Design of Circle-To-Circle Optimal Low Thrust Transfer Trajectories Using Coordinate Transformation

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Abstract: *The ultimate purpose of this study is to obtain approximate analytical solution of low thrust trajectory for the spacecraft which continuous thrust system is equipped with such as electric propulsion. Generally, analytical solution of low thrust trajectory is not exist because input is continuous. Approximate Analytical solution of low thrust trajectory is so valuable that it is used not only as rough estimate of fuel consumption and flight time in early mission analysis but also as initial solution of high cost numerical calculation for low thrust trajectory such as non-linear programming. In order to obtain approximate analytical solution coordinate transformation is valid as some previous studies. However, these studies ignore the optimality of low thrust trajectory based on variational method which introduces adjoint valuables. The method changes optimization problem to solving the two boundary problem including state and adjoint vector. Therefore, this study aims to obtain approximate analytical solution using coordinate transformation of state and adjoint valuables and this paper proposes that analytical solution in linearized region is extended to non-linear region in circle-to-circle transfer problem as the first step for achieving this purpose.*

Keywords: *Low Thrust Trajectories, Adjoint Variables*

1. Introduction

Recently, many deep space exploration missions have been enabled by electric propulsion systems. Japan Aerospace Exploration Agency (JAXA) succeeded in Hayabusa mission thanks to the electric propulsion systems which obtain much higher specific impulse than chemical propulsion systems. It is expected the electric propulsion systems will play an important role in future deep space exploration because they are useful for large energy which is required to change trajectories and for enrichment of observation equipment enabled by reduction of fuel consumption.

In order to design optimal low thrust trajectories by electric propulsion systems numerical calculation is inevitable even if the spacecraft (S/C) is influenced only by central body because continuous thrust is given to the S/C. Generally, non-linear programming is widely used to obtain the optimal solution of low thrust trajectories with continuous thrust such as Direct Collocation with Non Linear Programming (DCNLP) [1]. Although non-linear programming can optimizes with high versatility and find the solution for various problem with uniform method, convergence time is so long that it is not fit to when analyzed space is very large such as first step analysis for missions. Almost all non-linear programming method depends on the initial solution largely and can find local minimum solution if the initial solution is not appropriate. Then, the approximate analytical solution of low thrust trajectories is very valuable because it is useful for approximate estimation of fuel consumption and flight time in early mission analysis which has large search space. Moreover, if the approximate analytical solution is used as initial solution for numerical optimization, it can contribute to find global minimum solution remarkably. The ultimate purpose of this study is to obtain approximate analytical solution of low thrust trajectories.

It can be expected that coordinate transformation is valid to obtain approximate analytical solution of low thrust trajectories because some previous studies assumed that low thrust trajectories are designed using alternating rotational coordinates [2, 3]. These studies make designing of low thrust trajectories easy using coordinate transformation and shape-based method which determines the trajectory shape before deriving thrust history [4, 5]. However, these methods ignore optimality of trajectories based on variational method. Variational method which introduces adjoint valuables can change optimization problem to two boundary problem including state vector and adjoint vector. Therefore, this study aims to obtain approximate analytical solution using coordinate transformation of state and adjoint valuables and this paper proposes that analytical solution in linearized problem is extended to non-linear problem as the first step for achieving this purpose. In this paper circle-to-circle transfer problem is considered.

2. Extension of Linear Analytical Solution

2.1. Equation of Motion and Linear Analytical Solution

Circle-to-Circle transfer problem is considered (Fig. 1). The distance between S/C and central body and the phase of S/C are represented by r and θ respectively. Equations of motion are Eq. 1~3 where the velocity is (u, v) respect to (r, θ). It is presumed that the magnitude of thrust is constant and mass change is negligible during navigation. Ψ indicates the angle of thrust direction which is input.

$$\dot{r} = u \tag{1}$$

$$\dot{u} = \frac{v^2}{r} - \frac{\mu}{r^2} + T\sin\psi \tag{2}$$

$$\dot{v} = -\frac{uv}{r} + T\cos\psi \tag{3}$$

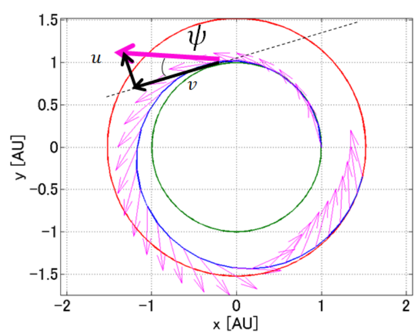


Figure 1. Circle–to-Circle Transfer with continuous thrust (Earth to Mars)

