A density-based approach to the propagation of re-entry uncertainties

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Abstract

The present study proposes a methodology based on the use of the continuity equation in the analysis of re-entry scenarios under the presence of uncertainty in the initial conditions and parameters. The prediction of re-entry trajectories is an extremely complex task. The accuracy of these predictions is influenced by the uncertainty on many factors, such as the initial conditions, the ballistic coefficient, the modelling of the atmosphere, etc. Quantifying such uncertainties is important and can be very useful in multiple circumstances. Re-entering spacecraft and rocket bodies can pose risk for people and properties on the ground. Therefore, it is important to quantify such uncertainties as to properly assess the probability of impact on areas of interest and evaluate the risk on the ground.
The traditional procedure to assess uncertainties is based on Monte Carlo method, where through a large number of simulations over randomly sampled initial conditions, the joint PDF is estimated through a frequentist approach. However, Monte Carlo simulations are not particularly suited for re-entry uncertainty analyses as they require a large number of simulations. Consequently, for high-dimensional and non-linear dynamics such as the ones associated with re-entry scenarios they become tremendously expensive. On the other end, the continuity equation allows the propagation of the probability density along with the dynamics of the system thus obtaining a step-by-step evolution of the actual density distribution. While with MC methods we propagate individual realisation of the initial PDF, with a density-based approach we propagate the ensemble of realisations conserving the total probability mass in the phase space. As the propagation involves the actual values of the density in the phase space, the density-based method, allow for a lower number of simulations with respect to a MC method.
The paper studies the effects of uncertainties in the initial conditions and in the ballistic coefficient for the test case of a Mars re-entry. The exact quantification of the ballistic coefficient at re-entry can be difficult to compute with high accuracy as it depends on the mass, cross-section and drag coefficient of the spacecraft. Consequently, it has been considered interesting to study the evolution of a re-entry test case where both the initial conditions and the ballistic coefficient are subject to uncertainties.

Keywords: re-entry, uncertainty, continuity equation, interpolation, marginal density, density reconstruction

Introduction

The present study proposes a methodology based on the use of the continuity equation for the analysis of re-entry scenarios under the presence of uncertainty in the initial conditions and parameters.
The study and prediction of re-entry trajectories is an extremely complex task. The accuracy of these predictions is influenced by many factors, such as the initial conditions (in the form of re-entry velocity, altitude, flight path angle, etc.), the ballistic coefficient of the object, the

modelling of the atmosphere, the aerothermal heating experienced by the object, etc. Uncertainties are associated to these parameters; whether the uncertainties are related to the measurements or to the modelling methodology, they produce an effect on the evolution of the re-entry trajectory, which, in turns, translates into an uncertainty in the prediction of the re-entry corridor, impact location, and casualty area. Quantifying such uncertainties is important and can be very useful in multiple circumstances. Re-entering spacecraft and rocket bodies can pose risk to people and properties on the ground. Consequently, it is important to quantify uncertainties related to re-entry scenarios as to properly assess the probability of impact on areas of interest. In addition, the design of exploration probes on other planets such as Mars can strongly benefit from uncertainty analyses in order to improve the robustness of the design of the mission by enabling the assessment of landing footprint and impact location uncertainty [1].
The traditional procedure to assess uncertainties is a Monte Carlo-based dispersion analysis, where through a large number of simulations over randomly sampled initial conditions and parameters, the joint Probability Density Function (PDF) is estimated through a frequentist approach. Monte Carlo (MC) simulations are not particularly suited for uncertainty analysis as they can only estimate the values of the uncertainties. In addition, the results they provide are only as good as the number of simulations performed. In general, in fact, MC dispersion analyses require many simulations. Consequently, for high-dimensional and non-linear dynamics, such as the ones associated with re-entry scenarios, they become tremendously expensive.
On the other hand, the continuity equation allows the propagation of the probability density along with the dynamics of the system thus obtaining a step-by-step evolution of the actual density distribution [1]. Such a methodology directly compares to Monte Carlo simulations where, instead, the distribution is only approximated through a large number of realizations. While with MC methods we propagate individual realisation of the initial PDF, with a density-based approach we propagate the ensemble of realisations conserving the total probability mass in the phase space throughout the simulation. As the propagation involves the actual values of the density in the phase space, the density-based method, allow for a lower number of simulations with respect to a MC method.
The paper studies the effects of uncertainties in the initial conditions and in the ballistic coefficient of satellites for a Mars re-entry test case. The uncertainty in the ballistic coefficient is included as it can strongly influence the shape and evolution of the re-entry trajectory and of the landing location. In addition, the exact quantification of the ballistic coefficient at re-entry can be difficult to compute with high accuracy as it depends on the mass, cross-section and drag coefficient of the spacecraft. The mass of the spacecraft at disposal may be not completely known, as the exact amount of residual propellant can be difficult to assess. The cross-sectional area of the satellite may be also uncertain, as it is difficult to predict the exact motion of an uncontrolled satellite when it enters in contact with the upper layers of the atmosphere. In addition, even the exact value of the drag coefficient can be of difficult computation. Once the propagation of the uncertainties is performed, the results are used to compute the impact probability on ground and the dispersion of both the impact location and the footprint.

