PROBABILISTIC ADMISSIBILITY IN ANGLES-ONLY INITIAL ORBIT DETERMINATION

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Abstract: *A probabilistic admissible-region analysis is proposed for track initiation using angles-only measurements. Given three observations and known sensor statistics, the unscented transform (UT) is used to map the uncertainty in the measurement space to uncertainty in the orbital coordinates space using Gauss' method. For a given region in orbital space, the probability that the computed uncertain candidate orbit is admissible is then defined. This definition provides a probabilistically-rigorous approach to assessing the degree to which the initiated track is admissible. However, the UT approach to track initiation assumes that the probability density functions in the measurement and orbital spaces are Gaussian. Therefore, a Monte Carlo (MC) analysis is used to map the uncertainty cloud in the measurement space to the full orbital space. This cloud is then used in order to visualize the true uncertainty cloud. Finally, the results from the probabilistic admissibility analysis are compared to those of the deterministic analysis, and both against the MC simulation. The results show that, while in some cases a deterministic analysis may result in rejecting (respectively, accepting) the hypothesis that the track is admissible, the computed probability of admissibility may be relatively high (respectively, low), indicating that the deterministic decision may be erroneous.*

Keywords: *Initial Orbit Determination, Probabilistic Analysis, Uncertainty Mapping, Unscented Transform, Monte Carlo Method*

1. Introduction

As new optical sensors come online and more and more optical observations become available for space objects previously too small or too far away to detect, the space surveillance community is presented with the computationally challenging problem of generating initial orbit solutions and determining association and admissibility for a large number of short-arc line-of-sight (angles-only) observations. Track association refers to the concept of determining whether or not a given subset of the angles-only observations was generated by the same space object. Admissibility refers to the concept of determining whether or not an uncertain candidate orbit lies within a specified subspace, referred to as the constrained admissible region (CAR) [1–16], of the space of all possible orbits. In this paper, we deal primarily with the latter problem and apply probabilistic techniques to rigorously determining the admissibility of uncertain candidate orbits.

The admissible region is a concept first introduced by Milani et al. [1, 2] to deal with the problem of identifying asteroids based on very short arc observations. Specifically, they referred to a region in the plane of possible ranges and range-rates defining those values for which a given line-of-sight observation produces an orbit solution that satisfies certain criteria. This concept has been extended by the space situational awareness (SSA) community [3–11, 13, 14, 16] to deal with the problem of tracking space objects in Earth orbit, for which the CAR refers to a region in the range, range-rate plane which produces orbit solutions with orbit elements satisfying some specified bounds. In previous work, Schumacher, Wilkins, and Roscoe [12, 15] extended this concept to include regions in the range, range plane satisfying orbit element bounds for pairs of observations. In this paper, we refer to the CAR not specifically in terms of range, range-rate or range, range, but more generally as a subspace of all possible orbit solutions for a given set of observations, independent of the coordinate system in which they are specified (similar to the projection concept described in [4] and [10]).

In order to perform any kind of probability-based analysis with these orbit solutions, we require an accurate representation of their uncertainty. Properly characterizing the uncertainty will allow us to more efficiently deal with large sets of sparse data by enabling the use of rigorous probabilistic techniques to, for example, asses probability of admissibility, perform data association, determine collision probabilities, or initialize a Bayesian estimation scheme. Unlike short-arc line-of-sight observations, optical observations of actively tracked space objects will contain long arcs of data, for which uncertainty is usually assumed to be Gaussian and the initial covariance is obtained from the error statistics of the differential correction algorithm used to fit the data. This paper builds on recent work on uncertainty propagation in initial orbit determination (IOD) and presents a probabilistic methodology for assessing the probability of admissibility, which contrasts with deterministic approaches that provide binary admissible/inadmissible type answers.

The subject of IOD dates back to the time of Gauss and Lambert [17] and has more recently been revisited by Gooding [18, 19], Karimi and Mortari [20], and others [21]. Uncertainty propagation has been investigated in detail in SSA research, particularly in how it applies to collision probability computation and Bayesian estimation. Junkins, Akella, and Alfriend [22] studied the general problem of nonlinear error propagation in orbital mechanics and showed that the choice of coordinates has a significant impact on how fast errors become non-Gaussian. Fujimoto, Scheeres, and Alfriend [23] developed analytical techniques to propagate uncertainty in the two-body problem using the concept of state transition tensors. Horwood and Poore [24] discussed the use of the Gauss von Mises distribution to better capture the evolution of the orbit uncertainty in angular coordinates. Several authors [25–27] have investigated the use of Gaussian mixture models for uncertainty propagation and Bayesian estimation. However, there has been very little attention paid in any of this research to the determination of prior uncertainty to initialize these methods.