In the case that the S/C is near the initial orbit Eq. 1~3 are linearized to Eq. 4~6 using initial orbit's rotation angular velocity.

$$\dot{r} = u \quad (4)$$
$$\dot{u} = 2nv - 3n^2 r + T\sin\psi \quad (5)$$
$$\dot{v} = -2nu + T\cos\psi \quad (6)$$

It is presumed that the flight time is minimized and cost function is $\int_0^{t_f} 1\,\mathrm{dt}$. Hamiltonian is given in Eq. 7 which satisfies the condition of optimality $\partial H/\partial\psi = 0$ according to variational method. Adjoint valuables are indicated by λ. Diffrential equations are given in Eq. 8 with optimal hamiltonian H_0, therefore, linearized optimal input is obtained by Eq.9 analytically.

$$H = 1 + \lambda_r u + \lambda_u(2nv + 3n^2 r + T\sin\psi) + \lambda_v(-2nu + T\cos\psi) \quad (7)$$
$$\lambda_r = -\frac{\partial H_0}{\partial r}, \lambda_u = -\frac{\partial H_0}{\partial u}, \lambda_v = -\frac{\partial H_0}{\partial v} \quad (8)$$
$$\tan\psi = \frac{\lambda_u}{\lambda_v} = \frac{\sin(nt+\phi)}{2\cos(nt+\phi)+C} \quad (9)$$

Optimal trajectories are obtained by solving two boundary problem that constant ϕ and C in Eq.9 converge terminal velocity $u(t_f)$ and $v(t_f)$.

2.2. Functional Type of Analytical Solution

Subsequently, it is considered that linearized analytical solution in Eq. 9 is extended to non-linear region and the method of the extension is explained in this section. The change from constant ϕ and C in Eq.9 to functions can extend linearized solution. The patched conic method in which two different trajectories which are calculated separately are connected is one of the most famous methods of deigning trajectories [6]. In this paper patched conic method of low thrust trajectories are used for determining function type of extended analytical solution. After two low thrust trajectory are calculated in in the region that the change of the distance to central body is small ($r_i \to r_i + \Delta r$ and $r_{i+1} + \Delta r$) with the angle of thrust direction in Eq.9, these adjacent two trajectories are combined. Thus, patched conic method is utilized continuously from initial orbit to terminal orbit like shown in Fig.2 and constant ϕ and C are determined as each boundary conditions of velocity are satisfied. We can guess the function type of new ϕ and C included in extended analytical solution cased on the history of the constant ϕ and C. Note that the calculation of low thrust trajectories with Ee.9 in $r \to r + \Delta r$ is correct precisely because Eq.9 is valid in linearized system.

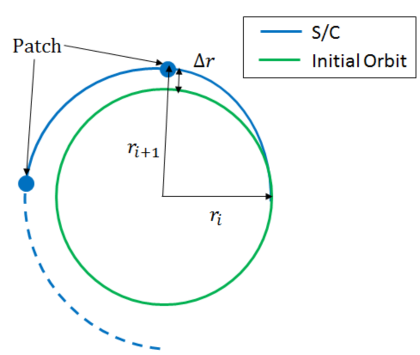


Figure 2. Patched Conic of Low Thrust Trajectories

Calculation conditions are shown in Tab. 1. Transfer trajectories from the Earth to the Mars are calculated and each boundary condition at patch point is referred to the optimal trajectories calculated by DCNLP. Figure. 3 shows the relation between constant ϕ, C and the distance from the Sun r. It is found that ϕ is decreasing function against r except for T=100 mN case in Fig. 3. ϕ is also decreasing function against time t similarly. Some tendencies of C and r that is united against different thrusts cannot be found. Therefore, the function types of ϕ and C are presumed Eq.10. ϕ is a linear decreasing function and C is a constant value.

$$\phi = \phi_0 + \phi_1 (r - r_0), \qquad C = C_0 \qquad \text{where } \phi_1 < 0 \qquad (10)$$

In Eq. 10 r_0 indicates the radius of the initial orbit. In order to design low thrust trajectories Eq.9 and 10 are used as the angle of thrust. The two boundary problem of which initial values are ϕ_0 and C_0 is solved using the Matlab function *fminsearch* for unconstrained minimization of the error of the terminal velocity. The flight time as cost function is evaluated when ϕ_1 varies. The condition whether the boundary condition is satisfied or not is that the error of the terminal velocity is less than 10 [m/s] and it is confirmed that the error of the terminal velocity can be converged to 0 if this condition is satisfied and the calculation is conducted with shorter time width of integral. In this paper the angle of thrust which is Eq. 9 that Eq.10 is substituted to is called extended analytical solution.

