RELATIVISTIC ACCELERATION OF PLANETARY ORBITERS

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Abstract: *This paper deals with the relativistic contributions to the gravitational acceleration of a planetary orbiter. The formulation for the relativistic corrections is different in the solar-system barycentric relativistic system and in the local planetocentric relativistic system. The ratio of this correction to the total acceleration is usually orders of magnitude larger in the barycentric system than in the planetocentric system. However, an appropriate Lorentz transformation of the total acceleration from one system to the other shows that both systems are equivalent to a very good accuracy. This paper discusses the steps that were taken at ESOC to validate the implementations of the relativistic correction to the gravitational acceleration in the interplanetary orbit determination software. It compares numerically both systems in the cases of a spacecraft in the vicinity of the Earth (Rosetta during its first swing-by) and a Mercury orbiter (BepiColombo). For the Mercury orbiter, the orbit is propagated in both systems and with an appropriate adjustment of the time argument and a Lorentz correction of the position vector, the resulting orbits are made to match very closely. Finally, the effects on the radiometric observables of neglecting the relativistic corrections to the acceleration in each system and of not performing the space-time transformations from the Mercury system to the barycentric system are presented.*

Keywords: *Relativistic Reference Systems, Lorentz transformation, Gravitation, Mercury Dynamic Time, Interplanetary Orbit Determination*

1. Introduction

The Geocentric and Barycentric Celestial Reference Systems (GCRS and BCRS) are two commonly used relativistic systems of space-time coordinates for spacecraft orbit propagation and determination. Despite their names, these systems do not just differ from each other by a geometric translation, but also by a generalized Lorentz transformation. When propagating the orbit of a planetary orbiter in the BCRS, it is often convenient for numerical as well as for modeling reasons to integrate the trajectory relative to the planet center, that is to integrate the difference between the spacecraft acceleration in the BCRS (direct acceleration) and the planet acceleration in the BCRS (opposite of the indirect acceleration). In that case, the relativistic reference system is the BCRS with a geometrically translated frame of reference for the space coordinates. In this paper, when it is said that a planetary orbiter trajectory is integrated in the BCRS, it actually means in the translated BCRS.

For most Earth orbiters, orbit integration and determination is performed in the GCRS. In that system, the main relativistic corrections to the Newtonian formulation of the acceleration are a relativistic correction in the central gravitational acceleration from the Earth, the Coriolis accelerations due to the Lense-Thirring precession (from Earth rotation) and due to geodesic precession (mainly due to the Sun). These effects are usually nine or ten orders of magnitude smaller than the total acceleration and at least one order of magnitude smaller than non-gravitational accelerations (e.g. atmospheric drag for lower orbits and solar radiation pressure) whose accurate modeling is challenging. Hence if there is no experiment on-board that allows to reduce the uncertainty in the non-gravitational acceleration (e.g. accelerometers or drag compensation system) and if there is no requirement for orbit reconstruction accuracy approaching the centimeter level, it is usual to neglect these relativistic corrections to the acceleration altogether.

The GCRS is only suitable in the immediate vicinity of the Earth. Therefore orbit propagation and determination of interplanetary spacecraft is usually performed in the BCRS. This requires that space-time events at a ground station on Earth appearing in the modeling of the observables are Lorentz-transformed from the geocentric to the barycentric system. For planetary orbiters, it is also possible to introduce a local relativistic planetocentric reference system with an associated planetocentric coordinate time when propagating the trajectory, however this requires a second Lorentz transformation from the planetocentric to the barycentric system in modeling the observables.

In the BCRS, the relativistic correction to the Newtonian gravitational acceleration is given by the Einstein-Infeld-Hoffman (EIH) equations, possibly adding the Lense-Thirring acceleration of a nearby planet. In the vicinity of a planet, the ratio of the corrective relativistic terms to the total acceleration can easily be two orders of magnitude higher in the BCRS than in the planetocentric relativistic system. However, an appropriate Lorentz transformation of the total acceleration from one system to the other shows that both systems are equivalent to a very good accuracy [1], [2].

Section 2. of this paper is an introduction to the BCRS and GCRS reference systems, coordinate time and scaling of coordinates. It presents the formulations for the relativistic corrections to the acceleration in both the BCRS and the GCRS as well as the space-time transformation relating the two systems.

ESOC's interplanetary orbit determination software uses the BCRS system to propagate trajectories and model observables. Until recently, however, only a simple relativistic corrective term due to the Sun was introduced in the acceleration. This term does not model the main relativistic correction when in the vicinity of a planet. Hence, it was decided to implement the full EIH equations. For the sake of validation, a comparison of the total acceleration of a spacecraft (Rosetta during its first Earth swing-by) in the vicinity of the Earth has been performed between the barycentric and geocentric systems using Lorentz transformations on position, velocity and acceleration. This is the subject of section 3..

The Mercury Planetary Orbiter (MPO) of the ESA BepiColombo mission is to be launched in 2015 for a Mercury orbit insertion in 2022, targeting a 2840x3948 km polar orbit. The pericenter altitude is 400 km and the orbital period is 2.325 hours. For navigation purposes in Mercury orbit, the spacecraft will be tracked daily by ESA deep space stations providing two-way range and Doppler observables with X-band uplink and dual-band X and Ka downlink. Routine orbit determination will not solve for Mercury orbital parameters and thus it will not make use of the range residuals in deriving the solution. Orbit propagation will be performed in the BCRS using ESOC navigation software.

Using additionally 2-way Doppler and range in Ka band (both uplink and downlink) and accurate accelerometers [3] to measure non-gravitational accelerations, the radio-science experiment intends to solve for Mercury gravity field and tides parameters [4, chapter 17] to improve Mercury and the Earth-Moon barycenter ephemerides, and to test general relativity [5]. For this purpose, the propagation of MPO orbit will be performed in the local Mercury-centric relativistic system and converted to the barycentric system for observable modeling [6]. In the Mercury system, the relativistic correction to the acceleration can be neglected, however the space-time transformation between the two systems cannot.

Section 4. introduces a local Mercury relativistic system with an associated coordinate time. In this system the orbit of the MPO is propagated and compared with the orbit resulting from the propagation in the barycentric system. The effects of the relativistic corrections to the acceleration in both systems, of the space and time transformations from one system to the other are separately assessed. The 2-way range and Doppler residuals resulting from failing to take into account these corrections are then evaluated. The goal of this analysis is to assess to which accuracy the difference in propagation and observable modeling between the ESOC navigation team and the radio science team can be made to match.

The relativistic correction to the light time due to the reduction of the coordinate velocity of light and to the bending of the light path nearby massive bodies for observable modeling [7], [8], [9], [10, section 8] is not discussed in this paper.

2. The Barycentric and Geocentric Celestial Reference Systems and their scaled versions

2.1. Space Time Reference Systems and the Metric Tensor

Space-time events are described by their coordinates in a space-time coordinate system. Let (t, x^1, x^2, x^3) and $(t+dt, x^1+dx^1, x^2+dx^2, x^3+dx^3)$ be the coordinates of infinitesimally separated events on the trajectory of a particle. Using, the metric tensor, written in this coordinate system $g_{\alpha\beta}$ (α and β indices running from 0 to 3), at (t, x^1, x^2, x^3), we can compute the square of the space-time interval:

$$ds^2 = g_{00}c^2dt^2 + 2g_{0i}cdtdx^i + g_{ij}dx^idx^j \tag{1}$$

where c is the speed of light. In the above equation, the Einstein summation convention is used [11], which in this case means that in the second term of the right-hand side the i index running from 1 to 3 is summed over and in the third term both i and j indices running from 1 to 3 are summed over. The space-time interval between two events is invariant by change of coordinate system.

When the space time interval is imaginary, the two events cannot influence each other. Therefore on the trajectory of a real particle $ds^2 \geq 0$. On the trajectory of light, ds is zero.

The proper time of the particle (the time kept by an ideal atomic clock comoving with the particle) changes from τ at (t, x^1, x^2, x^3) to $\tau + d\tau$ at $(t + dt, x^1 + dx^1, x^2 + dx^2, x^3 + dx^3)$ such that:

$$d\tau = ds/c \tag{2}$$

The trajectory of a particle (not submitted to any non-gravitational acceleration) between two space-time events A and B is a 4-dimensional curve called a world-line linking these two events which extremalize the integral:

$$\int_A^B \sqrt{|ds^2|} \tag{3}$$

If the particle is massive, there is a contribution from the particle itself in the metric tensor.

The parameter t is called coordinate time. The vector $(dx/dt, dy/dt, dz/dt)$ is called coordinate velocity.

2.2. The Barycentric Celestial Reference System

Defined by IAU 2000 Resolution B1.3 [12], the Barycentric Celestial Reference System (BCRS) is a system of space-time coordinates for the solar system in the framework of general relativity [13]. The coordinate system spatial origin is the barycenter of the solar system. The metric tensor is specified by the resolution. The orientation of the axes is however not specified, but according to IAU 2006 Resolution B2 [14], for all practical applications, unless otherwise stated, the BCRS is assumed to be oriented according to the International Celestial Reference System (ICRS) axes. The BCRS can thus be seen as a dynamic relativistic realization of the International Celestial Reference System. The coordinate time of this system is called Barycentric Coordinate Time (TCB). The BCRS is used to study the motion of bodies in the solar system: planetary ephemerides, interplanetary spacecraft.

2.3. The Geocentric Celestial Reference System

Defined by IAU 2000 Resolution B1.3 [12], the Geocentric Celestial Reference System (GCRS) is a system of space-time coordinates for the Earth in the framework of general relativity [13]. The coordinate system spatial origin is the center of the Earth. The metric tensor is specified by the resolution. The orientation of the axes is such that the transformation between BCRS and GCRS spatial coordinates contains no rotation components and thus for practical applications the axes are usually aligned with the ICRS axes. The coordinate time of this system is called Geocentric Coordinate Time (TCG). The GCRS is a local coordinate system to study the motion of a body in the vicinity of the geocenter: satellite laser ranging, Earth rotation, nutation and precession.

