Attitude and Orbit Control of a Spinning Solar Sail by the Vibrational Input on the Sail Membrane

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An attitude and orbit control method for a spinning solar sail utilizing the active shape control of the sail membrane is investigated. Spinning type membrane space structures easily deform because they have no supporting structures. This may lead to the unexpected change of the effect of solar radiation pressure (SRP) on the membranes. Since the SRP is a dominant factor of the dynamics of membrane space structures, especially for solar sails, the knowledge of the deformation is vital to them. However, it is almost impossible to precisely predict and design the actual deformation of the membrane before deployment. In this study, an active shape control method for membrane space structures is proposed, and is applied to the attitude and orbit control of a spinning solar sail.

Key Words: Solar Sail, Solar Radiation Pressure, Shape Control, Vibration Mode

Nomenclature

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Meaning</th>
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<tbody>
<tr>
<td>$t$</td>
<td>time</td>
</tr>
<tr>
<td>$x, y, z$</td>
<td>Cartesian coordinate</td>
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<tr>
<td>$r, \theta, z$</td>
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<td>$r$</td>
<td>position vector</td>
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<td>$M$</td>
<td>mass</td>
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<tr>
<td>$w$</td>
<td>out-of-plane deformation</td>
</tr>
<tr>
<td>$\Omega$</td>
<td>spin rate (angular velocity)</td>
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<tr>
<td>$r_a$</td>
<td>inner radius</td>
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<tr>
<td>$r_b$</td>
<td>outer radius</td>
</tr>
<tr>
<td>$h$</td>
<td>thickness</td>
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<td>density</td>
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<td>Poisson’s ratio</td>
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<td>$R, \Theta$</td>
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<td>moment of inertia</td>
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<td>$s$</td>
<td>sun vector</td>
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<td>$C_{abs}$</td>
<td>absorptivity</td>
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<td>Lambertian coefficient</td>
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<td>$A, B, a, b$</td>
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<td>$p, \lambda, N, Q, \gamma$</td>
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<td>$\delta_{x,y}$</td>
<td>Dirac’s delta</td>
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Superscripts

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<td>-</td>
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<td>forced response</td>
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<td>$O$</td>
<td>orbit-fixed frame</td>
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<tr>
<td>$SF$</td>
<td>spin-free-fixed frame</td>
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Subscripts

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<td>direction in cylindrical coordinate</td>
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<tr>
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<td>circumferential order of vibration</td>
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<tr>
<td>$n, m$</td>
<td>radial order of vibration</td>
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<tr>
<td>$0$</td>
<td>input by forced vibration</td>
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<tr>
<td>$SRP$</td>
<td>solar radiation pressure</td>
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<tr>
<td>$M$</td>
<td>membrane</td>
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<td>$B$</td>
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1. Introduction

Recently, the use of flexible membrane structures is gathering attention in the space field. They allows carrying lightweight structures with large area to space, and are expected to be widely used in the future. One of the membrane structures is solar sail. Solar sail is a spacecraft which is driven by solar radiation pressure (SRP), so that it can be propelled basically without any propellant. Fig. 1 shows the small solar sail demonstrator IKAROS launched in 2010 by Japan Aerospace Exploration Agency.\(^1\)

Membrane structures including solar sails can be classified roughly into two types depending on the deployment method of the membrane. One is mast-extension type, which deploys its membrane by use of inflatable mast structures. Another is spinning type, which deploys by centrifugal force. Since spinning type ones do not require any supporting structure, they are preferable in terms of system weight. In case of solar sail, spinning type is still more preferable because the propulsion performance is better when the ratio of area to mass is larger. However, spinning type membrane structures easily deform due to the lack of supporting structures. This may cause the unex-
Fig. 1.: Spinning solar sail demonstrator IKAROS with its 14m × 14m large sail membrane. 1) Photographed by a deployable camera on the orbit.

Expected change of the SRP on its membrane, which largely affects the attitude motion (and orbital motion in case of solar sails) of the spacecraft. 2)

Reflectivity control device (RCD) 3) is one solution to deal with the problem. It can change optical property of the surface electrically to control the SRP on it. Hence, switching multiple RCDs distributed on the membrane can cause the unbalance of the solar radiation force to generate the attitude control torque. However, RCDs have a problem of space environmental effects. Performance of RCDs is also affected by the deformation of the membrane. Therefore, knowledge of deformation of the membrane is vital not only to solar sailing, but to achieving general spinning type membrane space structures. Nevertheless, deformation of the flexible membrane structures is complex, unpredictable, and few studies have dealt with the active control of it.

This paper proposes an active shape control method for a spinning type membrane space structures by the vibrational input. In concrete, the method aims to actively vibrate the membrane by use of mechanical actuators to induce a certain vibration mode on it. In particular, we show that the shape control method works even if the vibrational input is limited to one-dimensional one.