Table 1. Calculation Condition

Departure Planet	Earth
Target Planet	Mars
Mass of S/C	500 kg
Thrust	50, 65, 75, 100 mN

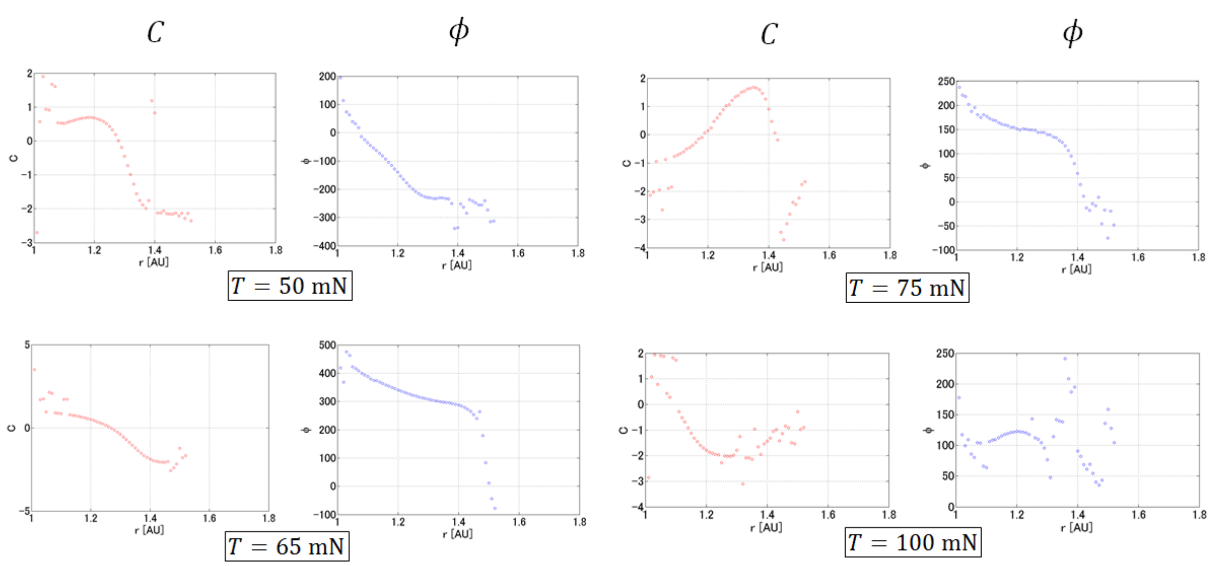
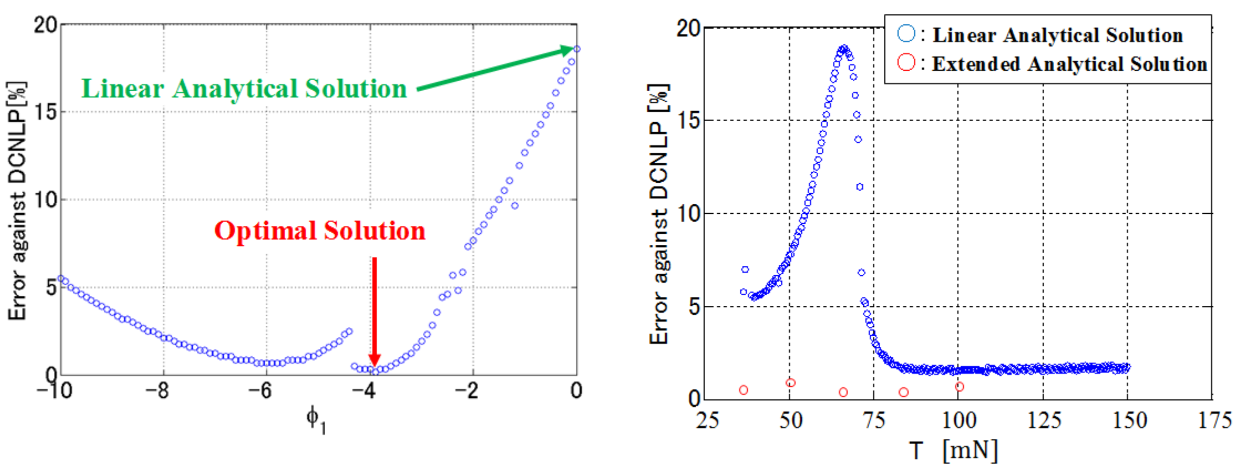