2.4. Relation between proper time and coordinate time

The proper time along the world-line of an observer and the coordinate time are related by the following differential equation:

$$\frac{d\tau}{dt} = \left(-g_{00}(t, x_{obs}(t)) - \frac{2}{c} g_{0i}(t, \mathbf{x}_{obs}(t))\dot{x}^i_{obs}(t) - \frac{1}{c^2} g_{ij}(t, \mathbf{x}_{obs}(t))\dot{x}^i_{obs}(t)\dot{x}^j_{obs}(t)\right)^{1/2} \tag{4}$$

where c is the speed of light, dot denotes the differentiation with respect to coordinate time t, τ is proper time of the observer, i and j indexes run from 1 to 3, Einstein summation convention is used, $g_{\alpha\beta}$, α and β running from 0 to 3, is the metric tensor and $\mathbf{x}_{obs} = (x^1_{obs}, x^2_{obs}, x^3_{obs})$ is the spatial component of the observer's coordinates. Then $\dot{x}^i_{obs}$ are the coordinate velocity components that are functions of coordinate time. The metric tensor components are functions of coordinate time directly but also indirectly through their dependence in the observer's spatial position. Thus we have:

$$\frac{d\tau}{dt} = f(t) \tag{5}$$

This equation can be integrated given an initial condition of the form

$$\tau(t_0) = \tau_0 \tag{6}$$

In the BCRS and GCRS, Eq. 4 simplifies approximatively to:

$$\frac{d\tau}{dt} = 1 - \frac{U}{c^2} - \frac{v^2}{2c^2} \tag{7}$$

where v is the magnitude of the observer's coordinate velocity in the considered system and U is the opposite of the observer's gravitational potential per unit mass taken to be zero at infinity, a quantity which from now on we will call potential.

In the BCRS, the potential U, neglecting the spherical potential, is computed from:

$$U = \sum_i \frac{\mu_i}{r_i} \tag{8}$$

where the sum is over all the planets and the Moon, μ_i is the mass parameter of body i that is the product Gm_i of the mass of body i, m_i and the universal constant of gravity G. r_i is the spatial distance between body i and the observer.

In the BCRS, the potential U, neglecting the spherical and third-body tidal potentials, is computed from:

$$U = \frac{\mu_E}{r_E} \tag{9}$$

where μ_E is the mass parameter of the Earth and r_E is the spatial distance between the geocenter and the observer.

2.5. Scaling of coordinate time

In general, coordinate time and proper time do not tick at the same rate. For practical reasons, it is usual to scale coordinate time, so that along the world-line of an observer at the surface of the Earth it would tick on average at the same rate as proper time. The spatial coordinates are also scaled by the same factor. Coordinate velocity is thus conserved. The metric tensor must also be scaled in such a way to conserve the proper time, that is if:

$$(\hat{t}, \hat{\mathbf{x}}(t)) = K \cdot (t, \mathbf{x}(t)) \tag{10}$$

then:

$$\hat{g}_{\alpha\beta}(\hat{t}, \hat{\mathbf{x}}(t)) = g_{\alpha\beta}(t, \mathbf{x}(t))/K^2 \tag{11}$$

where the hat denotes the scaled coordinate system and K is the constant scaling factor. Similarly, mass parameters of any gravitational bodies and accelerations are to be rescaled when switching between scaled and unscaled coordinates. The equations of motion in the scaled coordinates have the same form as the ones in the unscaled coordinates.

Two scaled coordinates were introduced by the IAU in 1976. They were later (1979) named: Terrestrial Dynamical Time (TDT) and Barycentric Dynamical Time (TDB). TDB is intended to serve as an independent time argument of barycentric ephemerides and equations of motion. TDT is a time scale for apparent geocentric ephemerides and the equation of motion of a satellite in the vicinity of the Earth. Their scaling factors were chosen so that their respective coordinate time would tick at the same rate as proper time along the word-line of an observer at the surface of the Earth. However as such TDB and TDT are ill-defined because the scaling factors would then depend on the actual position of the observer on Earth and on the time period used for averaging the rate difference between proper time and coordinate time. If the scaling factor depends on time then the equations of motion in the scaled system will have a different and more complex form. Therefore TDB was redefined as a linear function of TCB by IAU 2006 Resolution B3 [15]. The scaling factor is a defining constant whose value is such that to a good accuracy TDB is on average ticking at the same rate as the proper time along the word-line of an observer on the rotating geoid. In the same way, TDT was made obsolete and replaced by Terrestrial Time (TT) a linear function of TCG where the scaling factor is a defining constant (IAU 2000 Resolution B1.9 [12]). For practical purpose TT is realized by International Atomic Time (TAI) according to $TT = TAI + 32.184s$.

2.6. Scaled BCRS

Let $(t = TCB, \mathbf{x})$ be the coordinates of an event in the BCRS, then we define $(t^*, \mathbf{x}^*)$:

$$\begin{aligned} t^* &= F_B t \\ \mathbf{x}^* &= F_B \mathbf{x} \end{aligned} \tag{12}$$

where:

$$F_B = 1 - L_B \tag{13}$$

and L_B is the defining constant:

$$L_B = 1 - \frac{dTDB}{dTCB} = 1.550519768 \times 10^{-8} \tag{14}$$

$(t^* = TDB, x^*)$ are the space-time coordinates of the event in the scaled BCRS which we will from now on refer to as BCRS*.

The mass parameter $\mu^*$of a gravitational body in BCRS*differs from its mass parameter μ in BCRS by the scaling factor F_B:

$$\mu^* = F_B\mu \tag{15}$$

μ^*, $\mathbf{x}^*$ and t^* are TDB-compatible quantities whereas μ, $\mathbf{x}$ and t are TCB-compatible quantities [16] [17]. μ is also the physical mass parameter, whereas μ^* is an artificial mass parameter introduced by the scaling of the coordinate system.

Equation 7 can be adjusted with the scaling factor to get an approximate differential equation for the rate of the proper time of an observer with respect to TDB.

$$\frac{d\tau}{dTDB} = 1 - \frac{U}{c^2} - \frac{v^2}{2c^2} + L_B \tag{16}$$

2.7. Scaled GCRS

Let $(T = TCG, \mathbf{X})$ be the coordinates of an event in the GCRS, then we define $(T^*, \mathbf{X}^*)$:

$$\begin{aligned} T^{**} &= F_G T \\ \mathbf{X}^{**} &= F_G \mathbf{X} \end{aligned} \tag{17}$$

where:

$$F_G = 1 - L_G \tag{18}$$

and L_G is the defining constant:

$$L_G = 1 - \frac{dTT}{dTCG} = 6.969290134 \times 10^{-10} \tag{19}$$

$(T^{**} = TT, X^{**})$ are the space-time coordinates of the event in the scaled GCRS which we will from now on refer to as GCRS**.

The mass parameter μ of a gravitational body in GCRS is the same as the mass parameter of that same body in BCRS. This is the physical mass parameter. The mass parameter μ^{**} of a gravitational body in GCRS**differs from its mass parameter μ in GCRS by the scaling factor F_G:

$$\mu^{**} = F_G\mu \tag{20}$$

μ^{**}, $\mathbf{X}^{**}$ and T^{**} are TT-compatible quantities whereas μ, $\mathbf{X}$ and T are TCG-compatible quantities [16] [17].

Equation 7 can be adjusted with the scaling factor to get an approximate differential equation for the rate of the proper time of an observer with respect to TT.

$$\frac{d\tau}{dTT} = 1 - \frac{U}{c^2} - \frac{v^2}{2c^2} + L_G \tag{21}$$

2.8. Transformation between scaled BCRS and scaled GCRS

IAU 2000 Resolution B1.3 [12] provides the coordinate transformation between BCRS and GCRS that is derived from their metric tensors. Using Eq. 16 and Eq. 21, we can derive an approximate differential equation relating TT and TDB on any world-line ([10, section 4.3.2]):

$$\frac{dTT}{dTDB} = 1 - \frac{U_E}{c^2} - \frac{v_E^2}{2c^2} + \tilde{L} - \frac{\mathbf{v}_E \cdot \mathbf{V}}{c^2} - \frac{\mathbf{a}_E \cdot \mathbf{R}}{c^2} \tag{22}$$

where $\tilde{L} = L_B - L_E$, U_E is the potential at the Earth due to all other solar system bodies, $\mathbf{v}_E$ and $\mathbf{a}_E$ are the barycentric velocity and acceleration vectors of the Earth, $\mathbf{R}$ and $\mathbf{V}$ are the geocentric position and velocity vectors of the observer. The last two terms depend on the position and velocity of the observer in the geocentric system. The last term is much smaller than the other. The integrated version of Eq. 22 [10, section 4.3.3] can be separated in two parts, the first being the difference $TDB - TT$ at the geocenter, the second being a correction for the position of the observer. Accurate numerically integrated time ephemerides [18] provides the difference $TDB - TT$ along the world-line of the geocenter. For a spacecraft in Earth orbit or a ground station on the surface of the Earth, because of the values chosen for the scaling factors, TDB and TT differ mainly by periodic terms and the secular drift can be ignored. The main period in the difference between TDB and TT is the 1.7 millisecond amplitude due to the motion of the Earth around the Sun.