This method allows launching various missions with membrane structures. For example, the method makes it possible to control the thrust and torque that originate in the SRP. This can be applied to the attitude and orbit control of a solar sail. In another example, thin-film devices that are attached on the membrane can change their orientation; e.g. performance of thin-film solar cells can be enhanced by changing their orientation to the sun, even if the main body of the spacecraft has some pointing requirements. In the same way, not only can the reflection direction of RCDs be controlled, combination of torque control via shape control and the one via RCDs can realize the novel control of attitude motion.

The dynamics of membrane space structures is complex, in which structure, attitude, and orbit are coupled. The method proposed in this study allows controlling this dynamics, which has an infinite degree of freedom, only by a one-dimensional vibrational input.

In the following sections, the concrete way of the shape control is proposed first. Next, vibration mode of spinning membrane structures is derived analytically assuming a simple model. Then, a numerical analysis is conducted to show the validity and applicability of the analytical solution to general membrane structures. Finally, an optimal attitude and orbit control of a spinning solar sail is performed as an example of the application of the developed shape control method.

2. Shape Control of Membrane Structures

2.1. Shape control method

The flexible membrane structures are easily deformed with vibrations. The vibrations are theoretically described by a superposition of multiple vibration modes. When a certain periodic input is given on the membrane, and its frequency is enough close to the natural frequency of the structure, the corresponding vibration mode is then induced on the membrane. When the input wave consists of multiple waves superposed, the membrane can form various complex shapes.

The periodic input can be, for example, given as a forced vibration by use of mechanical actuators. In case of IKAROS, the sail membrane and the main body of the spacecraft are connected via tethers (Fig. 2). This paper proposes to drive the tethers for the shape control. Two ways are possible to give the vibrational input by use of tethers (Fig. 3). One is to shift the joint parts between the tethers and the main body up and down (Ex. 1). Another is to pull up tethers alternately which are attached at the top and bottom part, respectively, of the main body (Ex. 2).
brane, tensile stress for radial and circumferential direction is, considering a plane stress state of a spinning circular membrane structure.

3. Analytical Vibration Mode Analysis of Spinning Membrane Structures

Vibrations of spinning membrane structures are analytically derived in this section. Here, a uniform circular membrane is assumed for the simplicity. Vibrations of membrane structures consist of in-plane and out-of-plane ones, and they can be treated independently in the linear region. Hence, the out-of-plane vibration, which is the target of the shape control, is modeled. Response to the vibrational input is also analyzed. Finally, the conditions to form a static wave are discussed on the basis of the analytical solution.

3.1. Vibration mode analysis

3.1.1. Derivation of the mode function

Assume a spinning, uniform, circular membrane structure as shown in Fig. 5. The deformation of the membrane is described in the rotating cylindrical coordinate \( r - \theta - z \), whose origin is at the center of the membrane. The spin rate of the system is constant. The force for \( \hat{z} \) direction exerted on the microelement \( dM = \rho hr dr d\theta \) is described as follow.

\[
F_{\hat{z}} = \left\{ \frac{\partial}{\partial r} \left( \sigma_r \frac{\partial w}{\partial r} \right) + \frac{\partial}{\partial \theta} \left( \sigma_\theta \frac{\partial w}{\partial \theta} \right) \right\} hr dr d\theta \tag{1}
\]

Hence, equation of motion is written as follow.

\[
\ddot{w} dM = F_{\hat{z}} \Rightarrow \rho r^2 \ddot{w} = \frac{\partial}{\partial r} \left( \sigma_r \frac{\partial w}{\partial r} \right) + \frac{\partial}{\partial \theta} \left( \sigma_\theta \frac{\partial w}{\partial \theta} \right) \tag{2}
\]

Considering a plane stress state of a spinning circular membrane, tensile stress for radial and circumferential direction is,

\[
\sigma_r = \frac{3 + \nu'}{8} r \Omega^2 r_b \left( 1 - \frac{r^2}{r_b^2} \right)
\]

\[
\sigma_\theta = \frac{3 + \nu'}{8} r \Omega^2 r_b \left( 1 - \frac{1 + 3\nu' r^2}{3 + 3\nu' r_b^2} \right) \tag{3}
\]

Here, the following solution is assumed considering the separation of variables.

\[
w(r, \theta, t) = R(r) \Theta(\theta) q(t) \tag{4}
\]

Substituting Eq. (4) into Eq. (2), the following equations are obtained.

\[
\frac{d^2 q}{dr^2} + \omega^2 q = 0 \tag{5}
\]

\[
\frac{d^2 \Theta}{d\theta^2} + \nu^2 \Theta = 0 \tag{6}
\]

\[
r \frac{d}{dr} \left( \frac{dR}{dr} \right) + (\rho \omega^2 r - \nu^2 \sigma_\theta) R = 0 \tag{7}
\]

\( \omega \) and \( \nu \) appear as the constants of separation of variables. The boundary conditions are given as follows.

\[
\Theta(\theta) = \Theta(\theta + 2\pi) \tag{8}
\]

\[
R(r_b) < \infty \tag{9}
\]

