FAST AND ROBUST OPTIMIZATION OF HIGH FIDELITY CONTINUOUS THRUST TRANSFER ORBITS WITH CONSTRAINTS

Georges Krier(1), **David Mostaza-Prieto**(2)
(1) SES Engineering, Château de Betzdorf • L-6815 Betzdorf, Luxembourg. (352) 710725 217, georges.krier@ses.com
(2) SES Engineering, Château de Betzdorf • L-6815 Betzdorf, Luxembourg

Abstract: *In order to support operational low thrust transfer orbits to GEO, SES Engineering has identified the need for an optimization method with high levels of fidelity, robustness and speed. This paper presents the development and validation of an original method to optimize low and medium thrust trajectories between two orbits using a so called indirect method. The tool is able to support both studies and operations by solving any high fidelity minimum time or minimum fuel problem in a few seconds, without convergence issues and without the need for an initial guess. Several benchmark scenarios are provided together with conclusions and recommendations.*

Keywords: *Low thrust optimization. Electric Orbit Raising, Indirect method, satellite.*

1. Introduction

The solution to the low thrust transfer problem consists of finding a continuous thrust steering law that allows the spacecraft to follow an optimal trajectory from one orbit to another. Low thrust trajectories are characterized by long propagation times and many optimization variables. As a result, optimization programs are typically time-demanding and require a fair level of user expertise. Moreover, they normally need a “good” initial guess to converge, and it is not always obvious how close this guess needs to be from the solution. Sometimes, under similar conditions, the optimization algorithm fails to find a solution, making the initial guess generation a sort of art as demanding as the optimization itself. Adding constraints or discontinuities, such as allowing the trust to shut down, decreases the robustness and convergence of the algorithms. All these facts make the use trajectory optimization programs challenging for real time operations or trade off studies. Supporting operational transfers to GEO requires a method able to provide a true optimum in any scenario, without falling into convergence issues and within a reasonably short time, in the order of seconds-minutes.

In order to fulfill these requirements, comprehensive research has been conducted (see [5]). Almost all the existing methods can be classified as Direct or Indirect (or a combination of both). Direct methods transform the optimal problem into a linear programing problem, requiring to parametrize the control function (thrust steering law) by a discrete and large number of optimization variables. The problem is typically solved by means of an NLP solver [6], which adjust the control variables in order to reduce a given cost function. Although it is easier to include constraints and these methods are more robust, they are also slower and less accurate. On top, the discretization of the control sometimes produces several minima, so the problem may converge towards solutions that are quite far from the global optimum. Therefore, we believe that indirect methods are better suited for the orbit to orbit continuous thrust transfer problem. They are typically faster and more precise, however, quite difficult to converge. We have focused our efforts in enhancing the convergence properties of a problem formulated as an indirect method. The result is an optimization program, coded in Fortran (LOTTO), which is highly robust and able to provide results in a few seconds without an initial guess from the user. The following features can be included in the optimization:

- Minimum time problem (thrust always on)
- Minimum fuel problem (for fixed time, optimum coast periods)
- Eclipse shadowing (thrust reduction or thrust off)
- Non-spherical Earth potential (J2)
- Solar radiation pressure
- Third body gravitational perturbations by Sun and Moon.
- Drag force
- Altitude constraints via penalty functions.
- Slew rate constraints
- Intra-Orbit Averaging (optional)
- Longitude targeting

2. Methodology

LOTTO makes use of the indirect method for optimal control; this formulation has been reproduced in many references [1], and is just briefly presented here.