Characterization of the IOD uncertainty from sparse optical data requires relating astrometric measurement errors into a probability distribution in the orbit state space. DeMars et al. [28] determined first-order state uncertainties by approximating mean and covariance through the unscented transform (UT) of the measurement uncertainties. Garber [29] studied the impact of different measurement statistics and model assumptions on uncertainty and future probability computations for collision prediction. In pursuing this investigation for space surveillance, we take note of the similarities between this problem and the asteroid tracking problem investigated by Muinonen and Bowell [30], Virtanen, Muinonen, and Bowell [31] and authors such as Milani, Valsecchia, La Spina, Sansaturio, and Chesley [32–35]. Carpino, Milani, and Chesley [36] studied error statistics of optical observations of asteroids, specifically, and Ford [37] analyzed uncertainty in the orbits of extrasolar planets using Monte Carlo (MC) techniques.

Directly related to the present paper is the work of Weisman and Jah [38] and Binz and Healy [39, 40]. In [38], the authors apply a transformation of variables technique to map measurement space uncertainty into the angle-rate, angle acceleration, range and range-rate spaces using the system dynamics. In [39, 40], the authors use Gauss' angles-only method combined with the UT to empirically obtain the probability density function (pdf) in the orbital space. Unlike the purely probabilistic approach we pursue in the present paper, Binz and Healy employ the UT by mixing deterministic IOD criteria within a probabilistic framework.

In this paper we pursue a consistent and probabilistically rigorous approach to obtaining a statistical characterization of the uncertainty of the IOD (using the UT as well as the MC method) and assessing the probability of admissibility of the uncertain candidate orbit. The paper is organized as follows: in Section 2. we first state the general angles-only IOD problem and summarize the Gauss' method. In Section 2.1. we describe how uncertainty can be mapped from the measurement space to orbital space using the UT and MC methods. This then allows us to define the probability of admissibility that we discuss in Section 2.2. Finally, in Section 3. we demonstrate the main result of the paper using a numerically simulated example.

2. Probabilistic Angles-Only Initial Orbit Determination

2.1. Uncertainty Mapping from Measurement Space to Orbital Space

Let $\mathbf{z}_i = (\alpha_i, \delta_i)$ be the set of right ascension and declination measurements at time instant t_i. Given three distinct measurements $\mathbf{z} = (\mathbf{z}_1, \mathbf{z}_2, \mathbf{z}_3)$ taken at t_1, t_2, and t_3, Gauss' method produces a candidate orbit described by the six-dimensional state $\mathbf{x} = \mathbf{G}(\mathbf{z})$, where $\mathbf{G}(\cdot)$ is the function that maps a set of three angles-only measurements to orbital space according to the Gauss IOD solution method. For the details of Gauss' method, see, for example, [17] or [41]. The state may be specified in orbital elements, position-velocity coordinates, etc. Given bounds on the orbital coordinates, the CAR is constructed. These bounds may come from physical constraints such as the exclusion of orbits that intersect the Earth's surface, or geometric constraints imposed by the analyst on semi-major axis, eccentricity or inclination [1–3, 5–8, 12, 15, 16]. If a reconstructed orbit does not belong to the CAR, the orbit is considered inadmissible.

In the proposed probabilistic approach, we use the UT [42] to map the statistics of the measurements, described by the pdf $f_M(\mathbf{z})$, to statistics of the candidate orbit from the IOD solution, described by the pdf $\tilde{f}_O(\mathbf{x})$. The pdf $\tilde{f}_O(\mathbf{x})$ is then a UT Gaussian approximation of the unknown true non-Gaussian pdf denoted by $f_O(\mathbf{x})$. The measurement statistics are assumed to be Gaussian with mean $\boldsymbol{\mu}_\mathbf{z} = \mathbf{z}$ and covariance (assuming that the three measurements are statistically independent)

$$\mathbf{P}_\mathbf{z} = \begin{bmatrix} \mathbf{P}_\mathbf{z}^1 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{P}_\mathbf{z}^2 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{P}_\mathbf{z}^3 \end{bmatrix}, \tag{1}$$

where $\mathbf{P}_\mathbf{z}^i$ is the 2×2 measurement covariance matrix at each time t_i. In other words, $f_M(\mathbf{z}) = f_g(\mathbf{z}; \boldsymbol{\mu}_\mathbf{z}, \mathbf{P}_\mathbf{z})$. These measurement statistics are used to generate a set of sigma points $\mathcal{Z}_j$, $j =$