Application of the continuity equation to the re-entry problem

The proposed methodology applies the continuity equation to the propagation of the joint probability distribution representing the expression of the uncertainties related to the parameters of interest. Such an approach allows the propagation and prediction of the exact value of the uncertainty as a function of time. The expression for the continuity equations is as follows [2][3]

$$\frac{\partial n(\mathbf{x},t)}{\partial t} + \nabla \cdot \mathbf{f}(\mathbf{x}) = \dot{n}^{+} - \dot{n}^{-} \qquad 1$$

where $\nabla \cdot \mathbf{f}$ represents the forces acting on the system and takes into account slow, continuous phenomena such as gravity and atmospheric drag, and $\dot{n}^+ - \dot{n}^-$ represents the fast and discontinuous events (i.e. sources and sinks). For the case in exam, the source and sink terms have been neglected. Knowing the initial probability density distribution *n*, Eqn. 1 allow for the propagation of the density evolution with time [1]. This is a Partial Differential Equation (PDE) with the joint probability distribution function (PDF) $n(\mathbf{x}, t)$ being the dependent variable. Such an equation regulates the conservation of the total probability mass of the joint PDF in space and time due to the forces acting on the system. We solve Eqn. 1 using the Method of Characteristics (MOC) [4] by transforming the PDE into a system of ordinary differential equations (ODEs). Expressing the divergence in rectangular coordinates, the problem can be expressed in the phase space of the relevant parameters. For the general case of *m* variables, Eqn. 1 can be transformed into the following system of ODEs by applying the MOC.

$$\begin{cases} \dfrac{dt}{ds} = 1 \\ \dfrac{d\alpha_1}{ds} = v_{\alpha_1}(\alpha_1, \dots, \alpha_m) \\ \vdots \\ \dfrac{d\alpha_m}{ds} = v_{\alpha_m}(\alpha_1, \dots, \alpha_m) \\ \dfrac{dn}{ds} = -\left[\dfrac{\partial v_{\alpha_1}}{\partial \alpha_1} + \cdots, \dfrac{\partial v_{\alpha_m}}{\partial \alpha_m}\right] \cdot n(\alpha_1, \dots, \alpha_m) \end{cases} \quad (2)$$

where *s* is the independent variable, in this case the time *t*. The application of this methodology to the re-entry problem consists in specifying all the v_{α_m} and their partial derivatives, which describe the dynamics of the problem. For the present work, a three-state re-entry model has been adopted, which describes the evolution of the altitude (*h*), velocity (*v*), and flight path angle (*γ*) under the influence of gravity and atmospheric drag.

