A Control Theoretical Analysis of Formation Flight with Inter-satellite Lorentz Forces

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Abstract

Chief-deputy Lorentz Formation Flight (LFF) is a novel concept of formation flight in which an electrically-charged deputy satellite moves around a chief satellite equipped with electromagnetic coils. A steerable inter-satellite Lorentz force is thus created to control the relative motion. This work investigates the LFF problem from a nonlinear control perspective. The relative motion is described by a set of Clohessy-Wiltshire-like equations. A necessary and sufficient condition for the existence of two-dimensional plane motions is obtained. With the help of spherical coordinates, the plane motion is reformulated as an underactuated nonlinear control system in which the deputy's charge is the control variable. The tool of geometric control theory is used to analyse the controllability/accessibility issues of the system. It is found that LFF in the current scenario is not controllable. However, the existence of special periodic orbits shows to be promising in space applications.

Keywords: Formation Flight, Inter-satellite Lorentz Force, Nonlinear Controllability, Geometric Control, Invariant Motion.

I. Introduction

According to classical electrodynamics, an electrically-charged particle moving in a magnetic field experiences the Lorentz force [1]. Natural spacecraft charging at high altitudes has been observed [2,3]. The Lorentz force experienced by a charged satellite in magnetosphere fields of planets can be used to control satellite motion in a propellantless manner [4,5]. Recently this idea has been extended to develop a novel formation flight concept, namely, Lorentz Formation Flight (LFF) [6,7].

In this work, we mainly consider LFF with two satellites, namely, chief/deputy satellites. The chief satellite is equipped with electromagnetic coils to produce an electromagnetic field. The deputy spacecraft is electrically charged while moving in the vicinity of the chief [6]. A steerable intersatellite Lorentz force can thus be obtained to control the relative motion.

Though inter-satellite Lorentz force provides a promising concept of fuel-free formation flight, controlling LFF is quite challenging. The relative motion between LFF satellites is highly coupled. In addition, the system is always underactuated. Previous work mainly devotes efforts to find special formation configurations by analysing the dynamics behaviour [6,7]. Fundamental control issues, such as the controllability [8], reachability set [9], so far have not been discussed. It is still largely unknown that to what extent of control the Lorentz force is able to provide. Therefore, it is desirable to understand the system from a control-theoretical perspective. The current work takes the two-satellite LFF as an example. LFF is viewed as a control system, in which the charge level of the deputy is the control. The theoretical control issues such as the controllability, accessibility, feedback linearizability will be discussed.

To that end, a set of Clohessy-Wiltshire-like equations is developed to describe the relative motion. Due to coupling and nonlinearity of dynamics, the possibility of existence of simpler motion is first investigated. A necessary and sufficient condition for the existence of a two-dimensional plane motion is obtained. This type of special motion is then reformulated in a control-affine form by using spherical coordinates. The new dynamics that is an underactuated nonlinear control system is analysed by using the tool of geometric control theory [10,11].

II. Preliminaries

A. Magnetic Field and Lorentz Force

A magnetic dipole produces a spatial magnetic field. A vector potential is usually used to describe the distribution of the field [1]:

$$\mathbf{A} = \frac{\boldsymbol{\mu} \times \mathbf{r}}{r^3} \tag{1}$$

where $\boldsymbol{\mu}$ is the magnetic momentum of the dipole, $\mathbf{r}$ is the position vector from the dipole to the point of interests, and $r = |\mathbf{r}|$. Generally, $\mathbf{A}$ is a function of both time and position.

An electric charge q moving through a magnetic field is affected by the Lorentz force. The force is given by:

$$\mathbf{F} = q\mathbf{E} + q\mathbf{v} \times \mathbf{B} \tag{2}$$

where $\mathbf{E}$ is the electric field and $\mathbf{B}$ is the magnetic field. In the absence of an external electric field, the two fields are determined solely by $\mathbf{A}$:

$$\mathbf{E} = \frac{-\partial \mathbf{A}}{\partial t}, \quad \mathbf{B} = \nabla \times \mathbf{A} \tag{3}$$