$1,\ldots,13$, in the measurement space (there are $13 = 2\times 6+1$ sigma points since the measurement space has a dimension of 6). Each one of these measurement sigma points is then fed into the Gauss IOD algorithm to produce a set of 13 sigma points in the orbital space:

$$\mathcal{X}_j = \mathbf{G}(\mathcal{Z}_j),\ j = 1,\ldots,13. \tag{2}$$

These sigma points can then be used to obtain the transformed mean and covariance using the UT

$$\boldsymbol{\mu}_{\mathbf{x}} = \sum_{j=1}^{13} W_s^j \mathcal{X}_j \tag{3}$$

and

$$\mathbf{P}_{\mathbf{x}} = \sum_{j=1}^{13} W_c^j (\mathcal{X}_j - \boldsymbol{\mu}_{\mathbf{x}})(\mathcal{X}_j - \boldsymbol{\mu}_{\mathbf{x}})^{\mathrm{T}}, \tag{4}$$

where W_s^j and $W_c^j,\ j = 1,\ldots,13$, are the UT's sigma point and covariance weights, respectively. The resulting pdf is Gaussian with mean $\boldsymbol{\mu}_{\mathbf{x}}$ and covariance $\mathbf{P}_{\mathbf{x}}$: $\tilde{f}_O(\mathbf{x}) = f_g(\mathbf{x};\boldsymbol{\mu}_{\mathbf{x}},\mathbf{P}_{\mathbf{x}})$.

To visualize the true transformed distribution we perform a MC analysis. As before, we assume that the measurement process is Gaussian with distribution $f_M(\mathbf{z}) = f_g(\mathbf{z};\boldsymbol{\mu}_{\mathbf{z}},\mathbf{P}_{\mathbf{z}})$. We generate m samples $\mathbf{Z}_j,\ j = 1,\ldots,m$, from f_M and obtain a set of mapped points $\mathbf{X}_j = \mathbf{G}(\mathbf{Z}_j),\ j = 1,\ldots,m$, in the orbital space. These points can then be used to visualize the true transformed distribution $f_O(\mathbf{x})$.

Remark. We wish to highlight the main difference between the UT procedure we propose in this paper and the one developed by Binz and Healy in [39,40]. The approach proposed in this paper is a rigorous implementation of the UT in the following sense. The nonlinear mapping $\mathbf{G}(\cdot)$ is a map from the entire three-measurement space to the orbital space, and is *not* a map of an individual measurement $\mathbf{z}_i$, $i = 1,2,3$, to the orbital space. Hence, the sigma points should be drawn from the distribution in the six-dimensional measurement space defined with the global measurement variable $\mathbf{z}$ and not from the individual distributions defined on the individual measurement variables $\mathbf{z}_i$, $i = 1,2,3$. The result is a pdf $\tilde{f}_O(\mathbf{x})$ with a mean and covariance that are computed using the UT's standard formulation. Furthermore, determining whether or not an uncertain IOD solution is admissible should be based on probabilistic criteria that result in an assessment of the degree of admissibility of the uncertain IOD solution. Such a probabilistic measure, the probability of admissibility, will be discussed in the next section. In [39,40], on the other hand, each individual measurement $\mathbf{z}_i$ at time t_i, $i = 1,2,3$, is used to generate a set of 5 sigma points ($5 = 2\times 2+1$ due to the fact that the measurement space is two-dimensional) associated with the measurement. For each combination of sigma points, one from each of the 3 sets of sigma points, the IOD problem is solved. This results in $5\times 5\times 5 = 125$ combinations. Combinations that result in inadmissible solutions are discarded. The remaining admissible ones are averaged using a re-weighted version of the sigma point weights. The final mean value of the IOD solutions is then tested for admissibility. The rationale for the overall mixed probabilistic/deterministic approach seems to be a bit *ad hoc*.

2.2. Probability of Admissibility

Given $f_O(\mathbf{x})$, the probability of admissibility, p_A, over a given CAR Ω is defined as

$$p_A = \int_\Omega f_O(\mathbf{x})\mathrm{d}\mathbf{x}. \tag{5}$$

Even for a Gaussian pdf $f_O(\mathbf{x})$ with a hyper-cubic CAR Ω, the above integral does not have an analytic expression (except for the trivial one-dimensional case). Hence, one has to resort to numerical techniques to obtain p_A. In this paper we use the m orbital space points $\mathbf{X}_j$ to approximate the integral:

$$p_A = \int_\Omega f_O(\mathbf{x})\mathrm{d}\mathbf{x} = \int h_\Omega(\mathbf{x})\, f_O(\mathbf{x})\mathrm{d}\mathbf{x} \simeq \frac{1}{m}\sum_{j=1}^{m} h(\mathbf{X}_j), \tag{6}$$

where $h_\Omega(\mathbf{x})$ is an indicator function that is equal to 1 if $\mathbf{x} \in \Omega$ and zero otherwise, and where the second integral is over the entire state space. Since the rigorous modeling of the CAR is not the main focus in this paper, in the simulations we assume that Ω is hyper-cubic.