$$\begin{aligned} \dot{h} &= v \cdot sin(\gamma) \\ \dot{v} &= -\frac{1}{2} \cdot \frac{\rho}{\beta} \cdot v^2 - g \cdot \sin(\gamma) \\ \dot{\gamma} &= \frac{v \cdot \cos(\gamma)}{R_p + h} - \frac{g}{v} \cdot \cos(\gamma) + \frac{1}{2} \cdot \frac{\rho}{\beta} \cdot \frac{C_L}{C_D} \cdot v \end{aligned} \quad (3)$$

where ρ is the atmospheric density, g is the gravitational acceleration, β is the ballistic coefficient, R_p is the radius of the planet, C_L is the lift coefficient, and C_D is the drag coefficient. For the modelling of the atmosphere an exponential model has been adopted, while for the gravitational acceleration an inverse square model has been implemented [6]. In addition, for the present work, uncertainties are also considered for the ballistic coefficient (β) so that its contribution must be included in the dynamics described with the MOC. For this particular case, the ballistic coefficient is assumed to remain constant throughout the re-entry process so that its derivative is zero. Applying the MOC to the re-entry problem thus gives the following system of ODEs

$$\begin{cases} \dfrac{dt}{ds} = 1 \\ \dfrac{dh}{ds} = v \cdot sin(\gamma) \\ \dfrac{dv}{ds} = -\dfrac{1}{2} \cdot \dfrac{\rho}{\beta} \cdot v^2 - g \cdot \sin(\gamma) \\ \dfrac{d\gamma}{ds} = \dfrac{v \cdot \cos(\gamma)}{R_p + h} - \dfrac{g}{v} \cdot \cos(\gamma) + \dfrac{1}{2} \cdot \dfrac{\rho}{\beta} \cdot \dfrac{C_L}{C_D} \cdot v \\ \dfrac{d\beta}{ds} = 0 \\ \dfrac{dn}{ds} = \left[\dfrac{\rho}{\beta} \cdot v + \sin(\gamma) \cdot \left(\dfrac{v}{R_p + h} - \dfrac{g}{v}\right)\right] \cdot n(h, v, \gamma, \beta) \end{cases} \quad (4)$$

In general, it is difficult to find an analytical solution for Eqn. 4 (except for the trivial case of horizontal flight [1]), and the system has to be numerically integrated. The integration can be performed using a standard ODE solver such as the Runge-Kutta method with a specified time step. In such fashion, the time evolution of the density in the considered phase space can be obtained. As the solution for the re-entry problem is not analytical, it is necessary to sample the initial density distribution and to integrate the system of Eqn. 4 for each one of the sampled points. An important aspect that must be underlined is that with the presented method the value of the uncertainty correspondent to each sample is the actual uncertainty value and not an estimation of it, as it would be the case for a Monte Carlo based method.

Methodology for the density reconstruction and extraction of the marginal densities

Once the propagation is performed and the resulting coupling between the state of the satellite and the uncertainty density is obtained at each time step for all the initial samples, it is necessary to develop a methodology for the reconstruction of the density at each point in the domain defined by the phase space of the problem. The developed methodology consists in linearly interpolating between the sampled points by using Delaunay triangulations [7]. The first step is to generate the triangulation between the sampled points. Using the generated simplices a linear interpolation in an n-dimensional space can be carried out: for each interpolations point, the algorithm check to which simplex of the triangulation it belongs, then a linear interpolation with the density values associated with the vertices of the simplex is performed [8]. It is important here to highlight that during the re-entry evolution, not only the density changes but also the phase-space volume changes and deforms. In particular, the rate of change of the phase-space volume is related to the divergence of the vector field **f** (last line of Eqn. 4). Consequently, it is not only important the value of the density at a specific point but also the volume of the simplex to which it is associated. In the present work, the density reconstruction has been performed interpolating at every barycentre of the simplices generated by the Delaunay triangulation. To refine the density reconstruction, a recursive triangulation followed by a barycentric linear interpolation can be carried out. With this procedure, to each interpolated point, is also associated the respective volume of the simplex associated with the barycentre, so that the density can be properly reconstructed for the entire phase space volume.

Results

To present the application of the described density-based approach to the re-entry of satellites, two test cases have been selected. The chosen test case refers to the re-entry of the Pathfinder probe [10].