The transformation of the spatial coordinates from GCRS**to BCRS*is [2]:

$$(\mathbf{x} - \mathbf{x}_E) = \mathbf{X} - \left(\tilde{L} + \frac{U_E}{c^2} + \mathbf{X}\cdot\mathbf{a}_E\right)\mathbf{X} - \frac{1}{2c^2}(\mathbf{X}\cdot\mathbf{v}_E)\,\mathbf{v}_E + \frac{1}{2c^2}(\mathbf{X}\cdot\mathbf{X})\mathbf{a}_E \tag{23}$$

where $\mathbf{x}$ and $\mathbf{x}_E$ are the BCRS*spatial coordinates of respectively the observer and the geocenter, U_E is the potential at the location of the geocenter due to all bodies but the Earth, $\mathbf{X}$ is the spatial position of the observer in GCRS**, $\mathbf{v}_E$ and $\mathbf{a}_E$ are respectively the velocity and acceleration vectors of the Earth in BCRS*. The transformation of coordinate velocity and acceleration from GCRS**to BCRS*is [2]:

$$\begin{aligned}(\mathbf{v} - \mathbf{v}_E) =& \mathbf{V} - \frac{1}{c^2}\left(2U_E + \frac{{\mathbf{v}_E}^2}{2} + \mathbf{v}_E\cdot\mathbf{V} + 2\mathbf{a}_E\cdot\mathbf{X}\right)\mathbf{V} - \frac{1}{2c^2}(\mathbf{a}_E\cdot\mathbf{X} + \mathbf{v}_E\cdot\mathbf{V})\,\mathbf{v}_E \\ &- \frac{1}{c^2}\left(\frac{1}{2}\mathbf{v}_E\cdot\mathbf{X} - \mathbf{X}\cdot\mathbf{V}\right)\mathbf{a}_E - \frac{1}{c^2}\left(\mathbf{a}_E\cdot\mathbf{V} + \dot{U}_E\right)\mathbf{X}\end{aligned} \tag{24}$$

$$
\begin{aligned}
(\mathbf{a}-\mathbf{a}_E) = & \mathbf{A}+\tilde{L}\mathbf{A}-\frac{1}{c^2}\left(3U_E+{\mathbf{v}_E}^2+2\mathbf{v}_E\cdot\mathbf{V}+3\mathbf{a}_E\cdot\mathbf{X}\right)\mathbf{A}-\frac{1}{c^2}\left(\mathbf{a}_E\cdot\mathbf{A}\right)\mathbf{X} \\
& -\frac{1}{2c^2}\left(\mathbf{v}_E\cdot\mathbf{A}\right)\left(\mathbf{v}_E+2\mathbf{V}\right)-\frac{1}{c^2}\left(3\dot{U}_E+\mathbf{a}_E\cdot\mathbf{v}_E+4\mathbf{a}_E\cdot\mathbf{V}\right)\mathbf{V} \\
& -\frac{1}{c^2}\left(\mathbf{a}_E\cdot\mathbf{V}\right)\mathbf{v}_E-\frac{1}{c^2}\left(\left(\mathbf{v}_E\cdot\mathbf{V}\right)-\mathbf{V}^2-\mathbf{X}\cdot\mathbf{A}\right)\mathbf{a}_E
\end{aligned}
\tag{25}
$$

where $\dot{U}_E$ is the time derivative of the potential at the geocenter due to all other bodies, $\mathbf{V}$ and $\mathbf{A}$ are the coordinate velocity and acceleration of the spacecraft in GCRS**. The formulas are accurate to order $1/c^2$ and the corrective terms can indifferently be evaluated using for $\mathbf{X}$, $\mathbf{V}$ and $\mathbf{A}$ GCRS**or BCRS*quantities translated to the geocenter. The inverse transformation of Eq. 23, Eq. 24 and Eq. 25 is obtained by interchanging $(\mathbf{x}-\mathbf{x}_E)$ with $\mathbf{X}$, $(\mathbf{v}-\mathbf{v}_E)$ with $\mathbf{V}$ and $(\mathbf{a}-\mathbf{a}_E)$ with $\mathbf{A}$ in all 3 equations and inverting the sign of all corrective terms.

We have:

$$
\dot{U}_E = -\sum_{i\neq E}\frac{\mu_i(\mathbf{v}_i-\mathbf{v}_E)\cdot(\mathbf{r}_i-\mathbf{r}_E)}{r_{iE}^3}
\tag{26}
$$

where μ_i, $\mathbf{r}_i$ and $\mathbf{v}_i$ are the mass parameter and barycentric position and velocity vectors of body i, E is the index for Earth and r_{iE} is the geocentric range of body i.

Equation 23 is called the modified Lorentz transformation. It contains the special relativity Lorentz transformation due to the Earth barycentric velocity, the general relativity effects due to the gravitational potential and the scaling factors. Not only does it change the length of position vectors, it also changes the angle between position vectors and the Earth velocity vector (aberration). This equation contains more corrective terms than [10, equation 4-10] which is used in ESOC interplanetary orbit determination software for converting station position from GCRS**to BCRS*in modeling observables.

Finally, mass parameters of celestial bodies must also be transformed between GCRS**and BCRS*:

$$
\mu^{*} = (1-\tilde{L})\mu^{**}
\tag{27}
$$

2.9. Equation of motion in scaled BCRS

The equations of motion of massive bodies considered as mass monopoles in BCRS*are the Einstein-Infeld-Hoffman (EIH) equations [19, section 39.13]:

$$\ddot{\mathbf{r}}_i = \sum_{k\neq i} \frac{\mu_k \mathbf{r}_{ik}}{r_{ik}^3} +$$
$$\sum_{j\neq i} \frac{\mu_j \mathbf{r}_{ij}}{c^2 r_{ij}^3} \left\{ -4U_i - U_j + \dot{\mathbf{r}}_i \cdot \dot{\mathbf{r}}_i + 2\dot{\mathbf{r}}_j \cdot \dot{\mathbf{r}}_j - 4\dot{\mathbf{r}}_i \cdot \dot{\mathbf{r}}_j - \frac{3}{2} \frac{(\mathbf{r}_{ji} \cdot \dot{\mathbf{r}}_j)^2}{r_{ij}^2} + \frac{\mathbf{r}_{ij} \cdot \ddot{\mathbf{r}}_j}{2} \right\} \tag{28}$$
$$+ \frac{1}{c^2} \sum_{j\neq i} \frac{\mu_j}{r_{ij}^3} \left\{ \mathbf{r}_{ji} \cdot \left[4\dot{\mathbf{r}}_i - 3\dot{\mathbf{r}}_j \right] \right\} (\dot{\mathbf{r}}_i - \dot{\mathbf{r}}_j) + \frac{7}{2c^2} \sum_{j\neq i} \frac{\mu_j \ddot{\mathbf{r}}_j}{r_{ij}} + O(c^{-4})$$

where $\mathbf{r}_p$, $\dot{\mathbf{r}}_p$, $\ddot{\mathbf{r}}_p$, are the barycentric position, velocity and acceleration of body p, $\mathbf{r}_{pq}$ is the vector from body p to body q, r_{pq} the distance between bodies p and q, μ_p is the mass parameter of body p (zero for the satellite body which has negligible mass) and U_p is the potential at body p due to all other bodies. The first term is the Newtonian gravitational acceleration which must be computed using barycentric TDB-compatible quantities. Acceleration of all bodies $j \neq i$ in the right hand side of Eq. 28 can be computed with the Newtonian approximation to compute the relativistic gravitational acceleration of body i to order $1/c^2$.

For a planetary orbiter, it is usual to integrate the difference of the orbiter acceleration and the planet acceleration both evaluated from the EIH equations. Typically only the relativistic corrective terms with the j index matching either the central planet or the Sun have significant effects.

2.10. Equation of motion in scaled GCRS

In the local geocentric system, the following gravitational accelerations due to mass monopoles must be taken into account:

- the Newtonian and relativistic gravitational acceleration due to the Earth.
- the third body Newtonian acceleration due to all other bodies.
- the acceleration due to the De Sitter precession.

The central Newtonian and relativistic contribution of the Earth to the spacecraft acceleration is:

$$\mathbf{A} = -\frac{\mu}{R^3}\mathbf{R} + \frac{\mu}{c^2 R^3} \left\{ \left[\frac{4\mu}{R} - \mathbf{V} \cdot \mathbf{V} \right] \mathbf{R} + 4(\mathbf{R} \cdot \mathbf{V})\mathbf{V} \right\} \tag{29}$$

where $\mathbf{R}$, $\mathbf{V}$, $\mathbf{A}$, are the geocentric position, velocity and acceleration of the satellite, R the geocentric distance of the satellite, μ is the Earth mass parameter. All quantities (positions, accelerations, distances, mass parameters) are TT-compatible. The first term is the Newtonian gravitational acceleration. The other terms are the Schwarzschild correction to the acceleration. This equation is identical to the EIH equations with only one massive body placed at the center.

Computing accurately the third body acceleration is not straightforward since celestial bodies are outside the range of validity of the geocentric system. [10, section 4.5.1] recommends to:

- Convert time argument of integration from TT to TDB along the spacecraft world-line. Because this difference is essentially periodic and bounded by less than two milliseconds, this step can be neglected.
- Retrieve BCRS*state vector of Earth and third body from planetary ephemeris at TDB time.
- Compute spacecraft to third body vector and Earth to third body vector from BCRS*state of Earth and third body and GCRS**state of spacecraft.
- Compute third body acceleration using previously computed vectors and BCRS*mass parameter of third body.
- Rescale third body acceleration by multiplying by $(1 - \tilde{L})$

However these relativistic corrections in the third body perturbation are tiny and often neglected.

The gravity of mainly the Sun but also of the other planets makes the local inertial frames in the vicinity of the Earth precess slowly with respect to BCRS*. This effect is called De Sitter precession. However since GCRS**is kinematically non-rotating with respect to BCRS*, it is non inertial and the Coriolis acceleration must be taken into account.