Equations (8) and (9) mean, respectively, the continuity for circumferential direction, and the free edge boundary condition. Since the solution of \( \Theta(\theta) \) can be described sinusoidally, Eq. (8) gives the following condition.

\[
\nu = 0, 1 , 2, 3, \cdots \tag{10}
\]

Substituting Eq. (3) into Eq. (7), differential equation of \( R(r) \) is described as follow.

\[
\hat{r} \frac{d}{dr} \left( \hat{r} (1 - \hat{r}) \frac{dR}{dr} \right) + (\lambda^2 \hat{r}^2 - \nu^2) R = 0 \tag{11}
\]

where

\[
\hat{r} = \frac{r}{r_b} \tag{12}
\]

\[
\lambda^2 = \frac{8}{3 + \nu'} \left( \frac{\omega}{\Omega} \right)^2 + 1 + 3\nu' \geq 0 \tag{13}
\]

Assume the following solution for Eq. (11), which starts from the \( p \)-th order.

\[
R(\hat{r}) = \hat{r}^p \sum_{m=0}^{\infty} a_m \hat{r}^m \tag{14}
\]
Substituting Eq. (14) into Eq. (11), and evaluating the coefficient of \( \hat{r}^p \), \( p \) is derived as

\[
p = \pm \nu \tag{15}
\]

Since the solution of \( R(\hat{r}) \) diverges at \( \hat{r} = 0 \) when \( p = -\nu \), which conflicts with Eq. (9), only the case of \( p = \nu \) is enough to describe the actual deformation. Therefore, the following recurrence formula of \( a_m \) is derived by substituting Eq. (14) into Eq. (11) under \( p = \nu \).

\[
a_m = \frac{(m + \nu - 2)(m + \nu) - \nu^2}{m(m + 2\nu)} a_{m-2} \\
a_0 \neq 0 \\
a_{-1} = 0
\tag{16}
\]

The following condition is required to satisfy Eq. (9).

\[
\exists \in O, \hat{r}^2 = (n + \nu - 1)(n + \nu + 1), n \geq 1 \\
O = \{ x \in \mathbb{Z} | x \text{ is odd} \}
\tag{17}
\]

This is because \( a_m \) is zero in the \( n + 1 \) th or higher order, and the infinite series (14) becomes a polynomial of degree \( n - 1 \) when Eq. (17) holds. Otherwise the solution diverges at \( \hat{r} = 1 \). Let \( n \) denote the order of vibration for radial direction. Natural frequency is then expressed as the following equation, by substituting Eq. (17) into Eq. (13).

\[
\omega_{kn} = \Omega \sqrt{\frac{3 + \nu^2}{8}(\nu + n - 1)(\nu + n + 1) - \frac{1 + 3\nu^2}{8} \nu^2} \tag{18}
\]

Subscript means that the natural frequency depends both on circumferential and radial order of vibration. Eq. (16) can be rewritten as follow, by substituting Eq. (17).

\[
a_m = \frac{(m - n - 1)(m + n + 2\nu - 1)}{m(m + 2\nu)} a_{m-2} \tag{19}
\]

Therefore, the general term of \( a_m \) can be expressed as follow.

\[
a_{2k} = (-1)^{k+1} \frac{1}{2} \left( \frac{n + \nu}{2} + k \right) \left( \frac{n + \nu}{2} - k \right) \sqrt{\nu^2 + 2k + 1} \tag{20}
\]

Thus, the general solution of the radial mode function is described as follow.

\[
R_{\nu,\nu}(\hat{r}) = \sum_{k=0}^{\nu/2} (-1)^{k+1} \frac{1}{2} \left( \frac{n + \nu}{2} + k \right) \left( \frac{n + \nu}{2} - k \right) \sqrt{\nu^2 + 2k + 1} \hat{r}^{\nu+2k} \tag{21}
\]

Finally, the general solution of the vibration of a spinning circular membrane is expressed as follow.

\[
w(r, \theta, t) = \sum_{\nu} \sum_{n} R_{\nu,\nu} \left( \frac{r}{r_b} \right) \Theta_n(\theta) q_{\nu,\nu}(t) \\
= \sum_{\nu} \sum_{n} R_{\nu,\nu} \left( \frac{r}{r_b} \right) \left[ A_{\nu,\nu} e^{i(\nu + \nu_0 + a) + \nu_0} + B_{\nu,\nu} e^{i(\nu + \nu_0 + b)} \right] \tag{22}
\]

Figure 6 shows the examples of analytical vibration modes expressed by Eq. (22).