Table 1. Nominal initial conditions for the two re-entry scenarios considered.

Initial Conditions	Pathfinder
Altitude (km)	125
Velocity (km/s)	7.6
Flight Path Angle (deg)	-13.8
Ballistic Coefficient (kg/m^2)	62

After specifying the nominal initial conditions for the re-entry simulations, the relative uncertainties need to be specified. For the purpose of this work, a uniform uncertainty having a 10% variation from the nominal value has been considered for the three initial conditions (altitude, velocity, and flight path angle) and for the ballistic coefficient. The sampling of the initial conditions has been carried out using an algorithm based on the Halton sequence [11].
In general, re-entries are difficult to predict, especially in the latest stages, as the phenomenon is dependent from many factors, such as the solar activity, the atmospheric density, the attitude of the spacecraft, etc. Consequently, considering the uncertainties characterising the initial conditions at re-entry is a very important aspect of re-entry predictions. In fact, such the initial conditions affect the landing site of the spacecraft and the thermal and mechanical loads it is exposed to.

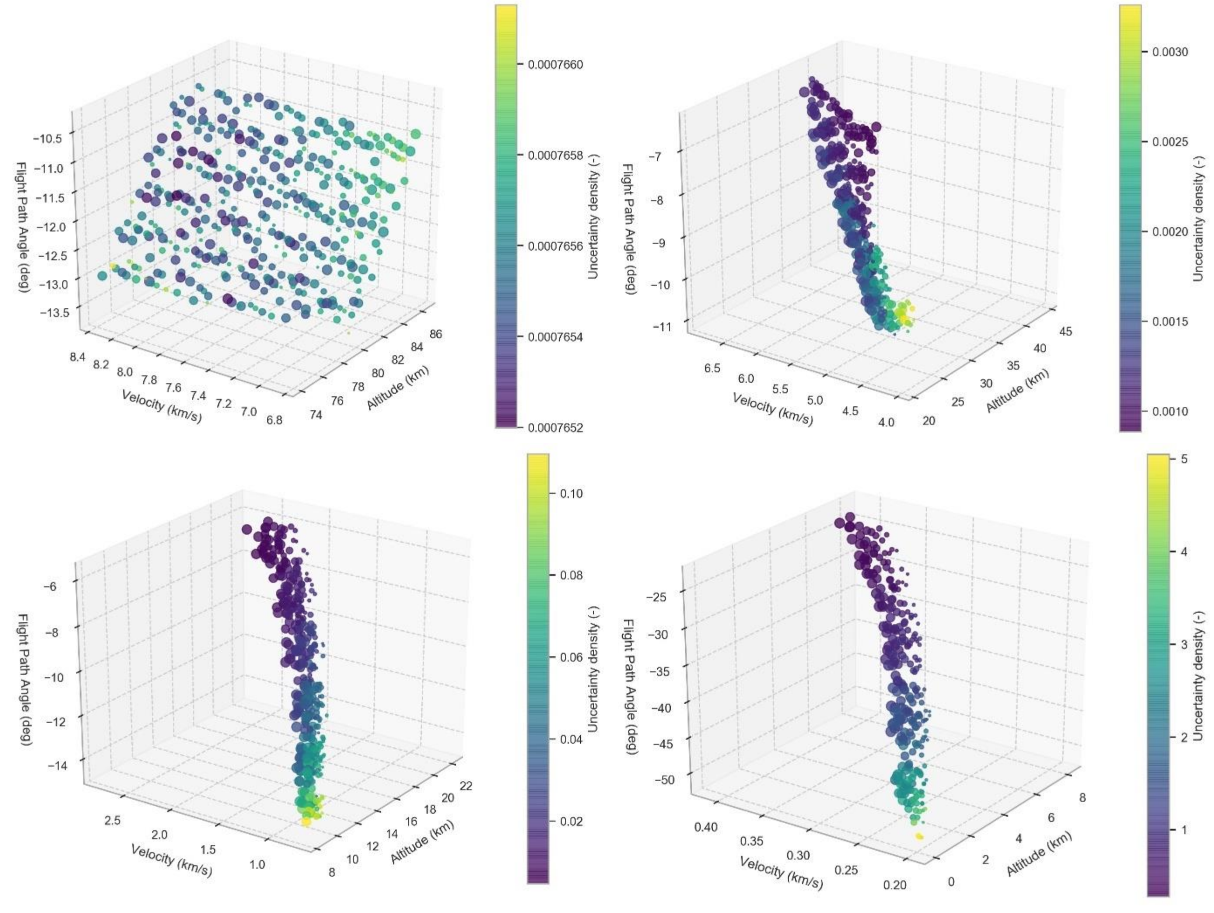

Figure 1: Scatter plot of the uncertainty evolution *for the test* case for the sampled initial conditions and parameters. The top-left plot represents the re-entry after 18 s, the top-right after 56 s, the bottom-left after 90 s, and the bottom-right after 162 s.

Alongside the initial conditions, a relevant parameter has been considered source of uncertainty that is the ballistic coefficient. In fact, it is usually difficult to effectively and precisely determine the ballistic coefficient of a satellite as it depends on its attitude and on the atmospheric conditions (through the drag coefficient). At the end-of-life of a spacecraft, its attitude may be known if the satellite is still controlled and operational; however, in other cases, the satellite is not operational anymore and its attitude cannot be controlled or known a priori. Similarly, the drag coefficient usually comes from engineering relations and it is estimated reducing the spacecraft structure to that one of simple geometrical objects [9] (e.g. boxes and cylinders). Consequently, the uncertainty related to it can be significant and should be considered.