The quantity p_A can then be used to assess the relative degree of admissibility of the uncertain candidate orbit solution. The limiting cases are $p_A = 1$, which corresponds to the IOD solution being admissible with 100% certainty, and $p_A = 0$, which corresponds to the IOD solution being inadmissible with 100% certainty. If it is determined that the solution is admissible (by, say, requiring that p_A be larger than a given threshold value p_A^* or some other decision rule[1]), then the mean value $\boldsymbol{\mu}_\mathbf{x}$ will be taken as the maximum likely solution to the IOD problem. The primary merit of our proposed approach is that it is completely probabilistic, by virtue of using p_A as the criterion for assessing the degree of admissibility of the IOD solution as opposed to employing a non-rigorous mixed deterministic/probabilistic approach. We expect that the statistically rigorous screening criteria for orbit hypotheses, which we have outlined here, are less likely to incorrectly reject/accept viable orbit hypotheses than are any *ad hoc* screening criteria.

Remark. The problem of how best to sample the CAR, and indeed the measurement space, is a challenging problem in its own right. The sampling method will have a direct impact on the computation of p_A and, if not performed properly, will lead to inaccurate results. As mentioned, in the present paper, we use a naive approach of taking random points from a uniform distribution over Ω to perform the MC integration in order to generate representative results for the numerical example. A detailed analysis of the sampling problem, drawing on the work of Tommei et al. [3] and Siminski et al. [13, 14], will be the subject of future work.

[1] Such an admissibility decision rule will then be amenable to a probabilistic analysis that quantifies the probabilities of errors of Type I (false positive, that is accepting admissibility when the IOD solution is indeed inadmissible) and errors of Type II (false negative, that is accepting inadmissibility when the IOD solution is indeed admissible). In such an analysis, the optimal choice of a threshold p_A^* can be derived. However, this is beyond the scope of the present paper. We are currently investigating the problem of optimal decision criteria.

3. Simulation Results

In this section we provide a numerical example that demonstrates the above ideas. We consider an orbit with the orbital element values shown in Tab. 1 at the initial simulation time. About 36 minutes from the start of the simulation, the first measurement is received from the object. Once three angles-only measurements are collected, the IOD algorithm is executed on the collected data. The measurement noise is assumed to be Gaussian with an angular standard deviation of 1 arcsec for both right ascension and declination.

The true orbital state at t_2 (time of the second measurement) is given by the following position and velocity:

$$\begin{aligned}&\text{True Position (km)} : (-231.9136,\ -10\,057.7426,\ 5\,496.8366)\\&\text{True Velocity (km/s)} : (2.7737,\ 0.7333,\ 4.8581).\end{aligned}$$

In Fig. 1 (for position) and Fig. 2 (for semimajor axis vs. eccentricity), the true state is represented by a solid square. The deterministic IOD algorithm was run on the measured data and the orbit was determined to have the following position and velocity:

$$\begin{aligned}&\text{Deterministic IOD Position (km)} : (-88.2427,\ -9\,927.4580,\ 5\,308.0606)\\&\text{Deterministic IOD Velocity (km/s)} : (2.6675,\ 0.6741,\ 4.6094).\end{aligned}$$

The deterministic IOD solution is represented by the green $+$ symbol. The error from this analysis was found to 2.35% of the true value for position and 4.91% for velocity.

The UT approach described in Section 2. was then implemented. The UT mean in the figures is represented by the blue circle and the 3-sigma ellipse is represented by the blue line (or surface in Fig. 1(a)). The UT mean position and velocity were found to be:

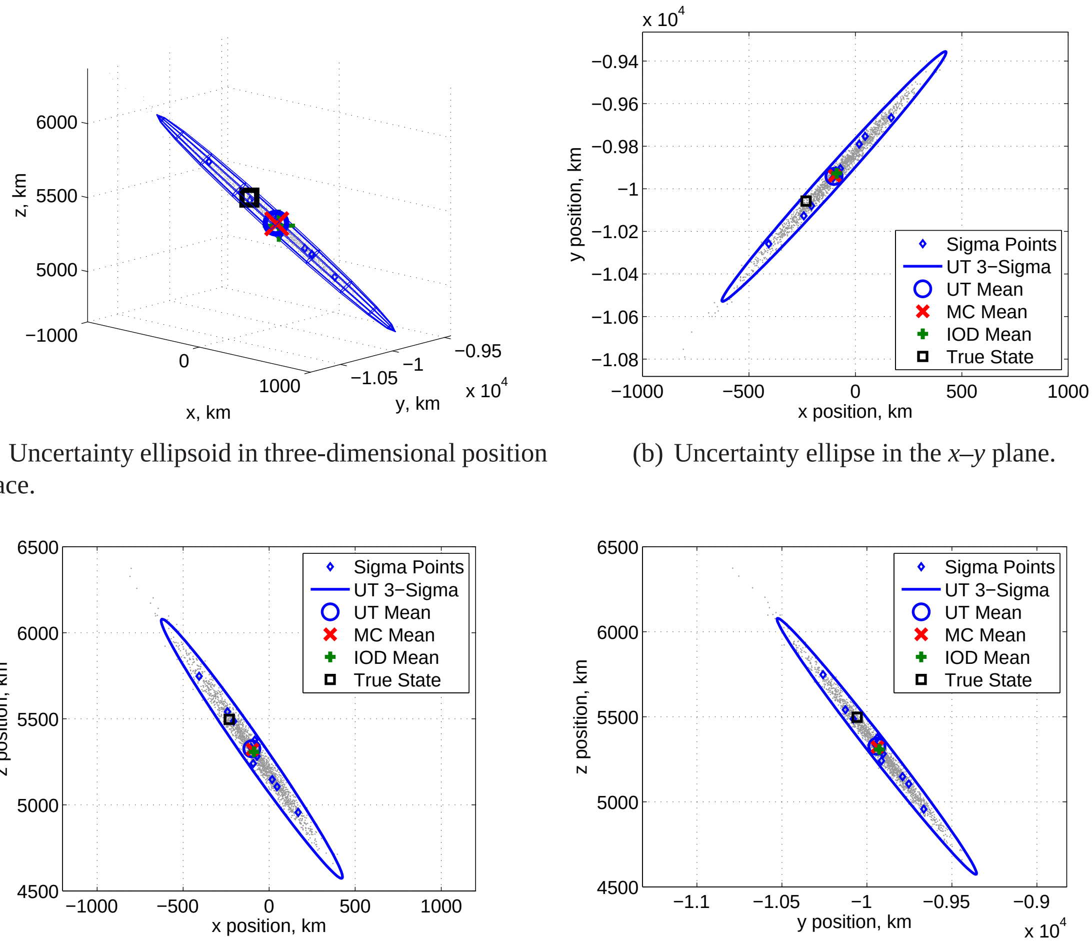
$$\begin{aligned}&\text{UT Mean Position (km)} : (-100.5409,\ -9\,941.9904,\ 5\,326.0415)\\&\text{UT Mean Velocity (km/s)} : (2.6754,\ 0.6751,\ 4.6265).\end{aligned}$$

The error from this analysis was found to be 2.13% of the true value for position and 4.58% for velocity. Note that the UT solution is slightly more accurate than the deterministic IOD solution.

Finally, the MC approach was implemented using 2000 samples. The MC points are represented by gray points in the figures. These form the MC-generated uncertainty cloud. The MC mean is

Table 1. Orbital elements of the true orbit.

Orbital Element	Value
Semimajor Axis (km)	10 571.4
Eccentricity	0.3
Inclination (deg)	63.4
Perigee (deg)	-90.0
Right Ascension of the Ascending Node (deg)	252.8
Initial True Anomaly (deg)	0.0



(a) Uncertainty ellipsoid in three-dimensional position space.

(b) Uncertainty ellipse in the x–y plane.

(c) Uncertainty ellipse in the x–z plane.

(d) Uncertainty ellipse in the y–z plane.

Figure 1. UT, MC, deterministic IOD results and true orbit in position space.

represented by the red $\times$ symbol. The MC mean position and velocity were found to be

$$\text{MC Mean Position (km)} : (-97.0272,\ -9\,938.6943,\ 5\,321.7201)$$
$$\text{MC Mean Velocity (km/s)} : (2.6708,\ 0.6724,\ 4.6247).$$

The error from this analysis was found to 2.19% of the true value for position and 4.65% for velocity. Note that the MC solution is slightly more accurate than the deterministic IOD solution, but the UT solution outperforms the MC and deterministic solutions. Note that this will not always be necessarily true. It is expected, for example, that the MC solution should outperform the UT solution when more MC points are used.

Next, we compute the probability of association. For the sake of simplicity, we assume that the CAR is a hyper-rectangle in the space of orbital elements. The lower and upper bounds on the elements are shown in Tab. 2. The bounds are shown in semimajor axis/eccentricity space in Fig. 2 using a dashed rectangle. We used MC integration, Eq. (6), with 18 million samples to compute the probability of association. As can be seen in Fig. 2, the deterministic, UT (defined by its mean) and MC (defined by its mean) solutions lie outside the CAR. In a purely deterministic framework, this