3.1.2. Orthogonality of the mode function

Equation (11) in the \((\nu, m)\)-th order can be written as

\[
\left( \lambda_{\nu,\nu}^2 \hat{r} - \frac{\nu^2}{\hat{r}} \right) R_{\nu,m} = -\frac{d}{d\hat{r}} \left( \hat{r}(1 - \hat{r}^2) \frac{dR_{\nu,m}}{d\hat{r}} \right) \tag{23}
\]

where

\[
\lambda_{\nu,\nu}^2 = (\nu + n - 1)(\nu + n + 1) \tag{24}
\]

from Eq. (17). By integrating the multiple of Eq. (23) and \( R_{\nu,m} \), the following equation is obtained.

\[
\int_0^1 \left( \lambda_{\nu,\nu}^2 \hat{r} - \frac{\nu^2}{\hat{r}} \right) R_{\nu,m} R_{\nu,m} d\hat{r} = \int_0^1 \hat{r}(1 - \hat{r}^2) \frac{dR_{\nu,m}}{d\hat{r}} \frac{dR_{\nu,m}}{d\hat{r}} d\hat{r} \tag{25}
\]

In the same way, the following equation is obtained about the \((\nu, m)\)-th order.

\[
\int_0^1 \left( \lambda_{\nu,\nu}^2 \hat{r} - \frac{\nu^2}{\hat{r}} \right) R_{\nu,m} R_{\nu,m} d\hat{r} = \int_0^1 \hat{r}(1 - \hat{r}^2) \frac{dR_{\nu,m}}{d\hat{r}} \frac{dR_{\nu,m}}{d\hat{r}} d\hat{r} \tag{26}
\]

Calculating Eq. (25) – Eq. (26), the following equation is derived.

\[
(\lambda_{\nu,n}^2 - \lambda_{\nu,\nu}^2) \int_0^1 \hat{r} R_{\nu,m} R_{\nu,m} d\hat{r} = 0 \tag{27}
\]

Since \( \lambda_{\nu,n}^2 \neq \lambda_{\nu,\nu}^2 \) when \( n \neq n \), the following equation holds.

\[
\int_0^1 \hat{r} R_{\nu,m} R_{\nu,m} d\hat{r} = 0 \quad (m \neq n) \tag{28}
\]

Thus, orthogonality of the radial function about its order \( n \) is confirmed. In addition to the case of circumferential mode function, the orthogonality is described by following equations.

\[
\int_0^{2\pi} \Theta_\phi \Theta_\phi d\theta = 2\pi \delta_{\mu,\nu} \tag{29}
\]
3.2. Forced response analysis

When the vibrational input is given to the membrane by tethers, the forced displacement at \( r = r_n \) can be expressed as follow.

\[
w(r_n, \theta, t) = w_0(\theta, t) = A_0 e^{i(\nu_0 t + \Omega t + \phi_0)} + B_0 e^{i(\nu_0 t + \Omega t + \phi_0)} \quad (30)
\]

When the deformation of the membrane is described as \( w(r, \theta, t) = \tilde{w}(r, \theta, t) + w_0(\theta, t) \), equation of motion (2) is rewritten as follow.

\[
\rho r^2 \ddot{\tilde{w}} + \frac{\partial}{\partial r} \left( \sigma_r \frac{\partial \tilde{w}}{\partial r} \right) + \frac{1}{r} \frac{\partial}{\partial \theta} \left( \sigma_\theta \frac{\partial \tilde{w}}{\partial \theta} \right) + F_0 = 0 \quad (31)
\]

where

\[
F_0 = -\rho r^2 \ddot{w}_0 + \frac{\partial}{\partial r} \left( \sigma_r \frac{\partial w_0}{\partial r} \right) \quad (32)
\]

\( F_0 \) denote a kind of inertial force which appears when considering the deformation via \( \tilde{w} \). From Eq. (30), \( F_0 \) can be rewritten as follow.

\[
F_0 = \left( \rho r \omega_0^2 - \frac{\sigma_r}{r} \frac{\partial^2}{\partial r^2} \right) w_0 \quad (33)
\]

Assuming that the vibration mode here is also described by the same mode function as in Eq. (22), \( \tilde{w} \) can be described as follow.

\[
\tilde{w}(r, \theta, t) = \sum_j \sum_n R_{j,n} \left( \frac{r}{r_n} \right) \Theta_j(\theta) \tilde{q}_{j,n}(t) \quad (34)
\]

From Eqs. (6), (7), (31) and (34), the following equation is obtained.

\[
\sum_j \sum_n R_{j,n} \sigma_j \left( \frac{\partial^2 \tilde{q}_{j,n}}{\partial r^2} \right) + \omega_n^2 \tilde{q}_{j,n} = \frac{F_0}{\rho r} \quad (35)
\]

Applying the orthogonality of mode functions described in Eq. (29) to Eq. (35), the following equation is derived.

\[
\frac{d^2 \tilde{q}_{j,n}}{dt^2} + \omega_n^2 \tilde{q}_{j,n} = \frac{1}{2 \pi N \rho r} \int_0^{2 \pi} \frac{R_{j,n} \Theta_j F_0}{\rho r} dr d\theta \quad (36)
\]

Substituting Eq. (33) into Eq. (36), the differential equation of the modal coordinate \( \tilde{q}_{j,n} \) is derived as follow.