Figure 1 shows the results for the re-entry simulation with 5000 initial samples. The images represent the evolution of the re-entry and of the uncertainties associated with it in the phase space of the problem (i.e. altitude, velocity, and flight path angle). The variation of the ballistic coefficient is represented by the size of the points in the scatter plot. The larger the point, the bigger is the ballistic coefficient. The four plots represent the evolution of the re-entry under uncertainties at different time instant: the top-left plot shows the uncertainties after 18 s, the top-right after 56 s, the bottom-left after 90 s, and the bottom-right after 162 s. As it is possible to observe from Figure 1, the volume of the considered phase-space starts as relatively compact and maintains this property even after 80 s from the beginning of the re-entry phase. However, as the re-entry progresses, the volume of the phase-space stretches and elongates. We mention this characteristic of the phase-space as it is important in the reconstruction of the density at each time step and in the computation of the marginal probabilities associated to the re-entry parameters.

The marginal probabilities are of interest for the re-entry simulations as they explicitly allow the visualization of the likelihood that a spacecraft has of being in a specific state. This state is represented by its position in the phase-space, i.e. altitude, velocity, and flight path angle but is not limited to it. In fact, the same information can be extracted for other useful quantities such as the impact range of the satellite that is its likelihood of landing in a certain location, or the heat load it must sustain during the different phases of the re-entry.