B. Geometric Control Theory[11]

Let $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ be two vector fields defined on a an n-dimensional differential manifold $\mathcal{M}$. The ad operator is iteratively defined by $\mathrm{ad}_{\boldsymbol{\alpha}}\boldsymbol{\beta} = \mathrm{ad}^1_{\boldsymbol{\alpha}}\boldsymbol{\beta} = [\boldsymbol{\alpha}, \boldsymbol{\beta}], \mathrm{ad}^{k+1}_{\boldsymbol{\alpha}}\boldsymbol{\beta} = \left[\boldsymbol{\alpha}, \mathrm{ad}^k_{\boldsymbol{\alpha}}\boldsymbol{\beta}\right]$, where $[\boldsymbol{\alpha}, \boldsymbol{\beta}]$ is the *Lie bracket*. A *distribution* Δ on a manifold $\mathcal{M}$ is a subspace of the tangent space $\Delta \subset T_x M$ for $\forall \mathbf{x} \in \mathcal{M}$. A *distribution* Δ is invariant under a vector field $\boldsymbol{\alpha}$ if $\forall \boldsymbol{\tau} \in \Delta \Rightarrow [\boldsymbol{\alpha}, \boldsymbol{\tau}] \in \Delta$. Furthermore, Δ is involutive if $\forall \boldsymbol{\alpha}, \boldsymbol{\beta} \in \Delta, [\boldsymbol{\alpha}, \boldsymbol{\beta}] \in \Delta$. Let Δ be a nonsingular distribution generated by vector fields $\boldsymbol{\alpha}_1, \cdots, \boldsymbol{\alpha}_r$. Then Δ is said to be *integrable* if there exist $n-r$ independent function $\lambda_1(\mathbf{x}), \cdots, \lambda_{n-r}(\mathbf{x})$ such that

$$\frac{\partial \lambda_j(\mathbf{x})}{\partial \mathbf{x}} \boldsymbol{\alpha}_i(\mathbf{x}) = 0, \forall i \in \{1, \cdots, r\}, \forall j \in \{1, \cdots, n-r\} \tag{4}$$

We consider the following control-affine system:

$$\dot{\mathbf{x}} = \mathbf{f} + \sum_{i=1}^{m} u_i \mathbf{g}_i \tag{5}$$

where $\mathbf{f}, \mathbf{g}_i \ (i = 1, \cdots, m)$ are smooth vector fields defined on an n-dimensional differential manifold $\mathcal{M}$, and $\mathbf{u} = [u_1, \cdots, u_m]^{\mathrm{T}} \in \mathbf{R}^m$ is the control.

Let $\mathcal{R}_t(\mathbf{x}_0) = \cup \mathcal{R}_{\tau \le t}(\mathbf{x}_0, \tau)$, where $\mathcal{R}(\mathbf{x}_0, t)$ is the *reachable set* from $\mathbf{x}_0$ at time *t*. The system is said to be **accessible** from $\mathbf{x}_0$ if $\mathcal{R}_t(\mathbf{x}_0)$ contains a non-empty open subset of $\mathcal{M}$.

Furthermore, a system is said to be **strongly accessible** from $\mathbf{x}_0$ if $\mathcal{R}(\mathbf{x}_0,t),\forall t>0$ contains a non-empty interior of $\mathcal{M}$. A system is said to be **controllable** from $\mathbf{x}_0$ if $\mathcal{R}_t(\mathbf{x}_0)=\mathcal{M}$. These notations can be also defined with respect to $\forall\mathbf{x}_0\in\mathcal{M}$, after which the **system** is called accessible, strongly accessible, and controllable, respectively.