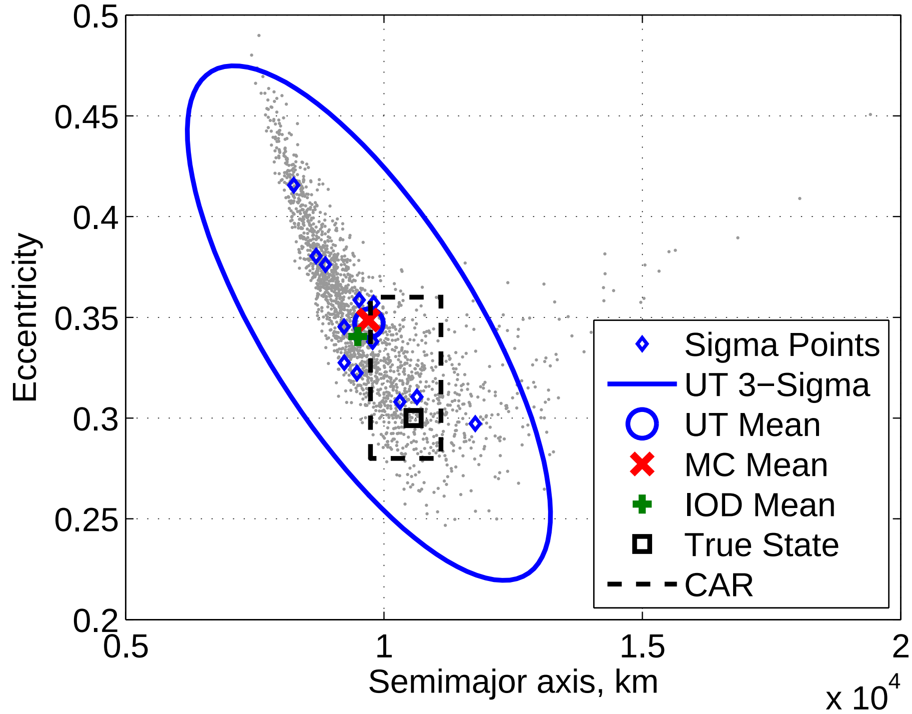


Figure 2. UT, MC, deterministic IOD results and true orbit in orbital element space.

would then lead to rejecting the admissibility of the solution. However, the computed value of p_A is 0.1913, which is significantly non-zero, suggesting that the solution may indeed be admissible. This example serves to show that binary admissible/inadmissible deterministic IOD tests fail to capture the uncertainty involved in such decisions. A notion such as the probability of admissibility can be used in a hypothesis testing framework to rigorously make admissibility decisions.

Table 2. CAR bounds in orbital elements.

Orbital Element	**Lower Bound**	**Upper Bound**
Semimajor Axis (km)	9 740.0	11 100.0
Eccentricity	0.28	0.36
Inclination (deg)	57.0	70.0
Perigee (deg)	-99.0	-81.0
Right Ascension of the Ascending Node (deg)	242.0	263.0
Initial True Anomaly (deg)	110.0	135.0

4. Conclusion

In this paper, we proposed a probabilistically consistent unscented transform (UT) based approach to quantify the uncertainty in Gauss' method for angles-only initial orbit determination (IOD). In addition, we introduced the notion of probability of admissibility over a specified constrained admissible region (CAR). We presented a numerical example to demonstrate the approach and the efficacy of the notion of the probability of admissibility. As shown in the selected example, the main conclusion of the paper is that deterministic *ad hoc* decision making can lead to erroneous admissibility decisions. The statistically rigorous screening criterion proposed in this paper is less likely to incorrectly reject/accept viable orbit hypotheses than are any *ad hoc* screening criteria. Currently, we are investigating statistically optimal choice for the threshold p_A^* (i.e., decision rule) for the analyst to use in order to minimize erroneous admissibility decisions. We are also exploring probabilistic information-based techniques for measurement-to-measurement and measurement-to-track association techniques in angles-only and other observation scenarios.

5. References

[1] Milani, A., Gronchi, G. F., Vitturi, M. D., and Knezevic, Z. "Orbit Determination with Very Short Arcs: I. Admissible Regions." Celestial Mechanics and Dynamical Astronomy, Vol. 90, pp. 59–87, 2004.

[2] Milani, A. and Knezevic, Z. "From Astrometry to Celestial Mechanics: Orbit Determination with Very Short Arcs." Celestial Mechanics and Dynamical Astronomy, Vol. 92, pp. 1–18, 2005.

[3] Tommei, G., Milani, A., and Rossi, A. "Orbit Determination of Space Debris: Admissibile Regions." Celestial Mechanics and Dynamical Astronomy, Vol. 97, pp. 289–304, 2007.

[4] Maruskin, J. M., Scheeres, D. J., and Alfriend, K. T. "Correlation of Optical Observations of Objects in Earth Orbit." Journal of Guidance, Control, and Dynamics, Vol. 32, No. 1, pp. 194–209, January–February 2009.

[5] Tommei, G., Milani, A., Farnocchia, D., and Rossi, A. "Correlation of Space Debris Observations by the Virtual Debris Algorithm." "Proccedings of the 5th European Conference on Space Debris," European Space Operations Center, Darmstadt, Germany, March 30–April 2 2009.