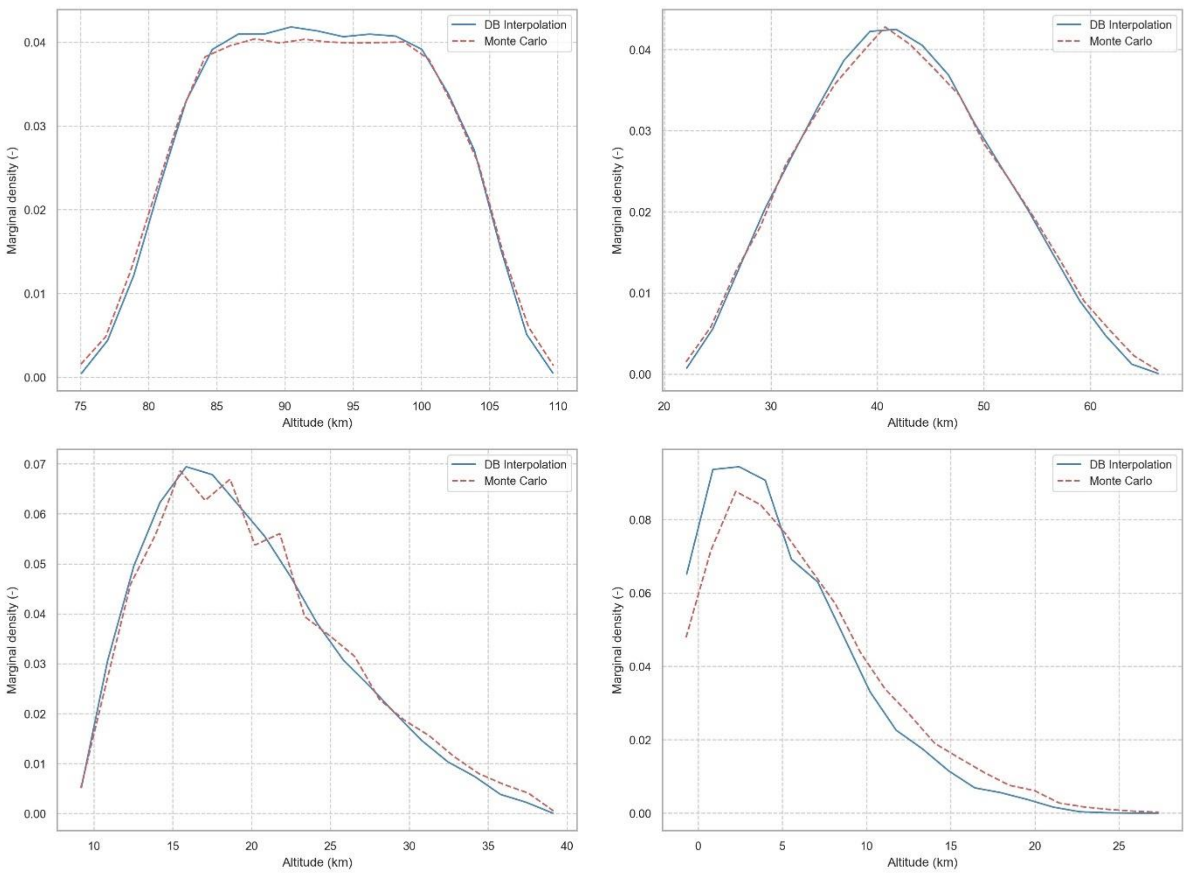

Figure 2: One-dimensional uncertainty marginal of the altitude. The top-left plot represents the re-entry after 18 s, the top-right after 56 s, the bottom-left after 90 s, and the bottom-right after 162 s. Comparison between the density-based triangulation (DB triangulation in the legend) and the Monte Carlo binning.

As these quantities are influenced by the state of the satellite, the uncertainty in the re-entry state and parameters propagates also on them. For example, knowing the uncertainty distribution of the heat load or the mechanical load on the satellite can help extracting the probability distribution of the altitude at which the satellite breaks-up. The plots of Figure 2 show examples of the one-dimensional marginals that can be extracted by first performing the interpolation on the scattered data and then integrating. The distribution obtained with the density-based method (blue line) is compared to the one obtained through Monte Carlo binning (red line). From the marginals in Figure 2, it is possible to observe that the distribution obtained through the density-based approach and the reconstruction through triangulation closely matches the one obtained with a Monte Carlo approach.

Figure 3 shows a marginal for the test case, which represents the probability distribution of the landing range of the spacecraft. The distribution represents the likelihood that the spacecraft would impact after the specified range. Both the density-based and the Monte Carlo methods predict that the peak probability corresponds to a landing range of 650 km. it can be noted that the overall range span is quite large, despite the steep re-entry angle, ranging from 500 km to 1000 km. Figure 3 clearly shows how the uncertainties in the initial conditions and parameters may significantly affect the re-entry of an interplanetary probe. As the finding and targeting a suitable impact location for the probe is one of the most important tasks, it is here clear the importance of considering the uncertainties when designing the re-entry phase.