Figure 3. The Relation between ϕ, C included in linear analytical solution and the distance from S/C to the Sun r

3. Flight with Extended Analytical Solution

In this section the result of the designing low thrust trajectory using extended analytical solution which is defined in prior section is shown. Figure. 4 shows that the error against DCNLP solution of flight time which is the cost function when the extended analytical solution is used as thrust profile. In the left hand side figure thrust is set in 65 [mN]. The plot of $\phi_1 = 0$ indicates linear analytical solution is utilized. The comparison between linear analytical solution case and extended analytical solution with which the flight time is minimized in $\phi_1 = -3.9$ case can find that the error against DCNLP solution of flight time is improved from 18% to 0.7%. The right hand side figure shows 5 cases thrusts are 35, 50, 65, 80, 100 [mN]. Blue plot indicates linear analytical solution and red plot indicates extended analytical solution of which the error against DCNLP solution is minimized. It is found that the errors are less than 1 % in all cases. This result shows this extended analytical solution of the angle of thrust can give more accurate optimal low thrust trajectory than linear analytical solution in various thrust magnitude cases. Figure 5 shows three states and the angle of thrust profiles compared between DCNLP and linear analytical and extended analytical solutions. This figure reveals extended analytical solution is closer to optimal DCNLP solution than linear analytical solution in all profiles. It is shown that linear analytical solution with small thrust cannot satisfy the boundary condition and there are no solution. Thus, this simple extended analytical solution that depends on r linearly can contribute not only optimality of trajectory but also extension solution space.



$T = 65$ [mN] $T = 35, 50, 65, 80, 100$ [mN]

Figure 4. The Error of Flight Time against DCNLP Solution
(Left Hand Side: Thrust is 65 [mN] Case
Right Hand Side: Optimal Solutions of 5 Thrust Cases using Extended Analytical solution are plotted (35, 50, 65, 80, 100 [mN]))