The rate of precession is given by [10, section 4.5.3]:

$$\Omega_{DeSitter} = \frac{3}{2c^2} \sum_j \mathbf{v}_{jE} \times \bigtriangledown \left(\frac{\mu_j}{r_{jE}} \right) \tag{30}$$

where the sum is over all bodies but the Earth (only the Sun produces a significant effect though), $\mathbf{v}_{jE}$ is the difference between the Earth and body j barycentric velocities, r_{jE} is the distance between the Earth and body j and $\bigtriangledown$ is the gradient operator.

The Coriolis acceleration of the spacecraft is:

$$\mathbf{a}_{DeSitter} = 2\Omega_{DeSitter} \times \mathbf{V} \tag{31}$$

The precession vector is essentially normal to the Earth orbital plane and almost constant in magnitude due to the low eccentricity of the Earth orbit. Equation 31 integrates to zero on a closed orbit when the precession vector is constant over the orbital period.

The centrifugal acceleration $\Omega_{DeSitter} \times \Omega_{DeSitter} \times \mathbf{R}$ is of order $1/c^4$ and therefore negligible.

The De Sitter acceleration is a relativistic correction to the third body acceleration which is implicit when using the EIH equations in the barycentric system [10, section 4.4.2]. Note that the Lense-Thirring precession [10, sections 4.4.3 and 4.5.4] is an independent effect which is due to the angular momentum of the Earth (or planet) and is not included in the EIH equations.

[20, section 6.8] provides a more complete formulation for relativistic effects in the geocentric system.

2.11. Acceleration due to the gravitational spherical harmonics of the Earth

The motion of the Earth around the Sun generates a relativistic distortion of the gravitational spherical harmonics in the barycentric frame. Because of this, computing accurately the acceleration due to the spherical harmonics in the barycentric system is not straightforward. A method called Relativistic Geopotential Correction (RGC) [2] was introduced to address this problem. This method consists in the following three steps [10, section 4.4.5]:

- Convert BCRS*position vector of spacecraft to GCRS**using a Lorentz transformation [10, equation 4-11].
- Compute acceleration due to spherical harmonics in GCRS**
- Convert this acceleration to the barycentric system using [10, equation 4-55]

This method can be generalized to other planets. Due to the distortion in the barycentric system, it should be more accurate to solve for the spherical harmonics expansion in the planetocentric system.

Additionally care has to be taken when using a spherical harmonic expansion, that the reference radius and mass parameter are given for the correct relativistic scaling of coordinate, otherwise they have to be rescaled.

The RGC is a relativistic correction due to the change of space-time coordinate system, but the accuracy of the formulation for the acceleration due to the geopotential in the geocentric system can also be improved by evaluating relativistic corrective terms. The relativistic correction to the acceleration due to the quadrupole moment (J_2) is given in [21, section 4.2.3].

3. Validation of an implementation of the EIH equations

This section describes the testing that was performed at ESOC to validate the implementation of the EIH equation in the interplanetary orbit determination system. The test method consists in comparing the acceleration computed in the barycentric system with the acceleration computed in the geocentric system. For this purpose, the acceleration in the geocentric system has to converted to the barycentric system using a Lorentz transformation. Hence, it was decided to first validate the implementation of the Lorentz transformation and assess its accuracy.

3.1. Validation of the Lorentz transformation on velocity and acceleration

To validate the velocity transformation, the following steps were performed:

- choose time TDB_1 perigee of Rosetta Earth Swing-By one
- choose small time interval dt and compute $TDB_2 = TDB_1 + dt$
- recompute dt as $dt = TDB_2 - TDB_1$
- from Rosetta orbit, get barycentric position of Rosetta at TDB_1 and TDB_2: $\mathbf{r}_1$ and $\mathbf{r}_2$
- from the ephemeris, get Earth barycentric position at TDB_1 and TDB_2: $\mathbf{r}_1^E$ and $\mathbf{r}_2^E$
- translate Rosetta position vector to Earth center $\mathbf{R}_1 = \mathbf{r}_1 - \mathbf{r}_1^E$ and $\mathbf{R}_2 = \mathbf{r}_2 - \mathbf{r}_2^E$

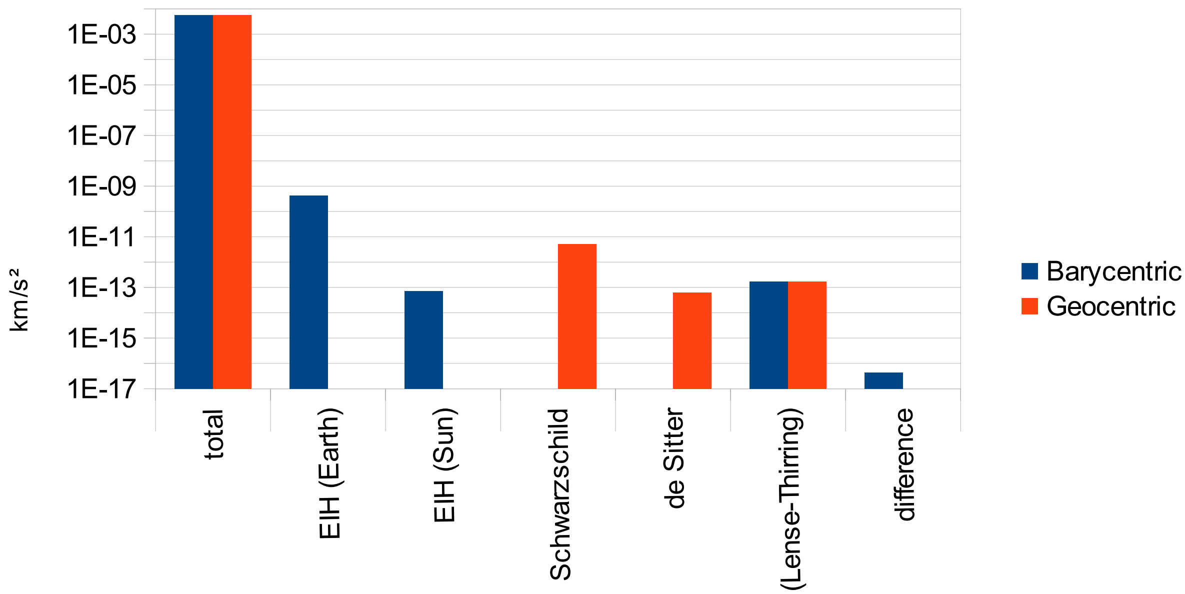

Figure 1. Comparison of acceleration contributions

- Lorentz transform these position vectors from the scaled barycentric system to the scaled geocentric system to get $\tilde{\mathbf{R}}_1$ and $\tilde{\mathbf{R}}_2$
- compute geocentric velocity in the barycentric system $\mathbf{V} = (\mathbf{R}_2 - \mathbf{R}_1)/(TDB_2 - TDB_1)$ at $TDB_3 = (TDB_1 + TDB_2)/2$
- transform geocentric velocity to the geocentric system using the Lorentz transformation of velocity to get $\tilde{\mathbf{V}}$ at TT_3, approximately but not exactly $(TT_2 + TT_1)/2$
- transform $TDB_2 - TDB_1$ to $TT_2 - TT_1$ along the spacecraft world-line. Note that $TT_2 - TT_1$ cannot be computed accurately from TT_2 and TT_1 in double precision, it must be computed as $(TDB_2 - TDB_1) - ((TDB_2 - TT_2) - (TDB_1 - TT_1))$
- compute velocity in geocentric system $\hat{\mathbf{V}} = (\tilde{\mathbf{R}}_2 - \tilde{\mathbf{R}}_1)/(TT_2 - TT_1)$ at $(TT_2 + TT_1)/2$.
- compare $\hat{\mathbf{V}}$ and $\tilde{\mathbf{V}}$

The acceleration transformation was validated in a similar way, replacing position by velocity and velocity by acceleration.

There are three main sources of errors which limits the accuracy to which the validation can be performed:

- Errors in the Lorentz transformation implementation itself. This is the main source of error in our test when using the simplified Lorentz transformation given by equations 4-10, 4-54 and 4-55 in [10]. In that case the difference in velocity is about $3.4 \times 10^{-11} km/s$ and the difference in acceleration $2.3 \times 10^{-14} km/s^2$. More than one order of magnitude can be gained by using Eq. 23, Eq. 24 and Eq. 25

- Error in the computation of the $TT_2 - TT_1$ time interval. If such an error would be by far the main limiting factor, then all components of $(\tilde{\mathbf{V}} - \hat{\mathbf{V}})/\hat{\mathbf{V}}$ (component-wise division) are almost equal to each other. This is not the case in our results.
- Error in computing the reference velocity as an average velocity and not an instantaneous velocity including especially truncation error in position vector differences when dt is tool small. This source of error can be slightly mitigated by adjusting dt. In our test case ten seconds is a good compromise.

To compute $TT_2 - TT_1$, assuming $dt = TDB_2 - TDB_1$ is small, we can adapt the algorithm of [10, section 11.4.2]:

$$TT_2 - TT_1 = dt - \frac{1}{2}(I(TDB_2) + I(TDB_1))dt + \frac{1}{12}\left(\dot{I}(TDB_2) - \dot{I}(TDB_1)\right)dt^2 \quad (32)$$

with:

$$I = \left(1 - \frac{dTT}{dTDB}\right) \quad (33)$$

The quantity $dTT/dTDB$ can be obtained from Eq. 22. The time derivative of I, $\dot{I}$ can be computed by differentiating the equation with respect to coordinate time using Eq. 26. With an initial dt of ten seconds, $TT2 - TT1$ is 26 nanoseconds shorter than $TDB2 - TDB1$.