\[
\frac{d^2 \tilde{q}_{j,n}}{dt^2} + \omega_n^2 \tilde{q}_{j,n} = Q_{j,n} \delta_{j,j'} Q_{j',n'} \left( A_0 e^{i(\nu_0 t + \Omega t + \phi_0)} + B_0 e^{i(\nu_0 t + \Omega t + \phi_0)} \right) \quad (37)
\]

where

\[
Q_{j,n} = \frac{1}{\rho N \rho r} \int_0^{2 \pi} \left( \rho r \omega_n^2 - \frac{\sigma_r}{r} \frac{\partial^2}{\partial r^2} \right) R_{j,n} dr \quad (38)
\]

Particular solution of Eq. (37) is

\[
\tilde{q}_{j,n} = \frac{Q_{j,n} \omega_n^2}{\omega_n^2 - \omega_0^2} \left( A_0 e^{i(\nu_0 t + \Omega t + \phi_0)} + B_0 e^{i(\nu_0 t + \Omega t + \phi_0)} \right) \quad (39)
\]

Therefore, the response to the vibrational input is expressed by the following equation.

\[
\ddot{w}(r, \theta, t) = \sum_n \frac{Q_{j,n} \omega_n^2}{\omega_n^2 - \omega_0^2} \left( \frac{r}{r_n} \right) \left( A_0 e^{i(\nu_0 t + \Omega t + \phi_0)} + B_0 e^{i(\nu_0 t + \Omega t + \phi_0)} \right) \quad (40)
\]

As can be seen from Eq. (40), only \( \nu_0 \)-th order mode appears for circumferential direction while the resonance magnification for radial direction is determined by the input frequency \( \Omega \) and the natural frequency \( \omega_{n,n} \).

3.3. How to induce the static wave

From Eq. (40), the vibration is expressed by the superposition of progressive wave and regressive wave for the circumferential direction.

progressive wave : \( B_0 e^{i(\nu_0 t + \Omega t + \phi_0)} \)

regressive wave : \( A_0 e^{i(\nu_0 t + \Omega t + \phi_0)} \)

Their velocity for \( \theta \) direction is \( \omega_0/\nu_0 \) and \( -\omega_0/\nu_0 \) respectively in the rotating frame. Considering that the angular velocity of the rotating frame is \( \Omega \), the velocity of the waves in the inertial frame is expressed as \( \Omega + \omega_0/\nu_0 \) and \( \Omega - \omega_0/\nu_0 \) respectively. When the following conditions are satisfied, the velocity of the regressive wave is zero and hence the induced waveform is static in the inertial frame.

\[
B_0 = 0 \quad \Omega - \omega_0/\nu_0 \Leftrightarrow \omega_0 = \nu_0 \Omega \quad (41)
\]

Equation (41) means that the input frequency must be synchronized with the spin rate to induce the static deformation, as described in section 2.2. In addition, the vibrational input must be given so that the progressive wave for \( \theta \) direction does not occur. The vibrational input which induces a static wave is, from the conditions, expressed as follow.

\[
w_0(\theta, t) = A_0 e^{i(\nu_0 t + \Omega t + \phi_0)} \quad (42)
\]

The corresponding static waveform of the membrane in the inertial frame is then derived as follow.

\[
\ddot{w}(r, \theta) = \sum_n \frac{A_0 Q_{j,n} \omega_n^2}{\omega_n^2 - \omega_0^2} \left( \frac{r}{r_n} \right) e^{i(\nu_0 t + \Omega t + \phi_0)} \quad (43)
\]

The static wave is essentially expressed as the superposition of vibrations of every radial order, but actually, only domestic order vibrations appear depending on the resonance magnification. Following equations show the representative shape of the static wave, in the low-frequency region from \( \nu_0 = 1 \) to 4.

\[
w_1(r, \theta) = A_1 \cos(\theta + \alpha_1) + A_1 R_{1,1} \cos(\theta + \alpha_1) \quad (44)
\]

\[
w_2(r, \theta) = A_2 \cos(2\theta + \alpha_2) + A_2 R_{2,1} \cos(2\theta + \alpha_2) \quad (44)
\]

\[
w_3(r, \theta) = A_3 \cos(3\theta + \alpha_3) + A_3 R_{3,1} \cos(3\theta + \alpha_3) \quad (44)
\]

\[
w_4(r, \theta) = A_4 \cos(4\theta + \alpha_4) + A_4 R_{4,1} \cos(4\theta + \alpha_4) \quad (44)
\]

In the following, they are called “\( \nu_0 \)-th order static wave”. Number of subscripts corresponds to the order of vibration.

4. Feasibility Evaluation

In section 3, vibration mode and forced response of a spinning membrane structure are analytically derived assuming a simple theoretical model. On the other hand, this assumption is not necessarily available to general membrane structures. For example, IKAROS has a square-shaped sail membrane due to the folding and packing method. The center of the membrane is empty to hold the main body for the same reason. Also, the deformation induced by the proposed method is expected to
In this section, feasibility of the theoretical model against general membrane structures is evaluated by numerical analysis. First, numerical analysis of a circular membrane validates the theoretical model in the non-linear region with \( r_a, 0 \). Next, deformation behavior of square-shaped membrane, which is based on the configuration of IKAROS sail membrane, is analyzed to be compared with that of the circular membrane. Thus, the effectiveness of applying the developed theoretical model to the general membrane structures is verified.