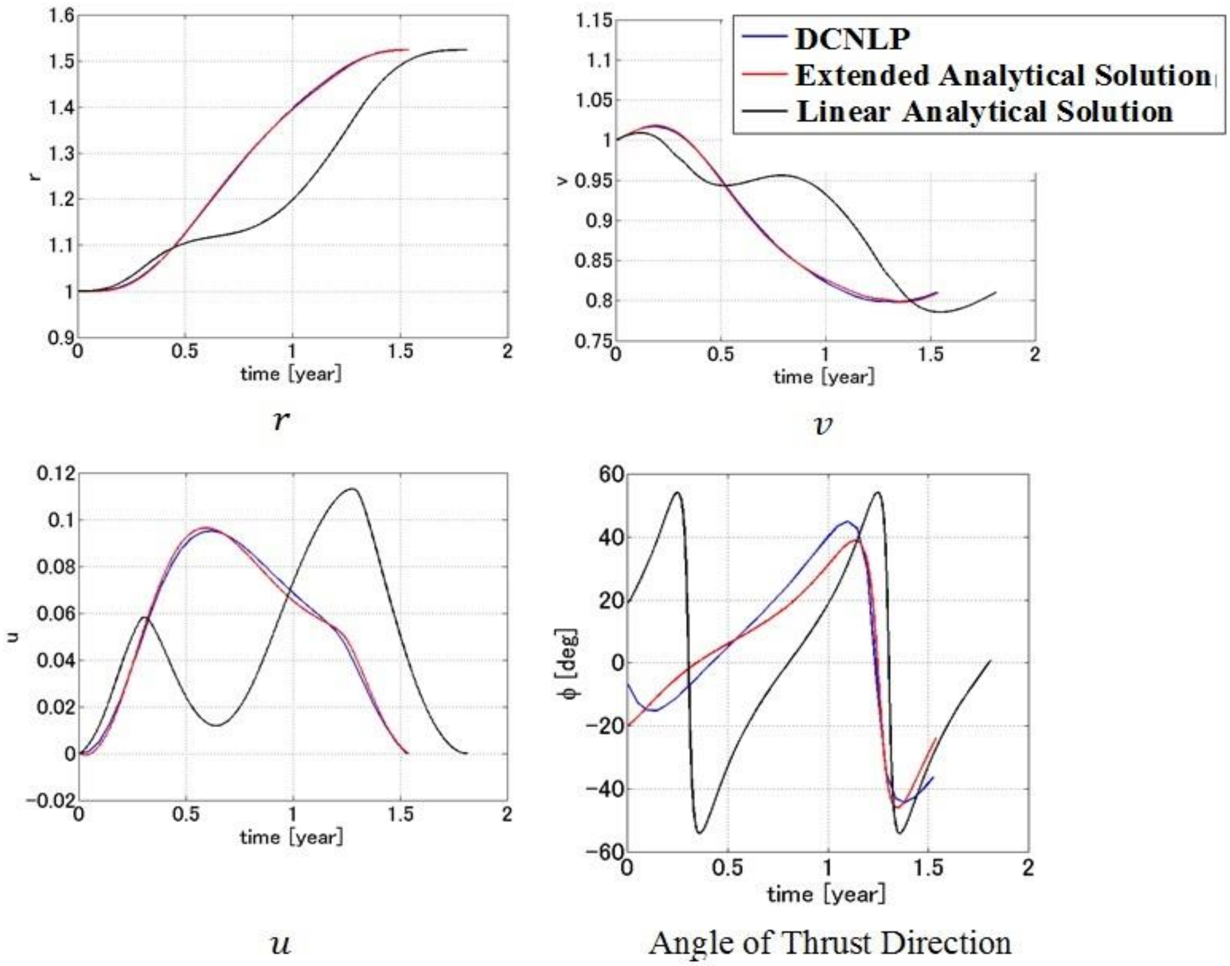


Figure 5. Comparison of State and Angle of Thrust (DCNLP, Extended Analytical Solution, Linear Analytical Solution)

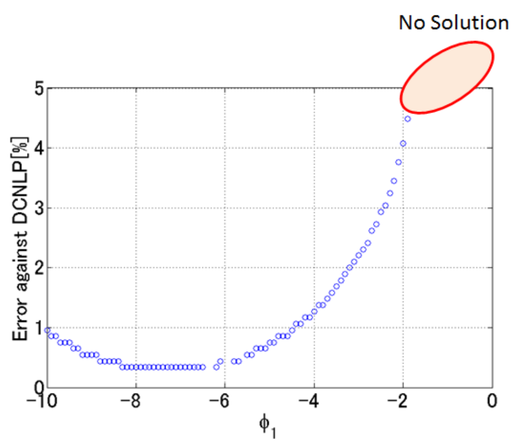


Figure 6. The Error of Flight Time against DCNLP Solution (Thrust is 35 [mN] Case)

4. Discussion

Although it is difficult to determine the function type of C included in Eq. 9 with patched conic of low thrust trajectories, it is presumed that C is given in Eq. 11 in the same way as ϕ.

$$\phi = \phi_0 + \phi_1(r - r_0), \qquad C = C_0 + C_1(r - r_0) \quad (11)$$

Figure 7 shows the change of angle of thrust when ϕ_1 and C_1 are changed respectively. It is found that ϕ_1 changes the period of the angle of thrust and C_1 changes the amplitude of the angle of the thrust. Figure 8 shows the angle of thrust profile compared between DCNLP solution and linear analytical solution $(\phi_1, C_1) = (0, 0)$ and $(\phi_1, C_1) = (-5.2, 0)$ which is optimized in terms of only ϕ_1 and $(\phi_1, C_1) = (-5.2, 1.5)$ which is optimized in terms of both ϕ_1 and C_1. Each case is named A~D shown in Tab. 2. In Fig. 8 D is closer to DCNLP case A than C and the error of amplitude of C is larger than D. However, because the flight times of C and D are hardly different shown in Tab. 2, the amplitude of angle of thrust is not related to the cost function at all. In the B case because not only amplitude but period are different from DCNLP largely, the time when the plus and minus of angle of thrust changes is also different and the flight time of B case is much longer than DCNLP shown in Tab. 2. Thus, In the view point of optimality the period of the angle of thrust, that is, the time when the plus and minus of angle of thrust changes is dominant. Therefore, the phase of Eq.9 ϕ is much more important than the constant of Eq.9 C in terms of optimality.