At the pericenter of Rosetta's first Earth swing-by, the spacecraft is at a geocentric distance of $8332km$ with a geocentric velocity of $10.5km/s$ and an acceleration of $5.7 \times 10^{-3} km/s^2$. In the test, the difference in the barycentric system velocity and the Lorentz-transformed geocentric system velocity is $5 \times 10^{-14} km/s$ and the difference in acceleration is $1.3 \times 10^{-16} km/s^2$.

3.2. Comparison of accelerations computed in the barycentric and geocentric system

The test procedure is:

- get Rosetta barycentric state at pericenter of Earth Swing-By one
- get barycentric states of 11 celestial bodies (Sun, 8 planets, Moon, Pluto) from DE405 ephemeris at that time
- compute with Eq. 28 Newtonian and relativistic accelerations on the spacecraft and on the geocenter
- compute the difference between spacecraft and geocenter accelerations to get acceleration A_B of the spacecraft in the geocentric frame, but in the barycentric system
- translate position and velocity of spacecraft to geocenter and perform Lorentz transformation of position and velocity
- compute direct and indirect Newtonian acceleration due to all bodies but Earth on spacecraft (see section 2.10.)
- rescale mass parameter of Earth to the geocentric system

- compute direct Newtonian and relativistic acceleration due to Earth on spacecraft in geocentric space-time using Eq. 29
- compute Coriolis acceleration due to De Sitter precession in the geocentric system using Eq. 30 and Eq. 31 but taking only into account the Sun contribution to the precession
- add all contribution to acceleration in geocentric system to get A_G
- convert A_G to $\hat{A}_B$ with Lorentz transformation of acceleration and compare to A_B

The test results are displayed in Fig. 1. The Lense-Thirring acceleration is just shown for comparison and was not included in the total. The acceleration labeled "EIH (Earth)" consists of the corrective term in Eq. 28 for the spacecraft body where index j is the Earth. The acceleration labeled "EIH (Sun)" consists of the difference of the correctives terms where index j is the Sun between the equation for the spacecraft and the equation for the Earth. The relativistic effects of the Sun are very roughly similar in both direction (not shown on the diagram) and magnitude between the geocentric system (De Sitter) and the barycentric system ("EIH (Sun)"). The relativistic correction due to the Earth are orders of magnitude larger in the barycentric system. The uncorrected total difference $A_G - A_B$ between the two systems (not shown in the figure) is $1.9 \times 10^{-10} km/s^2$ however the Lorentz transformation accounts for most of it. The corrected difference $A_G - \hat{A}_B$ (last bar of the histogram) is about $4 \times 10^{-17} km/s^2$.

4. Relativistic acceleration of a Mercury orbiter

4.1. BepiColombo orbit

Figure 2 shows, according to a reference trajectory for the launch on December 31 2015, the orbit of the Mercury Planetary Orbiter (MPO) of the BepiColombo mission on 2023 June 21. On that date, the Earth direction is almost contained in the orbital plane. This is desired for our analysis since in these conditions the effects of mismodeling on the range and range-rate residuals is maximized. On Fig. 2, the orbit is drawn in the orbital plane which almost contains the Mercury polar axis. The dashed part of the orbit represents the region where MPO is occulted by Mercury as seen from Earth. The orbital parameters are given in Tab. 1. On June 21 2023, Mercury is at -39 degrees true anomaly and its distance to the Sun is about 0.32 AU.

Table 1. MPO orbital parameters

epoch	June 21, 2023, 00:00:00 TDB
semi-major axis	3393.901 km
eccentricity	0.165003
inclination	90.097 deg
argument of ascending node	67.728 deg
argument of pericenter	4.849 deg
true anomaly	120.782 deg
pericenter distance	2833.896 km
apocenter distance	3953.906 km
orbital period	2.325 hours

The spacecraft is three-axis stabilized and it is expected that about two momentum desaturation

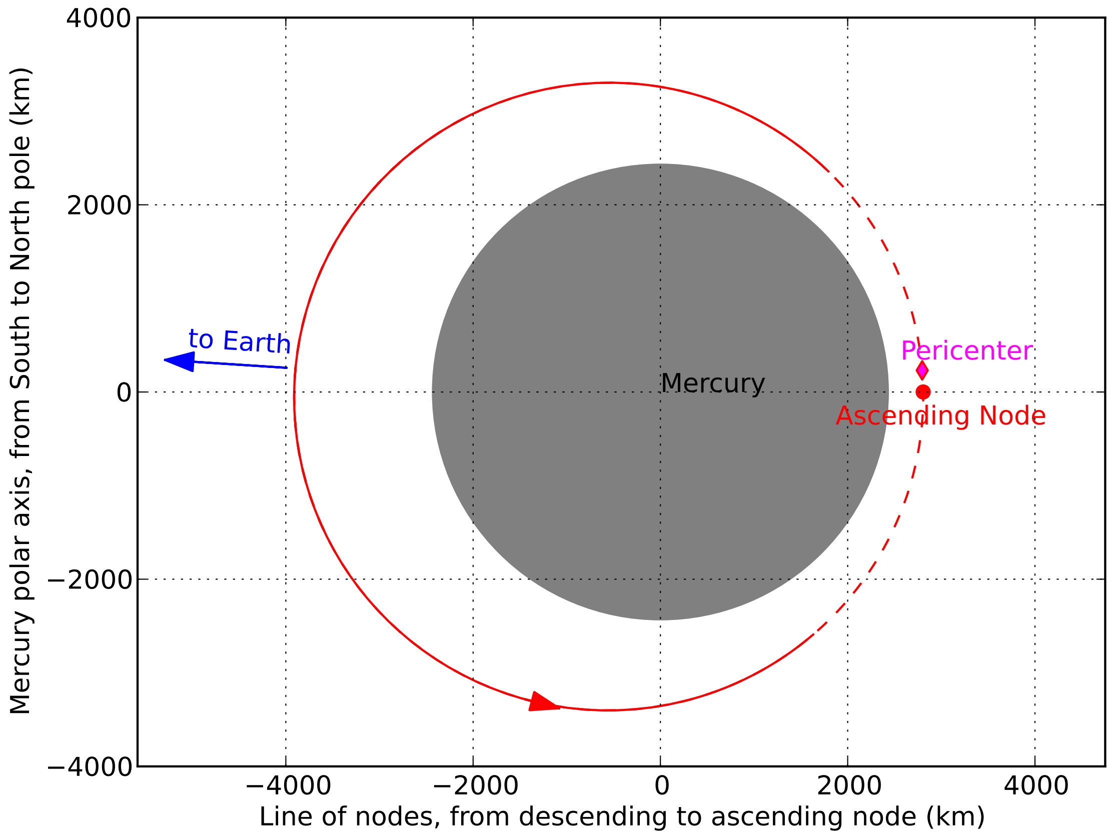

Figure 2. MPO orbit on 2023 June 21

maneuvers per 24 hours will performed. Desaturation maneuvers introduce uncertainty in the orbit determination process and the additional degrees of freedom provided by estimating scale factors to the maneuver acceleration may absorb part of the error introduced by mismodeling the relativistic acceleration over long arcs. Therefore in the following analysis we are only considering a propagation arc of twelve hours.

Navigation will be performed at ESOC. The ESOC interplanetary orbit determination software propagates the orbit in the barycentric system. A space-time transformation of the ground station events from the geocentric to the barycentric system is required to model radiometric observables. The radio-science team will perform the orbit propagation in a mercury-centric relativistic system, so that additionally a mercury-centric to barycentric space-time transformation will be required in modeling the observables.

The goal of the analysis presented in the next sections and performed by the ESOC navigation team is to compare the two formulations and to assess to which accuracy the results in the two systems can be matched with our presently implemented models for relativistic correction to the acceleration and space-time transformations.