### 4.1. Multi-particle model

A method called multi-particle model (MPM)\(^6\) is utilized in the numerical analysis. MPM simulates the thin-film membrane structure by spring-mass system. In this model, a number of particles are distributed across the entire surface of the membrane, and the particles are connected by springs and dampers. Since computational cost of the MPM is rather small than that of finite element method, the MPM is suitable for analyzing the global behavior of the membrane structures. Figure 7 shows the configuration of the MPM used in this study.

Spring constants in the MPM is determined on the basis of the principle of virtual work; so that the elastic energy as the

be large because of oscillation, while the theoretical model assumes an infinitesimal strain theory in the linear region. However, it is known that principal characteristics of vibrations of actual membrane structures tend to be similar to those of simple theoretical model.\(^5\)

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<table>
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<tr>
<th>Table 1.: Properties of the spacecraft.</th>
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spring-mass system is identical to the strain energy as the continuum. Table 1 shows the properties of the spacecraft used in the analysis. The main body is treated as a cylindrical rigid body. Tethers and bridges are also modeled as springs from their elastic properties. Parameters of the membrane derive from those of IKAROS sail membrane. Inner and outer radius of the circular membrane are determined so that the area of the membrane is identical to that of IKAROS sail.

### 4.2. Mode analysis

The membrane is replaced as a spring-mass system by applying the MPM. Vibration mode analysis is thereby available by linearizing equation of motion.\(^7\) Equation of motion of the system can be written in matrix form by setting up linearized simultaneous equations of all particles. Hence, vibration mode and its natural frequency can be calculated by the eigenvalue analysis of the mass matrix and the stiffness matrix.
Figures 8 through 11 show the vibration modes obtained by the MPM analysis, and their comparison with the analytical solutions. As can be seen from the figures, the numerical analysis result corresponds to the analytical solution in every order. Figure 12 shows the natural frequency of each vibration mode obtained by the MPM. Normalized by the spin rate.

Figures 14 and 15 show the simulation results against the 1st and 2nd mode input on the circular membrane respectively. Deformation states in the inertial frame at $t = 2000$ sec and $t = 3000$ sec are described as examples for each one. As can be seen from the figures, the waveform of the membrane is almost the same in each case since the tops and bottoms of the waves are in the same locations. In order to evaluate the accuracy of the shape control method against the analytical solution quantitatively, error analysis is conducted by means of least squares method; calculating mean error between displacement of all particles and the expected shape written in Eq. (45).

\begin{align}
\begin{align*}
  w_1^I(r, \theta) &= A_1 \cos(\theta + a_1) + \tilde{A}_1 \left( \frac{r}{r_a} \right) \cos(\theta + a_1) \\
  w_2^I(r, \theta) &= A_2 \cos(2\theta + a_2) + \tilde{A}_2 \left( \frac{r}{r_a} \right)^2 \cos(2\theta + a_2)
\end{align*}
\end{align}

Figures 14 and 15 show the simulation results against the 1st and 2nd mode input on the circular membrane respectively. Deformation states in the inertial frame at $t = 2000$ sec and $t = 3000$ sec are described as examples for each one. As can be seen from the figures, the waveform of the membrane is almost the same in each case since the tops and bottoms of the waves are in the same locations. In order to evaluate the accuracy of the shape control method against the analytical solution quantitatively, error analysis is conducted by means of least squares method; calculating mean error between displacement of all particles and the expected shape written in Eq. (45). Figure 16 shows the history of mean errors of the circular membrane. The figure confirms that the control error is almost within 20% through time. The possible reasons for the error are:

- discretization error of the membrane by the MPM
- transient response because of the small damping effect
- linearization error against the large deformation