The *accessibility algebra* $\mathcal{C}$ of system (5) are the linear combinations of repeated Lie bracket of the form $\left[\mathbf{w}_k,\left[\mathbf{w}_{k-1},\left[\cdots\left[\mathbf{w}_i,\mathbf{w}_j\right]\cdots\right]\right]\right]$, where $\mathbf{w}_i$ is a vector in the set $\{\mathbf{f},\mathbf{g}_1,\cdots,\mathbf{g}_m\}$. The *accessibility distribution* Δ_C is then $\Delta_C=\operatorname{span}\{\mathbf{w}\mid\mathbf{w}\in\mathcal{C}\}$. Let $\mathcal{C}_0$ be the *smallest* algebra that contains $\{\mathbf{g}_1,\cdots,\mathbf{g}_m\}$ and satisfies $[\mathbf{f},\mathbf{w}]\in\mathcal{C}_0,\forall\mathbf{w}\in\mathcal{C}_0$. Define the corresponding distribution as $\Delta_{C_0}=\operatorname{span}\{\mathbf{w}\mid\mathbf{w}\in\mathcal{C}_0\}$. Then $\mathcal{C}_0$ and Δ_{C_0} are called *strong accessibility algebra* and *strong accessibility distribution*, respectively.

For system (5), we have the following important theorems [11]:
Theorem 1. (*Frobenius*): A constant-dimensional distribution is integrable <u>if and only if</u> it is involutive.
Theorem 2. (*Strong Accessibility Rank Condition*): A <u>sufficient condition</u> is $\dim\Delta_{C_0}=n$.

An important tool in nonlinear control is feedback linearization. We consider a single-input system

$$\dot{\mathbf{x}}=\mathbf{f}+\mathbf{g}u \tag{6}$$

Theorem 3. (*Feedback Linearizable[12]*) The system (6) is feedback linearizable <u>if and only if</u> $\operatorname{rank}(G)=n$ and D is involutive, where

$$\begin{cases} G=\left[\mathbf{g},\mathrm{ad}_{\mathbf{f}}\mathbf{g},\cdots,\mathrm{ad}_{\mathbf{f}}^{n-1}\mathbf{g}\right] \\ D=\operatorname{span}\left\{\mathbf{g},\mathrm{ad}_{\mathbf{f}}\mathbf{g},\cdots,\mathrm{ad}_{\mathbf{f}}^{n-2}\mathbf{g}\right\} \end{cases}$$

III. Lorentz-Augmented CW Equations

A. System Description

We consider a formation of two satellite: the chief and the deputy. A constant magnetic dipole with magnetic moment is installed on the chief; and the deputy is charged. The deputy moving in the vicinity of the chief experiences a Lorentz force. Since the Lorentz force is an internal force that doesn't affect the motion of the barycenter, the motion of the barycenter follows a Keplerian orbit. We can then define a LVLH reference frame on the barycenter and we will study the relative motion of the deputy with respect to the barycenter. In the following, we assume the barycenter moves on a circular orbit.

Let $\mathbf{r}$ and $\mathbf{v}$ be the position and velocity of the deputy satellite in the LVLH frame. Then, the relative motion between the chief and the deputy is $k\mathbf{r}$ and $k\mathbf{v}$, where $k=(m_\mathrm{C}+m_\mathrm{D})/m_\mathrm{C}$ is the mass ratio. The motion of the deputy can be modelled by the classical CW equations:

$$\begin{cases} \dot{\mathbf{r}}=\mathbf{v} \\ \dot{\mathbf{v}}=A_1\mathbf{r}+A_2\mathbf{v}+\mathbf{u} \end{cases} \tag{7}$$

where $\mathbf{u}_L$ is the Lorentz acceleration, A_1 and A_2 are constant CW equations matrices [13].

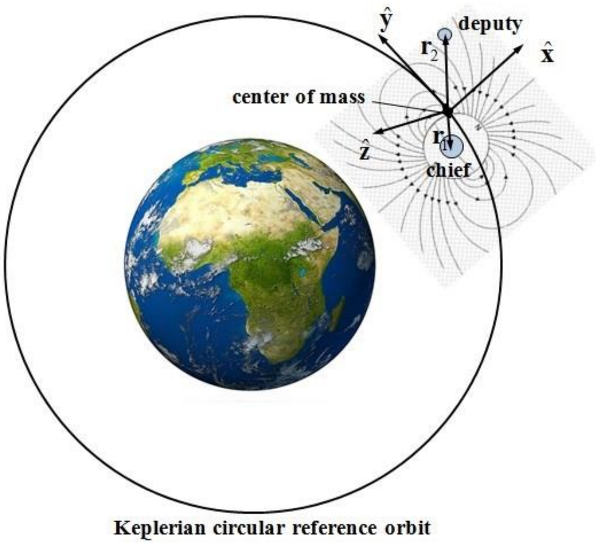


Fig. 1: Lorentz Formation Flight in the Barycenter LVLH frame[6]