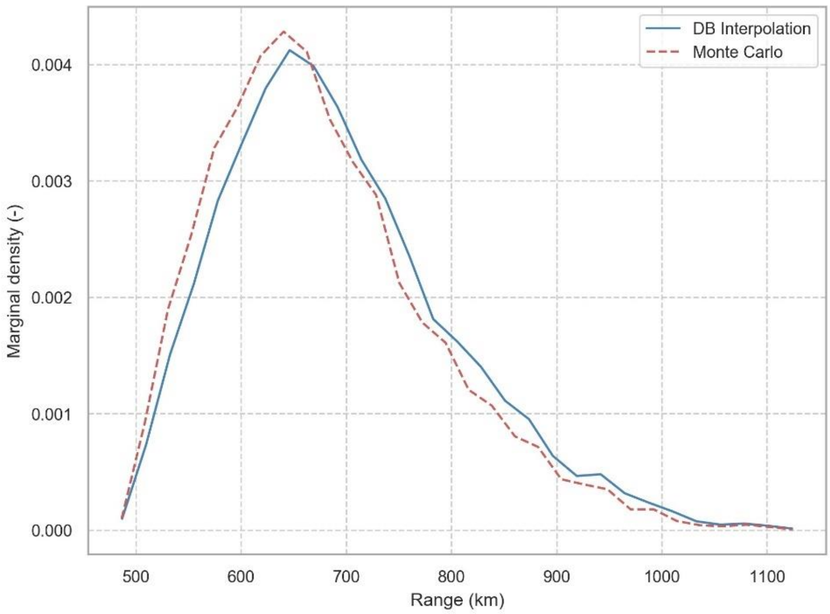

Figure 3: One-dimensional marginal for the landing range.

Conclusions ad future work

The presented work outlines a methodology to evaluate the effect of uncertainties in the re-entry of satellites without relying on computationally expensive Monte Carlo simulations, which become statistically meaningful only for high number of samples. The described methodology, instead, by directly propagating the probability density together with the equations of motion of the re-entry, allows the computation of the evolution of the actual uncertainty value for each of the sampled considered. Consequently, while the Monte Carlo approach estimates the uncertainties, the presented approach computes their actual values, thus allowing the reduction of the number of samples to be taken in order to have a statistically meaningful result. However, as we are interested in reconstruction the probability density and its marginal distributions on a thinner grid than the one used for the initial sampling, it is necessary to introduce a methodology that allows the computation of the density value in between the sampled points. This is achieved by linearly interpolating the set of scattered points obtained from the integration over the phase-space volume. With such a procedure, it is possible to extract the value of the density at any given point. For the purpose of this work, two test cases have been presented and the density-based method has been compared to the corresponding Monte Carlo method considering uncertainties in the initial conditions and in the ballistic coefficient of the satellites. This demonstrates the flexibility of the methodology, which allows the inclusion of both initial conditions and parameters of the re-entry as a source of uncertainty. The obtained results have been expressed in terms of one-dimensional distribution of the relevant variables, comparing the developed methodology with the equivalent results of a Monte Carlo based binning. Alongside the probability distributions for the standard re-entry variables (altitude, velocity, and flight path angle), even the uncertainties related to other important quantities in the re-entry process, such as the landing range have been presented.

The presented methodology can be very powerful and, if carefully developed and tuned, can be extend to more dimensions (the presented case has four) including a complete six-state re-entry integration and the uncertainty of other parameters. Its possibility to significantly reduce the number of samples with respect to equivalent Monte Carlo techniques for uncertainty estimation has a great appeal, as well as the fact that what we obtain from the simulations are the actual values of the uncertainties and not an estimation. For the future development of the methodology, the first step is to build a reliable algorithm that can isolate the alpha-shape of the set of points even when the volume of the phase-space is highly deformed. Such an algorithm must also support n-dimensional spaces, as the other main objective of the methodology is to include as many dimensions as possible by considering the uncertainties in the initial conditions and in the parameters of the re-entry.

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