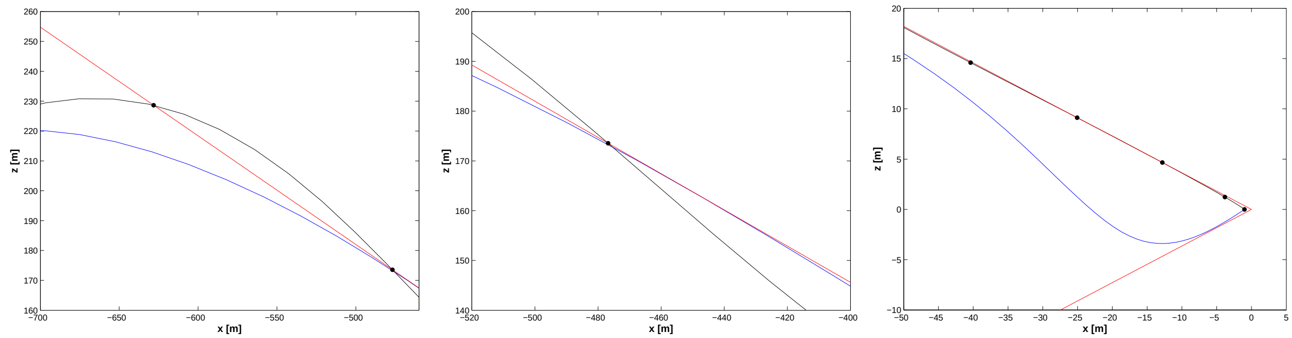
5 Concluding remarks

In this paper, the orbital rendezvous planning problem using continuous thrust is solved by means of a new approach based on the differential flatness and positive piecewise polynomials results. As opposed to most works on direct methods for optimal control problem reported in the literature, the developed methodology provides a new framework for satisfying constraints all along the path.

A Piecewise polynomial function spaces and B-splines basis functions

This appendix summarizes some results on B-splines basis functions on which the main contribution of this paper is built. The interested reader can refer to [3] for further details.

Definition 2 (piecewise polynomial function space) *Let k be a positive integer, $\xi = \{\xi_i\}_{i=1}^{l+1}$ be a strictly increasing sequence of points called breakpoints and $\nu = \{\nu_i\}_{i=1}^{l+1}$ be a non negative integer*



(a) (b) (c)

Figure 4: (a) Focus between the 8^{th} and 9^{th} collocation points, (b) focus around the 5^{th} collocation point, (c) focus on the vicinity of the RdV

sequence. $\mathbb{P}_{k,\xi}$ *denote the linear space of all piecewise polynomial functions denoted* $f(x)$ *of order* k *with breakpoint sequence* $\xi = \{\xi_i\}_{i=1}^{l+1}$ *and* $\{P_1, \ldots, P_l\}$ *a sequence of* l *polynomials of order* k *such that:*

$$f(x) = \begin{cases} P_1(x) & \text{if } \xi_1 < x < \xi_2, \\ \vdots & \\ P_l(x) & \text{if } \xi_l < x < \xi_{l+1}, \\ 0 & \text{if } x < \xi_1 \text{ or } \xi_{l+1} < x. \end{cases} \tag{39}$$

Then, $\mathbb{P}_{k,\xi,\nu}$ *is a linear subspace defined by the collection of the piecewise polynomial functions of* $\mathbb{P}_{k,\xi}$ *whose first* ν_i *derivatives are continuous at* ξ_i *(i.e. that are* C^{ν_i} *at* ξ_i*).*

Now let us describe a set of k order B-splines functions as a basis of $\mathbb{P}_{k,\xi,\nu}$

Definition 3 (B-splines) *Let* T *be a non decreasing sequence of points called knot sequence.* T *is defined on the sequence* ξ *such that:*

$$\begin{aligned} T &= \{t_1, \ldots, t_p\} \\ &= \{\underbrace{\xi_1, \cdots, \xi_1}_{\rho_1 \text{ times}}, \underbrace{\xi_2, \ldots, \xi_2}_{\rho_2 \text{ times}}, \ldots, \underbrace{\xi_{l+1}, \cdots, \xi_{l+1}}_{\rho_{l+1} \text{ times}}\}. \end{aligned} \tag{40}$$

In this formulation, $p = \sum_{i=1}^{l+1} \rho_i$ *and* $\rho = [\rho_1, \ldots, \rho_{l+1}]^T$ *is the vector of the* $\{\xi\}$ *breakpoints multiplicity in the knot sequence* T*. The set* $\mathcal{S}$ *of* k *order B-splines for the knot sequence* T *is defined by the iterative Cox-de Boor algorithm [3]:*

$$\begin{cases} B_{i,0}(x) = \begin{cases} 1 & t_i \le x \le t_{i+1} \\ 0 & otherwise \end{cases}, \\ B_{i,k}(x) = \frac{x-t_i}{t_{i+k-1}-t_i} B_{i,k-1}(x) + \frac{t_{i+k}-x}{t_{i+k}-t_{i+1}} B_{i+1,k-1}(x). \end{cases} \tag{41}$$

By virtue of the Curry-Schoenberg theorem [2], the set of k order B-splines defined on the knot sequence T is a basis of the subspace $\mathbb{P}_{k,\xi,\nu}$ if and only if the knot sequence T is structured such that the ρ_i involved in (40) satisfy the following property:

$$\rho_i = k - \nu_i, \; \forall i. \tag{42}$$

The interested reader can refer to [3] and [4] for an exhaustive description and detailed proof of the Curry-Schoenberg theorem.