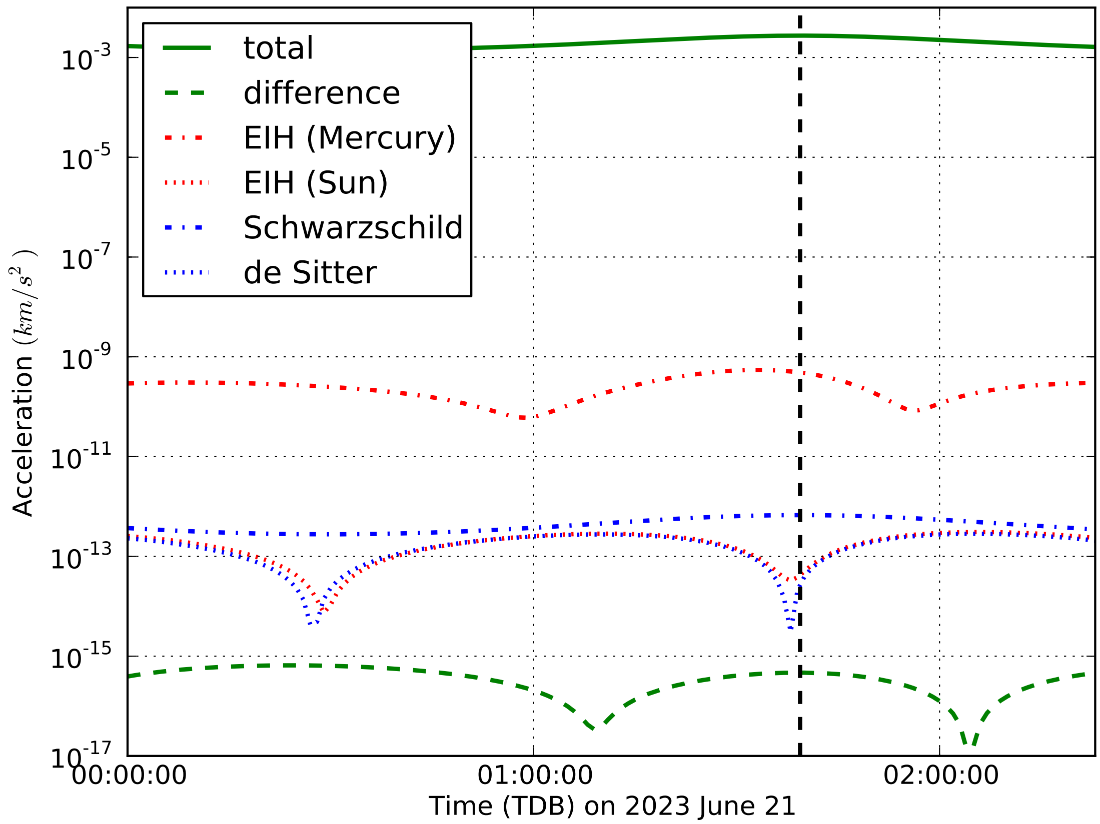

Figure 3. Comparison of acceleration contributions

4.2. A Mercury-centric relativistic system

In a similar way to how the GCRS is defined, we define the Mercury-centric Reference System (MRS) with coordinate time TCM (Mercury Coordinate Time). We also introduce a scaled coordinate system MRS$'$with scale factor L_M and coordinate time TDM (Mercury Dynamic Time). All formulas in section 2. for GCRS**and the transformation between BCRS*and GCRS**are valid for MRS$'$and the transformation between BCRS*and MRS$'$provided that the quantities for Earth are replaced by quantities for Mercury and that L_G is replaced by L_M. We choose $L_M = L_B$ so that $\tilde{L} = L_B - L_M = 0$. With this scale, the mass parameters of the celestial bodies are the same in BCRS*and MRS$'$.

Assuming a purely Keplerian motion for Mercury around the Sun with the Sun at the barycenter, Eq. 22 for the ratio of TDM to TDB along the world-line of the center of Mercury can be rewritten:

$$\frac{dTDM}{dTDB} = 1 - \frac{U_M}{c^2} - \frac{v_M^2}{2c^2} = 1 - \frac{\mu}{2ac^2(1-e^2)}\left(3 + e^2 + 4e\cos(\theta)\right) \tag{34}$$

where μ is the mass parameter of the Sun, $U_M = \mu/r$, r being the distance Mercury to Sun, v_M the

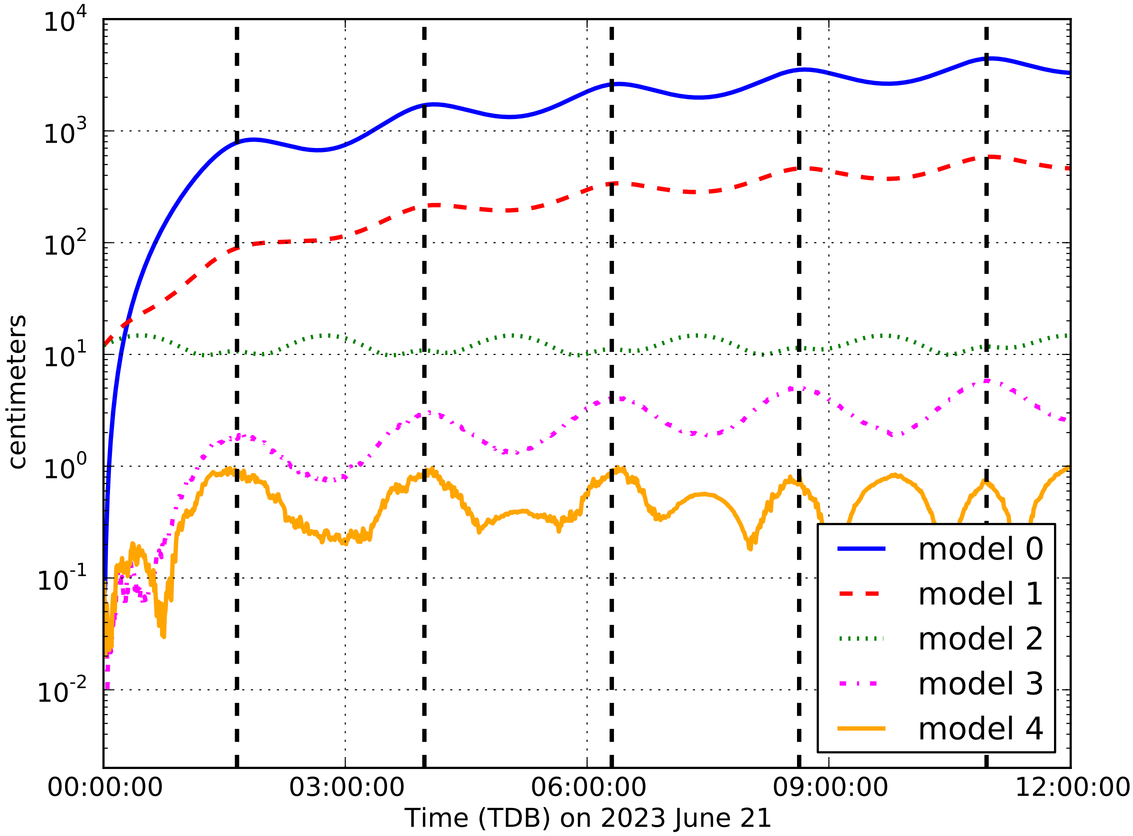

Figure 4. Position differences

heliocentric velocity of Mercury, a, e and θ are respectively the semi-major axis, the eccentricity and the true anomaly of Mercury's orbit around the Sun.

To order one in eccentricity, this becomes:

$$\frac{dTDM}{dTDB} = 1 - \frac{\mu}{2ac^2}\left(3 + 4e\cos(\eta(TDB - TDB_{peri}))\right) \tag{35}$$

η being the mean motion and TDB_{peri} a time where Mercury is at pericenter. This integrates to:

$$TDM = C + \left(1 - \frac{3\mu}{2ac^2}\right)TDB - \frac{2e\mu}{ac^2\eta}\sin(\eta(TDB - TDB_{peri})) \tag{36}$$

where C is a constant. Thus with our choice of scaling factor, TDM drifts with respect to TDB at a secular rate of approximatively $-3\mu/(2ac^2)$ that is 3.9×10^{-8}. TDM is losing 1.2 seconds per year with respect to TDB. The periodic term with the frequency of Mercury heliocentric orbit has an amplitude of approximatively $\frac{2e\mu}{ac^2\eta}$ that is about twelve milliseconds, much bigger than the

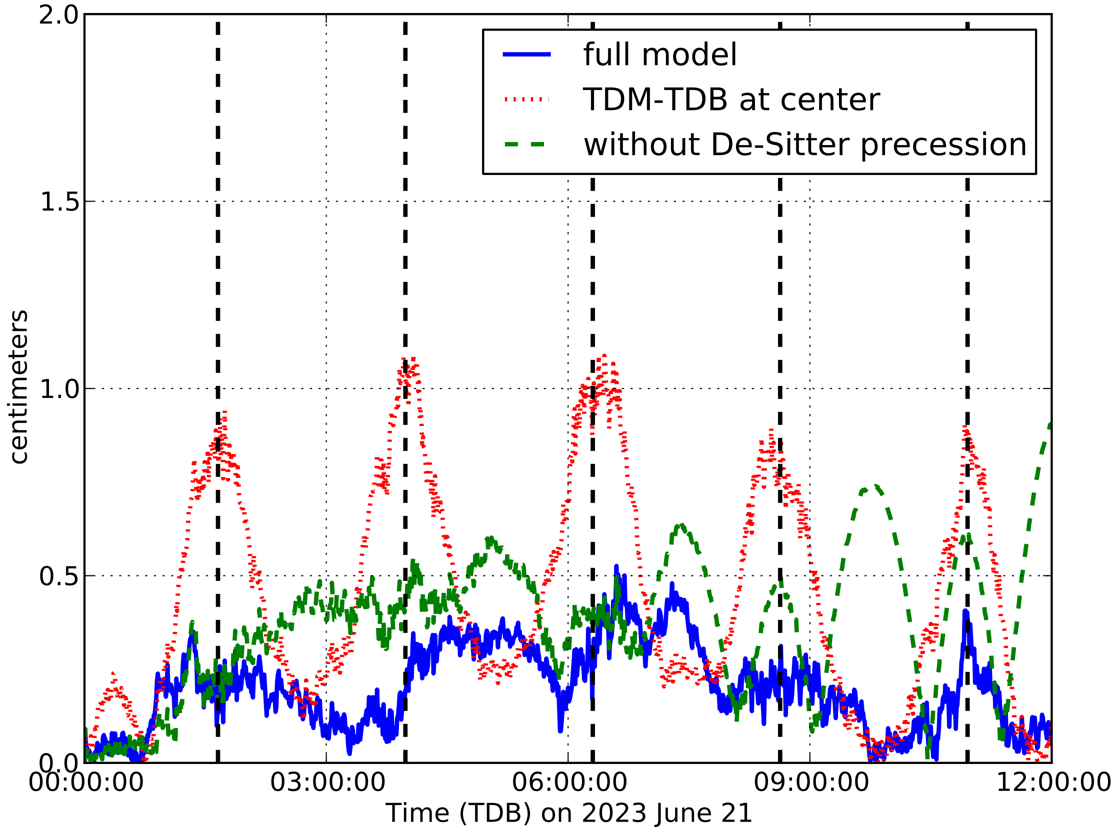

Figure 5. Position differences

main amplitude in the difference between TDB and TT. Numerically integrating Eq. 22 for both the orbiter and the center of Mercury and taking the difference, we get for the time period from 00:00 to 24:00 on 2023 June 21, a periodic signal with the period of the spacecraft orbit and an amplitude of 1.3 microsecond plus an offset of 0.6 microsecond. For the following analysis, we choose C so that $TDB = TDM$ on 2023 June 21 at 00:00.