[6] Fujimoto, K., Maruskin, J. M., and Scheeres, D. J. "Circular and Zero-Inclination Solutions for Optical Observations of Earth-Orbiting Objects." Celestial Mechanics and Dynamical Astronomy, Vol. 106, pp. 157–182, 2010.

[7] Farnocchia, D., Tommei, G., Milani, A., and Rossi, A. "Innovative Methods of Correlation and Orbit Determination for Space Debris." Celestial Mechanics and Dynamical Astronomy, Vol. 107, pp. 169–185, 2010.

[8] Gronchi, G. F., Dimare, L., and Milani, A. "Orbit Determination with the Two-Body Integrals." Celestial Mechanics and Dynamical Astronomy, Vol. 107, pp. 299–318, 2010.

[9] Milani, A., Tommei, G., Farnocchia, D., Rossi, A., Schildknecht, T., and Jehn, R. "Correlation and Orbit Determination of Space Objects based on Sparse Optical Data." Monthly Notices of the Royal Astronomical Society, Vol. 417, No. 3, pp. 2094–2103, 2011.

[10] Fujimoto, K. and Scheeres, D. J. "Correlation of Optical Observations of Earth-Orbiting Objects and Initial Orbit Determination." Journal of Guidance, Control, and Dynamics, Vol. 35, No. 1, pp. 208–221, January–February 2012.

[11] DeMars, K. J., Jah, M. K., and Schumacher, P. W., Jr. "Initial Orbit Determination using Short-Arc Angle and Angle Rate Data." IEEE Transactions on Aerospace and Electronic Systems, Vol. 43, No. 3, pp. 2628–2637, July 2012.

[12] Schumacher, P. W., Jr., Wilkins, M. P., and Roscoe, C. W. T. "Parallel Algorithm for Track Initiation for Optical Space Surveillance." "Proccedings of the 6th European Conference on Space Debris," European Space Operations Center, Darmstadt, Germany, April 22–25 2013.

[13] Siminski, J. A., Fiedler, H., and Schildknecht, T. "Track Association Performance of the Best Hypotheses Search Method." "Proccedings of the 6th European Conference on Space Debris," European Space Operations Center, Darmstadt, Germany, April 22–25 2013.

[14] Siminski, J. A., Montenbruck, O., Fiedler, H., and Weigel, M. "Best Hypothesis Search on Iso-Energy-Grid for Initial Orbit Determination and Track Association." Advances in the Astronautical Sciences, Vol. 148, pp. 605–617, 2013. (Proceedings of the 23rd AAS/AIAA Space Flight Mechanics Meeting, Kauai, HI, February 10–14 2013).

[15] Roscoe, C. W. T., Schumacher, P. W., Jr., and Wilkins, M. P. "Parallel Track Initiation for Optical Space Surveillance using Range and Range-Rate Bounds." Advances in the Astronautical Sciences, Vol. 150, pp. 989–1008, 2014. (Proceedings of the AAS/AIAA Astrodynamics Specialist Conference, Hilton Head, SC, August 11–15 2013, Paper AAS 13-767).

[16] DeMars, K. J. and Jah, M. K. "Probabilistic Initial Orbit Determination Using Gaussian Mixture Models." Journal of Guidance, Control, and Dynamics, Vol. 36, No. 5, pp. 1324–1335, September–October 2013.

[17] Escobal, P. R. Methods of Orbit Determination. John Wiley & Sons, Inc., 1965. (Reprint with corrections by Krieger Publishing Company, 1976).

[18] Gooding, R. H. "A New Procedure for Orbit Determination Based on Three Lines of Sight (Angles Only)." Technical Report 93004, Defense Research Agency, Farnborough, Hampshire, United Kingdom, April 13 1993. Available through Defense Technical Information Center and National Technical Information Service (USA).

[19] Gooding, R. H. "A New Procedure for the Solution of the Classical Problem of Minimal Orbit Determination from Three Lines of Sight." Celestial Mechanics and Dynamical Astronomy, Vol. 66, No. 4, pp. 387–423, 1997.

[20] Karimi, R. R. and Mortari, D. "Initial Orbit Determination using Multiple Observations." Celestial Mechanics and Dynamical Astronomy, Vol. 109, No. 2, pp. 167–180, 2011.

[21] Sarnecki, A. J. "A Projective Approach to Orbit Determination from Three Sight Lines." Celestial Mechanics and Dynamical Astronomy, Vol. 66, No. 4, pp. 425–451, 1997.

[22] Junkins, J. L., Akella, M. R., and Alfriend, K. T. "Non-Gaussian Error Propagation in Orbital Mechanics." Journal of the Astronautical Sciences, Vol. 44, No. 4, pp. 541–563, October–December 1996.