B Proof of theorem 2

To apply the positivity theorem, the inequality $a_i^T\overline{z}(\nu) \geq b_i$ must be expressed in a B-splines basis (we flipped the inequality (25) without loss of generality). Thus, by using (20), equation (25) is equivalent to:

$$\begin{aligned} &\sum_{j=1}^{n_B}(a_{i,1}B_{j,k}(\nu)+\cdots+a_{i,(r-1)m+1}B_{j,k}^{(r)}(\nu))C_{1,j}\ldots \\ &\quad+(a_{i,2}B_{j,k}(\nu)+\cdots+a_{i,(r-1)m+2}B_{j,k}^{(r)}(\nu))C_{2,j}+\ldots \\ &\quad+(a_{i,m}B_{j,k}(\nu)+\cdots+a_{i,rm}B_{j,k}^{(r)}(\nu))C_{m,j}\geq b_i. \end{aligned} \tag{43}$$

In inequality (43), the piecewise polynomial function is composed of a $\mathbb{P}_{k,\xi,\nu}$ piecewise polynomial and its r first derivatives. Considering that for a B-spline $B_{j,k}\in\mathbb{P}_{k,\xi,\nu}$, one has $\dot{B}_{j,k}\in\mathbb{P}_{k-1,\xi,\nu\ominus 1},\ldots,B_{j,k}^{(r)}\in\mathbb{P}_{k-r,\xi,\nu\ominus r}$ where the operator $\ominus$ is defined by $\nu\ominus r=(\max\{\nu_1-r,0\},\ldots,\max\{\nu_{l+1}-r,0\})$, with $r\leq\nu_i$ for $i=2,\ldots,l$. Then the sum (43), representing the gap, belongs to $\mathbb{P}_{k,\xi,\nu\ominus r}$ and, by virtue of the Curry-Schoenberg theorem [2], admits the B-splines representation basis $v(\nu)$ such that: (43) becomes:

$$\sum_{i=1}^{n_v}\left(\sum_{j=1}^{n_B}\alpha_{1,i,j}C_{1,j}+\cdots+\alpha_{m,i,j}C_{m,j}\right)v_{i,k}(\nu)\geq\ldots b=b\sum_{i=1}^{n_v}v_{i,k}(\nu). \tag{44}$$

with $\alpha_{p,j}\in\mathbb{R}^{n_v}$ $p=1,\ldots,m$ and $j=1,\ldots,n$. Vectors $\alpha_{l,i,j}$ are identified using the following system of equalities:

$$\begin{cases} E_1(\sum_{i=1}^{n_v}\alpha_{l,i,j}v_{i,k}(\nu)) = & E_1((a_{i,l}B_{j,k}(\nu)+\ldots \\ & +a_{i,(r-1)m+l}B_{j,k}^{(r)}(\nu))), \\ & \vdots \\ E_p(\sum_{i=1}^{n_v}\alpha_{l,i,j}v_{i,k}(\nu)) = & E_p((a_{i,l}B_{j,k}(\nu)+\ldots \\ & +a_{i,(r-1)m+l}B_{j,k}^{(r)}(\nu))), \end{cases} \tag{45}$$

with $l=1,\ldots,m$ and $j=1,\ldots,n$.
$E_p(f)=\int_0^{t_f}x^p\,f(x)\,dx$ denotes the p^{th} order moment of the function f. The index p is chosen such that equation (45) leads to a square linear matrix equality to obtain $\alpha_{l,i,j}$. Thus, inequality (25) is equivalent to the following positivity problem:

$$\kappa(\nu)=\sum_{i=1}^{n_v}\kappa_i v_{i,k}(\nu)\geq 0 \tag{46}$$

where $\kappa_i=\sum_{j=1}^{n_B}(\alpha_{1,i,j}C_{1,j}+\cdots+\alpha_{m,i,j}C_{m,j})-b$.
Then, determining the operators Λ and Λ^* is needed to recast the positivity problem (46) into an LMI problem by using theorem 1. These operators are built with a basis $u(\nu)$ satisfying the following inequality: $\nu\ominus r<\frac{k+1}{2}$ if k is odd. So, theorem 1 gives conditions on the κ coefficients so that inequality (B) holds:

$$\begin{cases}\kappa=\Lambda^*(Y), \quad Y\succeq 0, \\ \kappa=\alpha C-b.\end{cases} \tag{47}$$