4.3. Comparing the acceleration of the orbiter in the two systems

As we did in section 3. for Rosetta at pericenter of an Earth swing-by, we want to compare the acceleration of the MPO in the barycentric system with its acceleration in the Mercury-centric system. First we want to repeat the test performed in section 3.1. to assess the accuracy of the Lorentz transformation. The state of the MPO is taken on June 21, 2023 at a time close to its pericenter. The Mercury-centric velocity is about $3 km/s$ and the acceleration $2.7 \times 10^{-3} km/s^2$. The error in the validation of the transformation is $2.1 \times 10^{-14} km/s$ and $1.2 \times 10^{-16} km/s^2$.

Following the validation strategy presented in section 3.2., we compute the accelerations contribution in both systems, using an a-priori ephemeris for the position and velocity of the orbiter in BCRS*, every minute during a full orbit of the MPO on June 21 2023. The results are shown in Fig. 3. In this

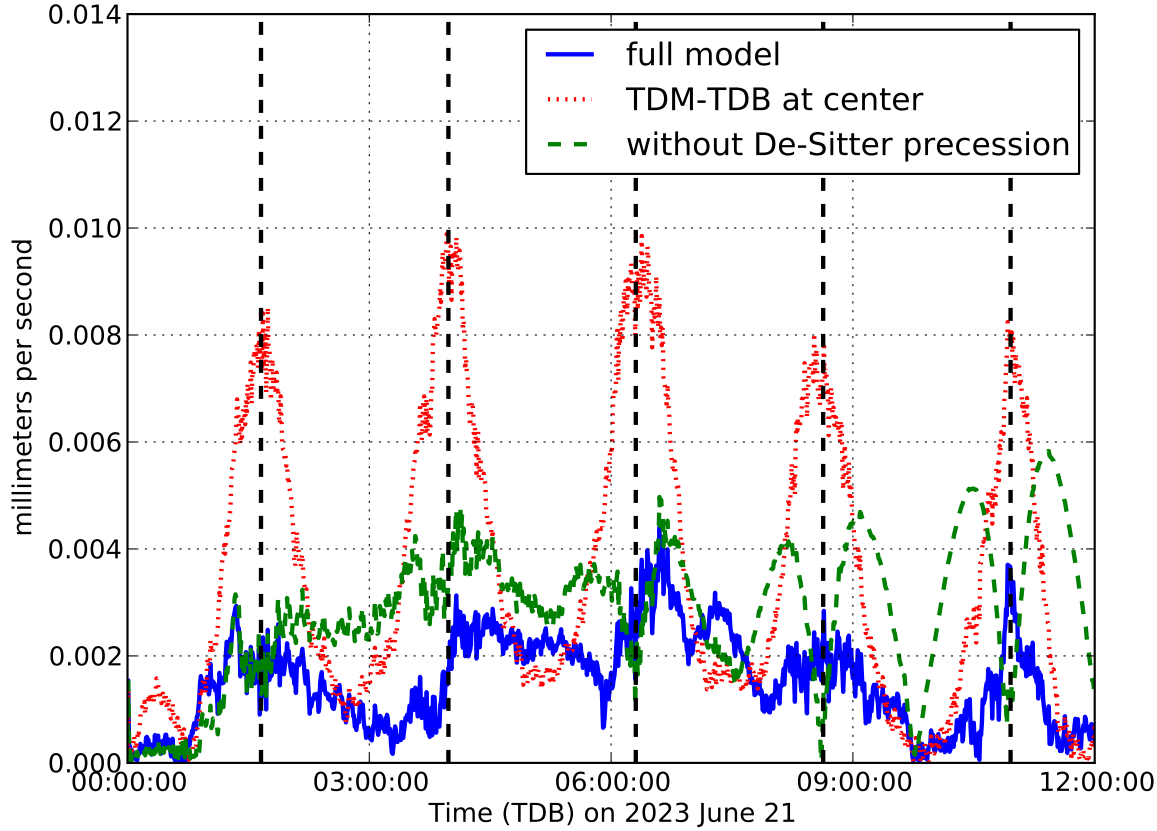

Figure 6. Velocity differences

plot, like in all subsequent plots, vertical dashed black lines denote pericenter times. The De Sitter acceleration is sweeping a large range of magnitude because the angle between the orbital plane and the precession vector is small. The relativistic effects of the Sun are very roughly similar in both direction (not shown in the plot) and magnitude between the Mercury-centric system (De Sitter) and the barycentric system ("EIH (Sun)"). The relativistic correction due to the Mercury are orders of magnitude larger in the barycentric system. The Lorentz-corrected difference in acceleration at pericenter is about $4.7 \times 10^{-16} km/s^2$. In the plot, the difference between the two models is higher around apocenter and pericenter which is when the De Sitter acceleration is smaller. The De Sitter precession has been computed using the Sun contribution only, however the EIH equations in the barycentric system is also including correction to the third-body acceleration due to the planets.

4.4. Propagation comparison

We propagate the orbit of the MPO relative to Mercury in the barycentric system with relativistic correction using the EIH equations. The resulting orbit is the reference to which all other propagations will be compared. The following propagations are performed in the Mercury-centric system:

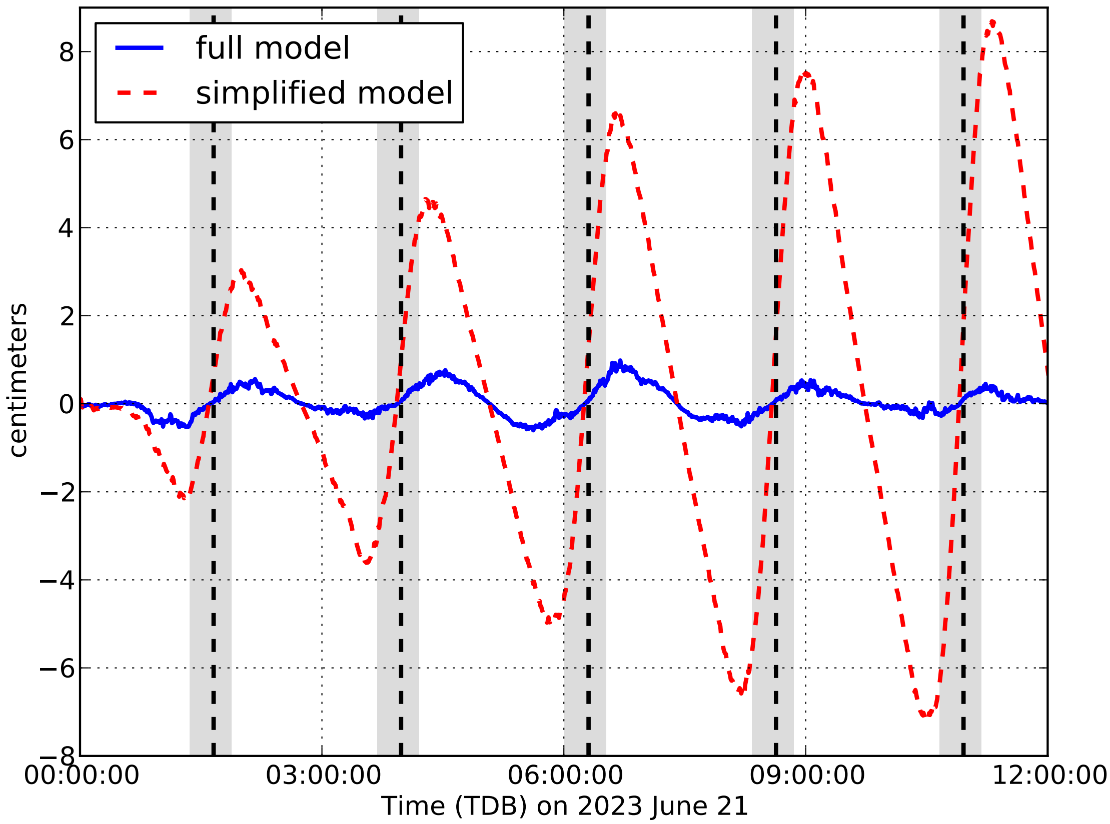

Figure 7. Two-way range residuals

Model 0 No relativistic acceleration are considered and no space-time transformation are performed (this model is similar to performing Newtonian only propagation in the barycentric system).

Model 1 Initial state (position and velocity) is Lorentz transformed in Mercury-centric system before starting propagation and third body ephemeris are accessed at a time corrected for the difference between TDB and the independent variable of integration TDM.

Model 2 As model 1 but the resulting orbit times are transformed from TDM to TDB before comparison.

Model 3 As model 2 but the resulting orbit states are Lorentz-transformed from MRS′to BCRS*before comparison.

Model 4 As model 3 plus Schwarzschild acceleration.

The difference between TDM and TDB is propagated along with the state of the spacecraft. Initially the differential equation for TDM-TDB is simplified to consider only the Mercury-center world-line.