[23] Fujimoto, K., Scheeres, D. J., and Alfriend, K. T. “Analytical Nonlinear Propagation of Uncertainty in the Two-Body Problem.” Journal of Guidance, Control, and Dynamics, Vol. 35, No. 2, pp. 497–509, March–April 2012.

[24] Horwood, J. T. and Poore, A. B. “Orbital State Uncertainty Realism.” “Proceedings of the AMOS Conference,” Wailea, HI, September 11–14 2012.

[25] Terejanu, G., Singla, P., Singh, T., and Scott, P. D. “Uncertainty Propagation for Nonlinear Dynamics Systems Using Gaussian Mixture Models.” Journal of Guidance, Control, and Dynamics, Vol. 31, No. 6, pp. 1623–1633, November–December 2008.

[26] Horwood, J. T., Aragon, N. D., and Poore, A. B. “Gaussian Sum Filters for Space Surveillance: Theory and Simulations.” Journal of Guidance, Control, and Dynamics, Vol. 34, No. 6, pp. 1839–1851, November–December 2011.

[27] DeMars, K. J., Bishop, R. H., and Jah, M. K. “A Splitting Gaussian Mixture Method for the Propagation of Uncertainty in Orbital Mechanics.” Advances in the Astronautical Sciences, Vol. 140, 2012. (Proceedings of the 21st AAS/AIAA Spaceflight Mechanics Meeting, New Orleans, LA, February 13–17 2011).

[28] DeMars, K. J., Jah, M. K., Giza, D., and Kelecy, T. “Orbit Determination Performance Improvements for High Area-to-Mass Ratio Space Object Tracking Using an Adaptive Gaussian Mixtures Estimation Algorithm.” “Proceedings of the 21st International Symposium on Space Flight Dynamics,” Toulouse, France, September 28–October 2 2009.

[29] Garber, D. D. “Uncertainty in Probability of Collision Calculations.” “Proceedings of the 3rd US-China Space Surveillance Technical Interchange,” Beijing, China, May 12–16 2013.

[30] Muinonen, K. and Bowell, E. “Asteroid Orbit Determination Using Bayesian Probabilities.” Icarus, Vol. 104, pp. 255–279, 1993.

[31] Virtanen, J., Muinonen, K., and Bowell, E. “Statistical Ranging of Asteroid Orbits.” Icarus, Vol. 154, pp. 412–431, 2001.

[32] Milani, A. “The Asteroid Identification Problem I. Recovery of Lost Asteroids.” Icarus, Vol. 137, pp. 269–292, 1999.

[33] Milani, A. and Valsecchi, G. B. “The Asteroid Identification Problem II: Target Plane Confidence Boundaries.” Icarus, Vol. 140, pp. 408–423, 1999.

[34] Milani, A., La Spina, A., Sansaturio, M. E., and Chesley, S. R. “The Asteroid Identification Problem III. Proposing Identifications.” Icarus, Vol. 144, pp. 39–53, 2000.

[35] Milani, A., Sansaturio, M. E., and Chesley, S. R. “The Asteroid Identification Problem IV: Attribution.” Icarus, Vol. 151, pp. 150–159, 2001.

[36] Carpino, M., Milani, A., and Chesley, S. R. “Error Statistics of Asteroid Optical Astrometric Observations.” Icarus, Vol. 166, pp. 248–270, 2003.

[37] Ford, E. B. "Quantifying the Uncertainty in the Orbits of Extrasolar Planets." Astronomical Journal, Vol. 129, pp. 1706–1717, 2005.

[38] Weisman, R. M. and Jah, M. K. "Uncertainty Quantification For Angles-Only Initial Orbit Determination." "Proceedings of the 24th AAS/AIAA Space Flight Mechanics Meeting," Santa Fe, NM, January 26–30 2014.

[39] Binz, C. R. and Healy, L. M. "Uncertainty Characterization for Angles-Only Initial Orbit Determination." Advances in the Astronautical Sciences, Vol. 150, pp. 1777–1792, 2014. (Proceedings of the AAS/AIAA Astrodynamics Specialist Conference, Hilton Head, SC, August 11–15 2013, Paper AAS 13-822).

[40] Binz, C. R. and Healy, L. M. "Utilization of Uncertainty Information in Angles-Only Initial Orbit Determination." "Proceedings of the 24th AAS/AIAA Space Flight Mechanics Meeting," Santa Fe, NM, January 26–30 2014. Paper AAS 14-431.

[41] Curtis, H. Orbital Mechanics for Engineering Students. Elsevier, 2005.

[42] Julier, S. and Uhlmann, J. "A New Method for the Nonlinear Transformation of Means and Covariances in Nonlinear Filters." IEEE Transactions on Automatic Control, Vol. 44, No. 3, pp. 477–482, March 2000.