In this work, we ignore the geomagnetic Lorentz force. The magnetic vector potential produced by the artificial magnetic moment at the deputy's position is

$$\mathbf{A} = \frac{\boldsymbol{\mu} \times \mathbf{r}}{k^2 r^3} \tag{8}$$

The magnetic flux density and the electric field are respectively

$$\mathbf{B} = \frac{1}{k^3 r^3}\left[3(\boldsymbol{\mu}\cdot\mathbf{r})\mathbf{r} - \boldsymbol{\mu}\right], \quad \mathbf{E} = -\frac{\dot{\boldsymbol{\mu}} \times \mathbf{r}}{k^2 r^3} \tag{9}$$

The acceleration produced by the Lorentz force is

$$\mathbf{u} = \frac{q}{m_{\mathrm{D}} k^2 r^3}\left[-k\dot{\boldsymbol{\mu}} \times \mathbf{r} + \mathbf{v} \times \left(3(\boldsymbol{\mu}\cdot\mathbf{r})\mathbf{r} - \boldsymbol{\mu}\right)\right] \tag{10}$$

In the following, we will replace q by $Q = q/\left(m_{\mathrm{D}} k^2\right)$ as a normalized charge.

As a time-varying magnetic field may induce a complicate electric field, we only consider the case with constant magnetic moment. Now the Lorentz acceleration is

$$\mathbf{u}_L = \frac{Q}{r^3}\left[(\mathbf{v} - \boldsymbol{\omega} \times \mathbf{r}) \times \left[3(\boldsymbol{\mu}\cdot\mathbf{r})\mathbf{r} - \boldsymbol{\mu}\right]\right] \tag{11}$$

In order to perform control-theoretical analysis, we rewrite it as the following control-affine form dynamics with Q being the control input:

$$\dot{\mathbf{x}} = \mathbf{f} + \mathbf{g}Q \tag{12}$$

where $\mathbf{f}$ and $\mathbf{g}$ are

$$\mathbf{f} = \begin{bmatrix} \mathbf{v} \\ A_1\mathbf{r} + A_2\mathbf{v} \end{bmatrix} \triangleq \begin{bmatrix} \mathbf{v} \\ \mathbf{u}_{\mathrm{cw}} \end{bmatrix}, \mathbf{g} = \begin{bmatrix} 0 \\ \dfrac{(\mathbf{v} - \boldsymbol{\omega} \times \mathbf{r}) \times \left[3(\boldsymbol{\mu}\cdot\mathbf{r})\mathbf{r} - \boldsymbol{\mu}\right]}{r^3} \end{bmatrix} \tag{13}$$

B. Plane Motion

Due to the complicity of the dynamics, it is desirable to examine the existence of simpler motions, such as invariant motion on a specific plane. We first notice a necessary and sufficient condition for the existence of a plane motion:

$$\dot{\mathbf{v}} \cdot (\mathbf{r} \times \mathbf{v}) \equiv 0 \tag{14}$$

After substituting the dynamics, the condition becomes a linear equation regarding Q:

$$c_0 + c_1 Q \equiv 0 \tag{15}$$

where c_0 and c_1 are coefficient functions:

$$\begin{cases} c_0 = \mathbf{u}_{\text{cw}} \cdot (\mathbf{r} \times \mathbf{v}) \\ c_1 = \dfrac{\left[(\mathbf{v} - \boldsymbol{\omega} \times \mathbf{r}) \times (3(\boldsymbol{\mu} \cdot \mathbf{r})\mathbf{r} - \boldsymbol{\mu})\right] \cdot (\mathbf{r} \times \mathbf{v})}{r^3} \end{cases} \tag{16}$$

It is necessary that $c_0 \equiv 0$ and $c_1 \equiv 0$. The Equation $c_1 \equiv 0$ yields:

$$\boldsymbol{\mu} \cdot \mathbf{r} = 0, \boldsymbol{\mu} \cdot \mathbf{v} = 0 \tag{17}$$