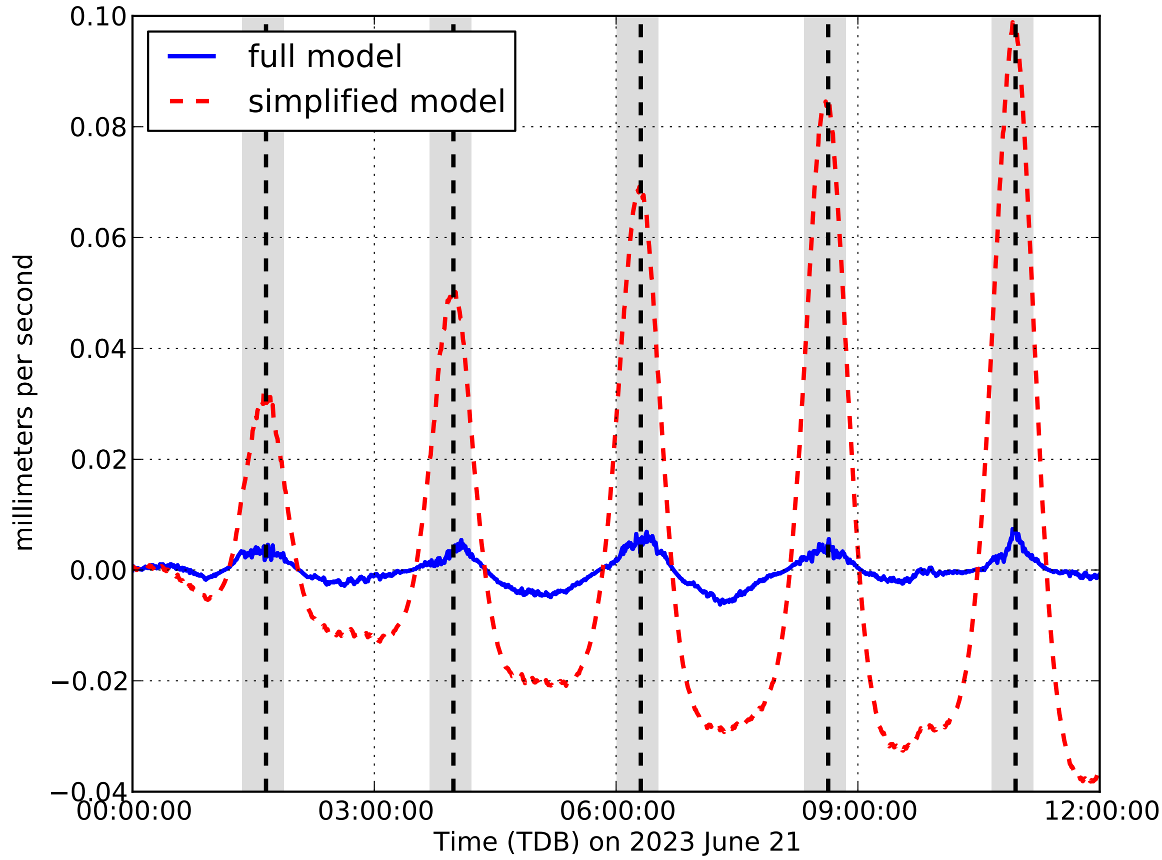

Figure 8. Two-way range-rate residuals

The comparison results are shown in Fig. 4. The adjustment of the orbit times from TDM to TDB (model 2) produces a significant improvement, but there is a residual periodic error which can be removed by Lorentz transforming the states to the barycentric system (model 3). The introduction of the Schwarzschild acceleration (model 4) allows to match the orbits at the centimeter level.

Figures 5 and 6 show the comparison results with the same reference orbit for a propagation including all corrections available to us and two other propagations in which only one correction was forgotten. The propagation labeled "full model" is similar to model 4 but includes additionally the De Sitter acceleration and the two terms due to the spacecraft in Eq. 22 are considered in the integration of $TDB - TT$. The propagation labeled as "TDM-TDB at center" is based on the differential equation for TDM at the Mercury center.

4.5. Range and range-rate residuals

Figures 7 and 8 show respectively the two-way range and range-rate residuals in the sense "reality minus modeled" assuming the reality is represented by the orbit propagated in the barycentric system with the EIH equations. The propagation labeled "simplified model" is not taking into account the Schwarzschild and De Sitter acceleration and is using the formulation for $TDM - TDB$ at the Mercury center not along the spacecraft orbit. Range and range-rate are computed here simply as geometric instantaneous quantities at a given time. The vertical grey stripes on the plots denote times when the spacecraft is occulted.

5. Conclusion

The numerical comparison that was performed between the Mercury-centric system and the barycentric system has shown that the resulting orbit states and observables can be made to match very accurately provided that the correct space-time transformations and relativistic corrections to the observables are applied. In the barycentric system, the relativistic correction to the acceleration should not be ignored. In the Mercury-centric system, the relativistic corrections to the acceleration are not so significant, but the space-time transformations to convert from and to the barycentric system are not to be neglected. For our test scenario, the difference in the propagation in the two systems with relativistic correction and after space-time transformations from one system to the other is 5 millimeters, 5 micrometers per second after twelve hours. Disabling all relativistic corrections results in a difference of about 40 meters, 4 centimeters per second.

The Mercury-centric system has the following advantages:

- More accurate treatment of the spherical harmonics expansion
- Simpler formulation for the relativistic corrections to the acceleration

It has however the following disadvantages:

- Requires space-time transformation of the orbit to the barycentric system for the purpose of observable modeling
- Third body acceleration computation not straightforward. However a simple adjustment of the time argument of the ephemeris of the third body makes this acceleration computation already far more precise than what is required for the achievable navigation accuracy

The main disadvantage of the barycentric system is the inaccurate modeling of the acceleration due to the spherical harmonics, but this problem has already been solved to a very satisfying accuracy [2] as discussed in section 2.11..

6. References

[1] S. Zhu, E. Groten, R. Pan, H. Yan, Z. Cheng, W. Zhu, C. Huang, M. Yao. “Motion of Satellites - The Choice of Reference Frames.” “The Few Body Problem, IAU Colloquium 96,” 1988. Also available online: `arXiv:0907.5100v2 [astro-ph.IM]`.

[2] C. Huang, J. Ries, B. Tapley, M. Watkins. “Relativistic Effects for Near-Earth Satellite Orbit Determination.” Celestial Mechanics and Dynamical Astronomy 48, 1990. Also available online: `http://articles.adsabs.harvard.edu/full/1990CeMDA..48..167H`.

[3] E.M. Alessi; S. Cicalò,; A. Milani. “Accelerometer Data Handling for the BepiColombo Orbit

Determination." 1st IAA Conference on Dynamics and Control of Space Systems. 2012.

[4] Andrea Milani and Giovanni F. Gronchi. Theory of Orbit Determination. Cambridge University Press, 2010.

[5] A. Milani, G. Tommei, D. Vokrouhlický, E. Latorre, S. Cicalò. "Relativistic models for the BepiColombo radioscience experiment." "Relativity in Fundamental Astronomy: Dynamics, Reference Frames and Data Analysis, IAU Symposium 261," 2010.

[6] Andrea Milani. "Relativistic reference frames for radio science experiments with a Mercury orbiter." `http://adams.dm.unipi.it/orbdet/relframemerc.pdf`, 2010.

[7] G. Tommei, A. Milani; D. Vokrouhlick. "Light-time computations for the BepiColombo Radio Science Experiment." Celestial Mechanics and Dynamical Astronomy, Vol. 107, No. 1-2, 2010. Also available online: `arXiv:0910.5818v2 [gr-qc]`.

[8] R.W. Hellings. "Relativistic Effects in Astronomical Timing Measurements." The Astronomical Journal, Vol. 91, No. 3, 1986. Also available online: `http://articles.adsabs.harvard.edu/full/1986AJ.....91..650H`.

[9] Neil Ashby and Bruno Bertotti. "Accurate light-time correction due to a gravitating mass." Classical and Quantum Gravity, Vol. 27, No. 14, 2010. Also available online: `arXiv:0912.2705 [gr-qc]`.

[10] T.D. Moyer. "Formulation for Observer and Computed Values of Deep Space Network (DSN) Data Types." Tech. rep., Jet Propulsion Laboratory (JPL), 2000. `http://descanso.jpl.nasa.gov/Monograph/series2/Descanso2_all.pdf`.

[11] "Wikipedia entry on Einstein notation." `http://en.wikipedia.org/wiki/Einstein_notation`.

[12] "IAU 2000 Resolutions." `http://www.iau.org/static/resolutions/IAU2000_French.pdf`.

[13] "Relativistic Glossary for Astronomers." `http://astro.geo.tu-dresden.de/RIFA/glossary.html`.

[14] "IAU 2006 Resolution B2." `http://www.iau.org/static/resolutions/IAU2006_Resol2.pdf`.

[15] "IAU 2006 Resolution B3." `http://www.iau.org/static/resolutions/IAU2006_Resol3.pdf`.

[16] S. Klioner. "Relativistic Scaling of Astronomical Quantities and the System of Astronomical Units." Astronomy and Astrophysics 478, 2008. Also available online: `arXiv:astro-ph/0508292v3`.

[17] S.A. Klioner, N. Capitaine, W. Folner, B. Guinot, T.Y. Huang, S. Kopeikin, G. Petit, E. Pitjeva, P.K. Seidelman, M. Soffel. "Units of Relativistic Time Scales and Associated Quantities." "Relativity in Fundamental Astronomy: Dynamics, Reference Frames and Data Analysis, IAU Symposium 261," 2010. Also available online: `arXiv:0907.5100v2 [astro-ph.IM]`.

[18] A. Fienga, J. Laskar, T. Morley, H. Manche, P. Kuchynka, C. Le Poncin-Lafitte, F. Budnik, M. Gastineau, L. Somenzi. "INPOP08, a 4-D planetary ephemeris: from asteroid and time-scale computation to ESA Mars Express and Venus Express contributions." Astronomy and Astrophysics 507, 2009. Also available online: `arXiv:0906.2860v2 [astro-ph.EP]`.

[19] C.W. Misner, K.S. Thorne, J.A. Wheeler. Gravitation. W. H. Freeman, 1973.

[20] ANSI/AIAA. "Astrodynamics - Propagation Specifications, Technical Definition and Recommended Practices." Tech. rep., ANSI/AIAA, 2010. `http://www.astrodynamicstandards.com/Resources/ANSI_AIAA_S-131-2010.pdf`.

[21] Michael H. Soffel. Relativity in Astrometry, Celestial Mechanics and Geodesy. Astronomy and Astrophysics Library, Springer Verlag, 1989.