Hence,

$$\alpha C-b=\Lambda^*(Y),\ Y\succeq 0. \tag{48}$$

Finally, the inclusion of a trajectory $t \mapsto \overline{z}(\nu)$ into the intersection of n_c half-spaces is written as the conjunction of the n_c membership problem defined in the theorem 2 i.e.

$$\begin{cases} \alpha_1 C - b_1 = \Lambda^*(Y_1), \quad Y_1 \succeq 0, \\ \vdots \\ \alpha_{n_c} C - b_{n_c} = \Lambda^*(Y_{n_c}), \quad Y_{n_c} \succeq 0. \end{cases} \tag{49}$$

which concludes the proof.

References

[1] L. Breger and J How. Safe trajectories for autonomous rendezvous of spacecraft. *Journal of Guidance, Control and Dynamics*, 31(5):1478–1489, 2008.

[2] H. Curry and I. Schoenberg. On pólya frequency functions: The fundamental spline functions and their limits. *J. d'Analyse Math.*, 17:71–107, 1966.

[3] C. de Boor. A pratical guide to spline. In *Applied Mathematical Sciences*, volume 27. Springer, 1978.

[4] C. de Boor and G Fix. Spline approximation by quasi-interpolants. *Journal of Approximation Theory*, 8:19–45, 1973.

[5] N. Faiz, S.K. Agrawal, and R.M. Murray. Differentially flat systems with inequality constraints: An approach to real-time feasible trajectory generation. *J. Guidance, Control and Dynamics*, 24, 2001.

[6] M. Fliess, J. Lévine, P. Martin, and P. Rouchon. On differentially flat nonliear systems. *Proc. 2nd. IFAC NOLCOS Symposium Bordeaux*, 1992.

[7] M. Fliess, J. Lévine, P. Martin, and P. Rouchon. Sur les systèmes non linéaires différentiellement plats. *C.R. Acad. Sci. Paris*, 1992.

[8] M. Fliess, J. Lévine, P. Martin, and P. Rouchon. a lie-bäcklund approach to equivalence and flatness of nonlinear systems. *IEEE Trans. Automat. Contr.*, 44(5):922–937, 1999.

[9] C.D. Hall and V.C. Collazo-Perez. Minimum-time orbital phasing maneuvers. *Journal of Guidance, Control, and Dynamics*, 26(6):934–941, 2003.

[10] D. Henrion and J.-B. Lasserre. Lmis for constrained polynomial interpolation with application in trajectory planning. *Systems & Control Letters*, 55(6), June 2006.

[11] D.F. Lawden. *Optimal trajectories for space navigation*. Butterworth, London, England, 1963.

[12] J. Lévine and D.V. Ngyuen. Flat output characterisation for linear systems using polynomial matrices. *Systems and Control Letters*, 48:69–75, 2003.

[13] J. Löfberg. Yalmip : A toolbox for modeling and optimization in MATLAB. In *Proceedings of the CACSD Conference*, Taipei, Taiwan, 2004.

[14] C. Louembet, F Cazaurang, and A Zolghadri. Motion planning for flat systems using B-splines parametrization:an LMI approach. *Automatica*, Submitted.

[15] C. Louembet, F Cazaurang, A Zolghadri, C. Pittet, and C Charbonnel. Design of algorithms for satellite slew manoeuver by flatness and collocation. In *26th American Control Conference (ACC)*, New York City, USA, 11-13 July 2007.

[16] C. Louembet, F Cazaurang, A Zolghadri, C. Pittet, and C Charbonnel. Path planning for satellite slew maneuvers: A combined flatness and collocation based approach. *IET Control Theory & Applications*, 2009.

[17] M.B. Milam, K. Mushambi, and R.M. Murray. A new computational approach to real-time trajectory generation for constrained mechanical systems. In *IEEE Conference on Decision and Control*, 2000.

[18] Yu Nesterov. *High Performance Optimization.*, chapter Squared functional systems and optimization problems, pages 405–440. Kluwer Academic Press, Dordrecht, 2000.

[19] P.A. Parrilo. *Structured Semidefinite Programs and Semialgebraic Geometry Methods in Robustness and Optimization*. PhD thesis, California Institute of Technology, Pasadena, California, 2000.

[20] J.F. Sturm. Using SeDuMi 1.02, a Matlab toolbox for optimization over symmetric cones. *Optimization methods and software*, 11-12:625–653, 1999.

[21] J. Tschauner. The elliptic orbit rendezvous. In *AIAA 4th Aerospace Sciences Meeting*, Los Angeles, Californie, USA, Juin 1966.

[22] J. Tschauner and P. Hempel. Optimale beschleunigungs-programme fur des rendezvous manover. *Astronautica Acta*, 5-6:296–307, 1964.