Also, the errors of the 2nd mode response is larger than those of the 1st mode one because the shape of the 2nd mode one is more
result confirms that the waveforms of the square membrane is almost static similarly as those of the circular one. Figure 19 shows the history of mean errors of the induced shape. As can be seen, the history shows the similar trends as that of circular one. Figure 19 shows the history of mean errors of the induced shape. As can be seen, the history shows the similar trends as that of circular one. Figure 19 shows the history of mean errors of the induced shape. As can be seen, the history shows the similar trends as that of circular one. Figure 19 shows the history of mean errors of the induced shape. As can be seen, the history shows the similar trends as that of circular one. Figure 19 shows the history of mean errors of the induced shape. As can be seen, the history shows the similar trends as that of circular one. Figure 19 shows the history of mean errors of the induced shape. As can be seen, the history shows the similar trends as that of circular one. Figure 19 shows the history of mean errors of the induced shape. As can be seen, the history shows the similar trends as that of circular one. Figure 19 shows the history of mean errors of the induced shape. As can be seen, the history shows the similar trends as that of circular one. Figure 19 shows the history of mean errors of the induced shape. As can be seen, the history shows the similar trends as that of circular one. Figure 19 shows the history of mean errors of the induced shape. As can be seen, the history shows the similar trends as that of circular one. Figure 19 shows the history of mean errors of the induced shape. As can be seen, the history shows the similar trends as that of circular one. Figure 19 shows the history of mean errors of the induced shape. As can be seen, the history shows the similar trends as that of circular one. Figure 19 shows the history of mean errors of the induced shape. As can be seen, the history shows the similar trends as that of circular one. Figure 19 shows the history of mean errors of the induced shape. As can be seen, the history shows the similar trends as that of circular one. Figure 19 shows the history of mean errors of the induced shape. As can be seen, the history shows the similar trends as that of circular one. Figure 19 shows the history of mean errors of the induced shape. As can be seen, the history shows the similar trends as that of circular one. Figure 19 shows the history of mean errors of the induced shape. As can be seen, the history shows the similar trends as that of circular one. Figure 19 shows the history of mean errors of the induced shape. As can be seen, the history shows the similar trends as that of circular one. Figure 19 shows the history of mean errors of the induced shape. As can be seen, the history shows the similar trends as that of circular one. Figure 19 shows the history of mean errors of the induced shape. As can be seen, the history shows the similar trends as that of circular one. Figure 19 shows the history of mean errors of the induced shape. As can be seen, the history shows the similar trends as that of circular one. Figure 19 shows the history of mean errors of the induced shape. As can be seen, the history shows the similar trends as that of circular one. Figure 19 shows the history of mean errors of the induced shape. As can be seen, the history shows the similar trends as that of circular one. Figure 19 shows the history of mean errors of the induced shape. As can be seen, the history shows the similar trends as that of circular one. Figure 19 shows the history of mean errors of the induced shape. As can be seen, the history shows the similar trends as that of circular one. Figure 19 shows the history of mean errors of the induced shape. As can be seen, the history shows the similar trends as that of circular one. Figure 19 shows the history of mean errors of the induced shape. As can be seen, the history shows the similar trends as that of circular one. Figure 19 shows the history of mean errors of the induced shape. As can be seen, the history shows the similar trends as that of circular one. Figure 19 shows the history of mean errors of the induced shape. As can be seen, the history shows the similar trends as that of circular one. Figure 19 shows the history of mean errors of the induced shape. As can be seen, the history shows the similar trends as that of circular one.
this orbit control method can not only be performed without any fuel, but achieve faster maneuver of the sail membrane than RCDs.

However, the sail membrane cannot keep its orientation toward a certain point forever because of the SRP disturbance torque; the total angular momentum drifts over the performance of the 1st order static wave. This problem can be solved by the superposition of the 1st and higher order static waves. The super position causes asymmetric deformation of the membrane, which results in additional SRP torque. In summary, thrust is produced. The orbit-fixed frame moves in accordance with the orbital motion. Its origin is at the center of mass of the spacecraft. Hence, the thrust and torque exerted on the spacecraft can be derived by SRP. Figure 20 shows the schematic chart of the method.

5.2. Formulation of thrust and torque caused by SRP

As for the derivation of the thrust and torque caused by SRP, orbit-fixed frame $\Sigma^O$ and spin-free-fixed frame $\Sigma^{SF}$ are introduced. The orbit-fixed frame moves in accordance with the orbital motion. Its origin is at the center of mass of the spacecraft. $z$ axis is parallel to the sun direction, $y$ axis is perpendicular to the orbital plane, and $x$ axis forms a right-handed system. The spin-free-fixed frame is an extended one of a body-fixed frame, in which only $z$ axis is fixed to the spin axis of the system.

The effect of the SRP on the sail membrane can be divided into specular reflection, diffuse reflection, and absorption. The force exerted on a microelement of the sail membrane by each effect is written as follows.\(^3\)

\[
\begin{align*}
df_{\text{spe}} &= -2PC_{\text{spe}}(s \cdot n)\eta n ds \\
\left| df_{\text{dif}} \right| &= -PC_{\text{dif}}(s \cdot n)s + B_1(s \cdot n)n ds \\
\left| df_{\text{abs}} \right| &= -PC_{\text{abs}}(s \cdot n)s ds
\end{align*}
\]

Hence, the thrust and torque exerted on the spacecraft can be derived by the following equations.

\[
\begin{align*}
F_{\text{SRP}} &= \int (df_{\text{spe}} + df_{\text{dif}} + df_{\text{abs}}) \\
T_{\text{SRP}} &= \int \vec{r}_M \times (df_{\text{spe}} + df_{\text{dif}} + df_{\text{abs}})
\end{align*}
\]

In order to the attitude and orbital motion properly, the thrust is described in the orbit-fixed frame and the torque is described in the spin-free-fixed frame.