The Equation $c_0 \equiv 0$ yields

$$(3nxy + 2v_x x + 2v_y y)v_z + (nv_x y - 4nv_y x - 2v_x^2 - 2v_y^2)z = 0 \tag{18}$$

which then reduces to the following

$$v_y \equiv -2nx, y \equiv \frac{2v_x}{n} \quad \text{or} \quad z = v_z \equiv 0 \tag{19}$$

The first case in Eq. (19) is trivial since it is the CW periodic condition and it further yields $\boldsymbol{\mu} = 0$. The second case in Eq. (19) allows for a nontrivial solution

$$\mu_x = \mu_y = 0, \mu_z \neq 0 \tag{20}$$

It is well known that the periodic relative motion in CW equations is a spatial ellipse. However, only the plane motion on the $x-y$ plane is preserved when the Lorentz force is involved. In the following context, we mainly consider the $x-y$ plane case.

IV. X-Y Plane Motion Analysis

A. Existence of A First Integral

To facilitate analysis, we change the state variable from Cartesian coordinates (x, y, z) to spherical coordinates (l, α, β) by

$$\begin{cases} x = l\cos\alpha\cos\beta \\ y = l\cos\alpha\sin\beta \\ z = l\sin\alpha \end{cases} \tag{21}$$

where l is the relative distance, α is the elevation measured from the $x-y$ plane and β is the azimuth measured from the x axis. The new full dynamics are

$$\begin{cases} \ddot{l} = l\left\{\left[3\cos^2\beta + (\dot{\beta}+1)^2\right]\cos^2\alpha + \dot{\alpha}^2 - 1\right\} - \dfrac{\mu_z \dot{\beta}\cos^2\alpha}{l^2}Q \\ \ddot{\alpha} = \dfrac{\left[3\cos^2\beta + 3(\dot{\beta}+1)^2\right] l\sin\alpha\cos\alpha - 2\dot{l}\dot{\alpha}}{l} - \dfrac{\mu_z \dot{\beta}\sin 2\alpha}{l^3}Q \\ \ddot{\beta} = \dfrac{\left[3l\sin\beta\cos\beta - 2\dot{l}(\dot{\beta}+1)\right]\cos\alpha + 2\dot{\alpha}(\dot{\beta}+1)l\sin\alpha}{l\cos\alpha} + \dfrac{\mu_z(2\dot{\alpha}l\sin\alpha + \dot{l}\cos\alpha)}{l^4\cos\alpha}Q \end{cases} \tag{22}$$

For the plane motion at hand, $\alpha \equiv 0$. Let $\alpha \equiv 0, \dot{\alpha} \equiv 0$, $\dot{l} = L$ and $\dot{\beta} = h$. We have the four-dimensional in-plane motion dynamics

$$\dot{\mathbf{x}}_p = \mathbf{f}_p + \mathbf{g}_p Q \tag{23}$$

where

$$\mathbf{f}_p = \begin{bmatrix} L \\ h \\ l\left(3\cos^2\beta + h^2 + 2h\right) \\ -\dfrac{3l\sin\beta\cos\beta + 2L\left(1+h\right)}{l} \end{bmatrix}, \mathbf{g}_p = \begin{bmatrix} 0 \\ 0 \\ -\dfrac{h}{l^2} \\ \dfrac{L}{l^4} \end{bmatrix}$$

The subscript p will be omitted in the following context.

In the following we will use differential geometry tools to investigate the integrability of motion. For system (23), we construct two distributions as follows:

$$\begin{cases} P = \text{span}\left\{\mathbf{g}, \text{ad}_{\mathbf{f}}\mathbf{g}, \text{ad}_{\mathbf{f}}^2\mathbf{g}\right\}, \\ R = \text{span}\left\{\mathbf{f}, \mathbf{g}, \text{ad}_{\mathbf{f}}\mathbf{g}, \text{ad}_{\mathbf{f}}^2\mathbf{g}\right\} \end{cases} \tag{24}$$