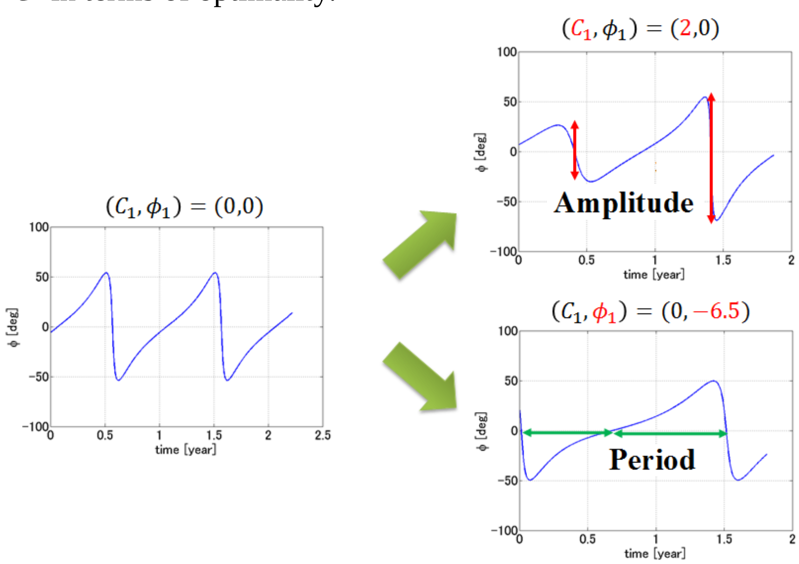


Figure 7. The Role of Constant Terms of Analytical Solution in Angle of Thrust

Table 2. Error of Flight Time against DCNLP in Different Conditions

	Error of Flight Time against DCNLP
A: DCNLP	0.0%
B: $(\phi_1, C_1) = (0, 0)$	7.6%
C: $(\phi_1, C_1) = (-5.2, 0)$	0.9%
D: $(\phi_1, C_1) = (-5.2, 1.5)$	0.5%

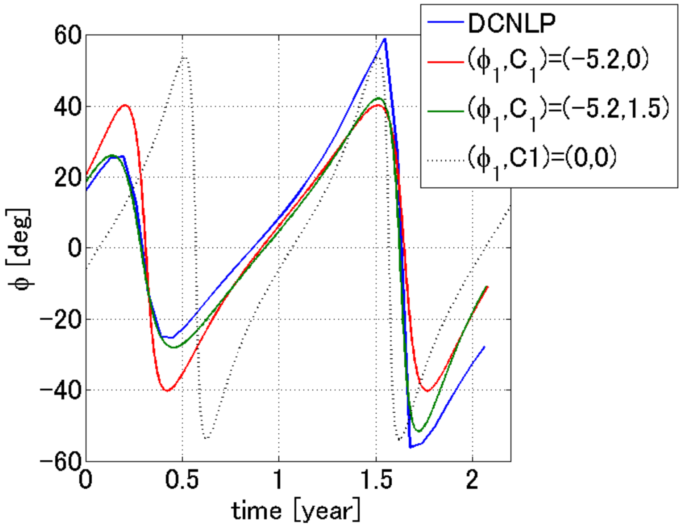


Figure 8. Comparison of Angle of Thrust in Different Conditions

5. Conclusion

In circle-to-circle transfer problem approximate analytical solution of the thrust profile of low thrust trajectory which is constrained to maintain thrust in constant which is obtained by linearizing one constant term of linear analytical solution in terms of radius r gives high accurate solution of which error of cost function against DCNLP solution is less than 1%. In the case that the magnitude of thrust is small if linear analytical solution of thrust profile is utilized, there is no solution which can satisfy the boundary condition in rendezvous problem. However, even if thrust is small, accurate extended analytical solution can be obtained by searching the optimal coefficient of linear function about r in angle of thrust profile.

6. References

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