In the actual way, only 2nd order static wave is used for the attitude control out of higher order ones. This is because the shape control accuracy deteriorates as the wave order increases. Replacing the superscript $I$ in Eq. (45) with $SF$, the shape of the sail membrane in $\Sigma^{SF}$ is expressed as follow.

\[
w^{SF}(r, \theta) = w_1^{SF}(r, \theta) + w_2^{SF}(r, \theta)
\]

Normal vector of the sail microelement can be derived by the following equation.

\[
n = \frac{\partial r_M}{\partial r} \times \frac{\partial r_M}{\partial \theta}
\]

From Eqs. (48), (49) and (50), the thrust and torque are expressed as follows, respectively.

\[
\begin{align*}
F_{\text{SRP}}^{SF} &= \begin{bmatrix} f_1(\psi, \phi) + f_2(\psi, \phi) \frac{A_1}{r_0} \cos a_1 + f_3(\psi, \phi) \frac{A_3}{r_0} \sin a_1 \\
f_4(\psi, \phi) + f_5(\psi, \phi) \frac{A_1}{r_0} \cos a_1 + f_6(\psi, \phi) \frac{A_3}{r_0} \sin a_1
\end{bmatrix} \\
T_{\text{SRP}}^{SF} &= \begin{bmatrix} \tau_1 \phi - \tau_2 \frac{A_1}{r_0} \sin a_1 - \frac{A_3}{r_0} \sin(a_2 - a_1) \\
\tau_1 \psi + \tau_2 \frac{A_1}{r_0} \cos a_1 + \frac{A_3}{r_0} \cos(a_2 - a_1) - \tau_4 \frac{A_1}{r_0} \phi \sin a_1 + \phi \cos a_1
\end{bmatrix}
\end{align*}
\]

where

\[
\begin{align*}
\tau_1 &= (C_{\text{dif}} + C_{\text{abs}})PS_l \\
\tau_2 &= \gamma M(C_{\text{dif}} + C_{\text{abs}}) + (B_1 C_{\text{dif}} + C_{\text{spe}})PS_l \\
\tau_3 &= \frac{1}{2} \gamma (B_1 C_{\text{dif}} + C_{\text{spe}})PS_l \\
\tau_4 &= \gamma M(C_{\text{dif}} + C_{\text{abs}})PS_l
\end{align*}
\]

\[
\gamma_I = \frac{I_1}{I_1 + I_M}, \quad \gamma_M = \frac{I_M}{I_1 + I_M}
\]

Formulations of $f_i(\psi, \phi)$ through $f_6(\psi, \phi)$ are written in Eq. (55). Thus, the thrust and torque can be controlled via $A_1$, $A_2$, $a_1$, and $a_2$.

In order to verify the validity of the analytical derivation of the thrust and torque, numerical simulation is conducted by the MPM. In addition to the evaluation of the shape control accuracy, which was conducted in section 4.3., the force and torque derived by SRP are calculated and output. Figure 21 shows one of simulation results. As can be seen from the figure, mean values of the numerical simulation correspond to the analytical values. Therefore, the formulations of the thrust and torque
expressed in Eqs. (51) and (52), which is based on the simple theoretical model described in section 3., can be used to the attitude and orbit control of a spinning solar sail.

### 5.3. Optimization problem of attitude and orbital motion

As the summary of this study, an optimization problem of attitude and orbital motion of a spinning solar sail is solved. The analysis aims to achieve maximum acceleration/deceleration by utilizing the attitude and orbit control based on the shape control of the sail membrane. Initial position and velocity of the spacecraft are identical to those of Earth, and the initial attitude is the sun direction. For the simplicity, the orbital motion is limited in two dimensional one. The optimum control rule and the resulting attitude/orbital motion is calculated by a method called DCNLP (direct collocation with non-linear programming).

Control inputs are \( A_1, A_2, a_1, \) and \( a_2 \). Note that the propulsion performance (area / mass) of the solar sail used in this analysis is set to be better than that of IKAROS because the propulsion performance of IKAROS is not sufficient for the orbit control in realistic time span.

Figure 22 shows the optimization result of deceleration maneuver. Figure 22a confirms that the spacecraft is continuously decelerated. Figure 22b shows that the sail membrane keeps orienting toward a certain point. This point is the optimum direction to decelerate the spacecraft by photon propulsion. A noteworthy fact is that the orientation of the sail moves from the initial point to the optimum point almost linearly, despite the presence of disturbance. This owes to the small time constant of the 1st order static wave. The attitude control by the 2nd order static wave (i.e. canceling disturbance out) also works.

### 6. Conclusion

An active shape control method for a spinning membrane space structure is proposed. The applicability of the shape control method to the attitude and orbit control problem of a spinning solar sail is also presented. Deformation behavior of flexible structures such as membranes is so complex, and this study provides an effective solution to handle it. Use of flexible membrane structures in the space field is essential to the future space exploitation since they have many advantages to conventional space structures. The proposed shape control method gives an effective insight to the development of them.

In this study, the optimization problem of attitude and orbital motion is taken as one possible application of the shape control method. However, the method has various applicability to the deep space exploration missions.

### References