It can be shown both P and R are invariant under $\mathbf{f}$ and $\mathbf{g}$; and $P = R$. According to Theorem 1, the system is integrable. Because $\dim(P) = \dim(R) = 3 = 4-1$, one integral can be found. Let the integral be $\lambda(l, \beta, L, h)$, then λ is the solution to the following PDE:

$$\nabla\lambda^{\mathrm{T}} P = 0^{\mathrm{T}} \tag{25}$$

where

$$\nabla\lambda = \left[\frac{\partial\lambda}{\partial l}, \frac{\partial\lambda}{\partial\beta}, \frac{\partial\lambda}{\partial L}, \frac{\partial\lambda}{\partial h}\right]^{\mathrm{T}}$$

Solving the PDE yields the following general solution of λ:

$$\lambda = \frac{l^2}{2}\left(2h^2 - 3\cos 2\beta - 3\right) + L^2 \tag{26}$$

The existence of integral (26) is a result of time-invariance of the vector potential $\mathbf{A}$. The Lorentz force in this case is conservative and it does no work. The integral is in fact the total energy of the system.

It is well known that CW equations with three control components is fully controllable. However, the CW equations augmented by the Lorentz force doesn't permit this behaviour. In other words, if it is required that the inter-satellite Lorentz force is used to fully control the relative motion, then it is necessary that the vector potential $\mathbf{A}$ must be time-varying.

B. Dynamics on The x-y Plane

Due to the existence of the first integral, we are able to reduce the dimension of the dynamics by one. Note that λ is a constant, we can solve l as:

$$l = \frac{\sqrt{-\left(3\cos^2\beta - h^2\right)\left(\lambda - L^2\right)}}{\left|3\cos^2\beta - h^2\right|} \tag{27}$$

The plot $3\cos^2\beta - h^2$ on the (β, h) plane is shown in Figure 1. Because $3\cos^2\beta - h^2$ shows up in the denominator of Eq. (27), it cannot be zero during the motion. Thus $3\cos^2\beta - h^2$ serves as a separatrix of two types of motion with the same constant λ. We can see that if initially (β, h) is inside the area enclosed by the curve, then the evolution of (β, h) cannot go beyond that region. In other words, if we desire a relative trajectory that makes a full revolution around the origin, i.e., β spreads $(0,2\pi)$, then initially (β,h) must be outside the enclosed zone in Fig. 2, i.e., $3\cos^2\beta - h^2 < 0$.

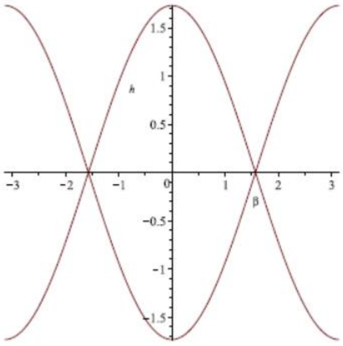


Fig.2 The (β,h) curve

The reduced dynamics regarding $x_r = [\beta, L, h]^{\mathrm{T}}$ are

$$\dot{\mathbf{x}}_r = \mathbf{f}_r + \mathbf{g}_r Q \tag{28}$$

where

$$\mathbf{f}_{\mathrm{r}} = \begin{bmatrix} h \\ -\dfrac{\left(3\cos^2\beta + h^2 + 2h\right)\sqrt{-\left(3\cos^2\beta - h^2\right)\left(\lambda - L^2\right)}}{3\cos^2\beta - h^2} \\ \dfrac{2L\left(L^2-\lambda\right)(h+1)\sqrt{-\left(\mathrm{h}^2 - 3\cos^2\beta\right)\left(L^2-\lambda\right)} - 3\left(L^2-\lambda\right)^2 \sin\beta\cos\beta}{\left(L^2-\lambda\right)^2} \end{bmatrix},$$

$$\mathbf{g}_{\mathrm{r}} = \begin{bmatrix} 0 \\ \dfrac{h\left(h^2 - 3\cos^2\beta\right)}{L^2-\lambda} \\ \dfrac{9L\cos^4\beta - 6Lh^2\cos^2\beta + Lh^4}{\left(L^2-\lambda\right)^2} \end{bmatrix}$$

For a nonlinear control system, the concept of controllability is usually quite strong, thus the notation of accessibility is always adopted to provide a weaker control-theoretical result. To that end, we consider the minimal strong accessibility Lie algebra of (28). It can be verified that the accessibility algebra and accessibility distribution are respectively

$$\begin{cases} \mathcal{C}_0 = \left[\mathbf{g}, \mathrm{ad}_{\mathbf{f}}\mathbf{g}, \mathrm{ad}_{\mathbf{f}}^2\mathbf{g}\right], \\ \Delta_{\mathrm{C}_0} = \mathrm{span}\left\{\mathbf{g}, \mathrm{ad}_{\mathbf{f}}\mathbf{g}, \mathrm{ad}_{\mathbf{f}}^2\mathbf{g}\right\} \end{cases} \tag{29}$$

The rank of Δ_{C_0} is $\mathrm{rank}\left(\Delta_{\mathrm{C}_0}\right) = 3$. According to Theorem 2, the reduced system (28) is strongly accessible. Therefore, the reachability set of system (28) is dense on the three-dimensional configuration manifold.

Note that system (28) is also in a control-affine form with a single input. Let $G \triangleq \left[\mathbf{g}, \mathrm{ad}_{\mathbf{f}}\mathbf{g}, \mathrm{ad}_{\mathbf{f}}^2\mathbf{g}\right]$ and $D \triangleq \mathrm{span}\left\{\mathbf{g}, \mathrm{ad}_{\mathbf{f}}\mathbf{g}\right\}$. It is not difficult to check

$$\begin{cases} \mathrm{rank(G)} = 3 \\ \left[\mathbf{g}, \mathrm{ad}_{\mathbf{f}}\mathbf{g}\right] \notin \mathrm{D} \end{cases} \tag{30}$$

for system (28). Thus, according to Theorem 3, the system (28) cannot be feedback linearized.

V. Circular Orbits

A. Periodic solution on X-Y Plane

Due to the lack of a general statement of controllability, in this section we turn to find special solutions, for example, equilibrium. Unfortunately, the system doesn't permit any non-trivial equilibrium. The trivial ones are just the formation of "a string of pearls" in the CW case

$$\left\{\pm\frac{\pi}{2}, 0, 0\right\}, \quad Q = 0 \tag{31}$$

However, circular trajectory exists. By letting $L = 0, \dot{L} = 0$, a circular trajectory can be obtained, in which the radius l is

$$l = \sqrt{\frac{\lambda}{h^2 - 3\cos^2\beta}} \tag{32}$$

We can find that the required charge is

$$Q = \frac{\left(3\cos^2\beta + h^2 + 2h\right)\lambda\sqrt{-\left(3\cos^2\beta - h^2\right)\lambda}}{h\left(3\cos^2 b - h^2\right)} \tag{33}$$

From Eq. (33), it can be seen $\lambda > 0, h^2 > 3\cos^2\beta$ is necessary to permit a circular trajectory.

After substituting Eq. (33) into (28), the dynamics of (β, h) system can be obtained

$$\begin{cases} \dot{\beta} = h \\ \dot{h} = -3\sin\beta\cos\beta \end{cases} \tag{34}$$

The solution to Equation (47) is expressed in elliptic integrals:

$$\beta = \mathrm{am}\left(\frac{t + C_2}{C_1}, \sqrt{3}C_1\right), \quad h = \frac{\mathrm{dn}\left(\frac{t + C_2}{C_1}, \sqrt{3}C_1\right)}{C_1} \tag{35}$$

where am and dn are two Jacobi elliptic functions (JCF): amplitude(am) function and delta amplitude (dn) function; and C_1 can be expressed by initial conditions:

$$C_1^2 = \frac{1}{3\sin_0^2\beta + h_0^2} \tag{36}$$

In the JCFs, the parameter is $\sqrt{3}C_1$. From the JCF theory, we know that if the parameter $\sqrt{3}C_1 < 1$, then β is unbounded and the motion is a full circle. This conditions gives $3\sin_0^2\beta + h_0^2 > 3$, which is the same as previously shown.

Note the angular rate $\dot{\beta}$ is not constant. The period is

$$T = C_1 \mathrm{am}^{-1}\left(\mathrm{am}\left(\frac{C_2}{C_1}, \sqrt{3}C_1 \right) + 2\pi, \sqrt{3}C_1 \right) - C_2 \tag{37}$$

If initially $\beta = 0$, then $C_1 = 1/h_0, C_2 = 0$. The period is

$$T = \frac{4}{h_0} K\left(\frac{\sqrt{3}}{h_0} \right) \tag{38}$$

which is a monotonically decreasing function with respect to h_0.

B. Transfer Between Circular Orbits

We parameterize the circular orbits by $[\lambda, \beta, h]$, where $\lambda > 0, h^2 > 3\cos^2\beta$. In order to transfer between two circular orbits, the orbits should have the same constant λ. Without loss of generality, we let $\beta = 0$ in the above representation. The transfer between circular obits can be described as $[\lambda, h_1] \to [\lambda, h_2]$, where $\lambda > 0, h_1^2 > 3, h_2^2 > 3$. Then l can be computed:

$$l = -\frac{\sqrt{-\left(3-h^2\right)J_c}}{3-h^2} \tag{39}$$

We write the dynamics of l as

$$\ddot{l} = f_L + g_L Q \tag{40}$$

A classical PD controller can be designed to accomplish the transfer. The controller is:

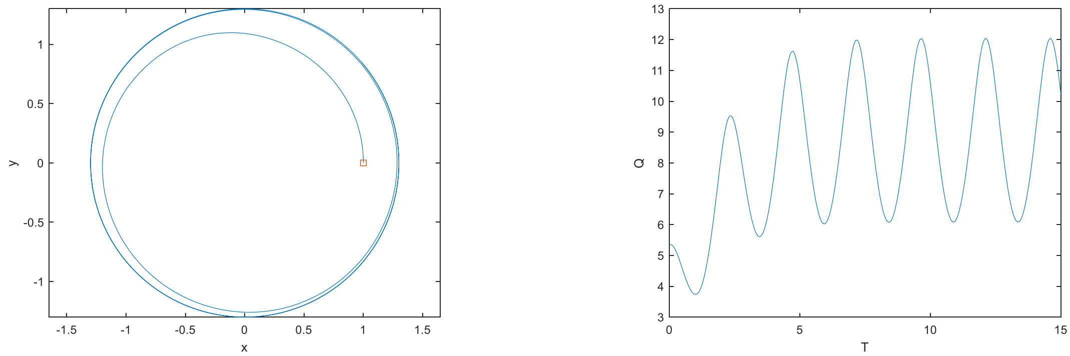
$$Q = -\frac{f_L}{g_L} - k_l\left(l - l_T\right) - k_L L \tag{41}$$

where k_l, k_L are control gains.

The following figures give a preliminary example of circle-to-circle transfer on x-y plane by using only inter-satellite Lorentz force. In the simulation, the objective is to transfer from $l = 1$ to $l = 1.3$. The other simulation parameters are:

$$\begin{cases} \lambda = 1, \mu_z = 1, k = 1.5, \\ k_l = 1, k_L = 2 \end{cases}$$

Here the example is only for an illustrative purpose. Therefore, the real implementation issues, such as the charge saturation, are not considered.



a) Transfer Trajectory   *b) Time history of Q*

Fig.3 Circle-to-circle Transfer

VI. Conclusion

Due to the fact that Lorentz force is always perpendicular to the velocity, the LFF with a time-invariant magnetic field is not controllable, which is also confirmed by the existence of a motion integral. Invariant plane motion only exists on the *x*-*y* plane in the barycentre LVLH frame. This special motion has a degree of freedom of three. It is shown that the reduced system is strongly accessible, thus the accessibility set is locally dense. The LFF permits *x*-*y* plane circular periodic orbits, along which the relative motion is not homogeneous. Local transfer between circular orbits is achievable.

Though the relative motion in LFF is highly coupled and is not fully controllable, the existence of special periodic orbits does not rule out all the promising space applications. The potential application includes fuel-free distributed space system design, as well as noncontracting space debris mitigation techniques.

Acknowledgments

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