The state vector consists of the six orbital elements ***x***, which in the current implementation can be either the Keplerian elements ($a, e, i, \Omega, \omega, M$) or the equinoctial (non-singular) elements (a, h, k, p, q, L) [3]. The six dynamic equations for the state vector are:

$$\dot{x}_i(t) = f_i(\mathbf{x}(t), \mathbf{u}(t), t) \tag{1}$$

The initial state vector at time t_0 is $\boldsymbol{x_0}$ and the target final state vector at time t_f is $\boldsymbol{x_f}$.
The thrust vector ***u***, which is the control variable, is characterized by the magnitude u and the unit vector $\hat{\boldsymbol{u}}$, which can be described by the in-plane pitch steering angle β and the out-of-plane yaw steering angle α:

$$\mathbf{u} = u \cdot \hat{\mathbf{u}} = u \cdot \begin{pmatrix} \sin\beta \ \cos\alpha \\ \cos\beta \ \cos\alpha \\ \sin\alpha \end{pmatrix} \tag{2}$$

The thrust is limited in magnitude to a maximum thrust level u_{max} and to a minimum thrust level u_{min} assumed to be zero. We also assume that the specific impulse I_{sp} is constant (constant exhaust velocity).

The aim is to find the steering law ***u(t)*** that minimizes a cost function C, which in a Lagrangian formulation, is an integral cost without terminal cost:

$$C = \int_{t_0}^{t_f} L(\mathbf{x}(t), \mathbf{u}(t), t)\, dt \tag{3}$$

For minimum time problems, $L=1$ and the final time t_f is unknown; for minimum mass problems $L=u$ and t_f is given.

With the introduction of the Hamiltonian H and the Lagrange multipliers λ_i, also called co-states,

$$H = L + \sum_{i=1}^{n} \lambda_i f_i \tag{4}$$

the optimal steering law $\boldsymbol{u^{*}=u(\lambda(t),x(t))}$ is derived from the well-known optimality conditions:

$$\frac{\partial H}{\partial u_j} = 0 \tag{5}$$

The adjoint differential equations are:

$$\dot{\lambda}_i = -\frac{\partial H}{\partial x_i} \tag{6}$$

and the dynamic equations can also be written as:

$$\dot{x}_i = \frac{\partial H}{\partial \lambda_i} \tag{7}$$

The major difficulty is to find initial values for the λ_i, that will lead to $\boldsymbol{x}=\boldsymbol{x_f}$ at $t=t_f$. There are six co-states if the final longitude is targeted and five if the final longitude does not matter.
The functions f_i can be split into three parts, a contribution from the thrust acceleration, which is linear in u; a contribution from the non-Keplerian perturbations (non-spherical Earth potential, third body perturbations, solar pressure radiation, atmospheric drag) and an extra term for the fast varying sixth orbital element (mean anomaly M when using Keplerian, mean longitude L when using equinoctial elements):

$$f_i = u \sum_{j=1}^{3} A_{ij}(\mathbf{x}) \cdot \hat{u}_j + f_i^{pert}(\mathbf{x}, t) + \delta_{i6} \sqrt{\frac{\mu}{a^3}} \tag{8}$$

μ is the gravitational parameter. The 6x3 matrix $\boldsymbol{A}$ can be derived with the help of the Gauss form of Lagrange's planetary equation. It depends on the frame type which can be either radial-transverse-normal *RSW* or tangential-velocity-normal *NTW*. It also depends on the type of elements which can be either Keplerian or Equinoctial. The expressions for A_{ij} can be found for instance in Battin [4].

Because the thrust only appears linearly in the Hamiltonian, the optimal solution for the minimum mass problem is either to maximize or minimize the thrust magnitude [1][2]. The decision is taken by the switching function S which is the proportionality factor of u in the Hamiltonian (for minimum mass problems):

$$S = 1 + \sum_{i=1}^{n} \sum_{j=1}^{3} \lambda_i A_{ij} \hat{u}_j \tag{9}$$

If S is negative then the thrust is u_{max} else it is u_{min}.

When the thrust is on, the optimality condition can be used to determine the thrust direction:

$$\hat{\mathbf{u}} = -\frac{\mathbf{A}^{\mathbf{T}}\boldsymbol{\lambda}}{\left|\mathbf{A}^{\mathbf{T}}\boldsymbol{\lambda}\right|} \tag{10}$$

Formulated this way, the problem consists of finding the initial set of co-states (λ_0) that allows deriving the control law that transfers the satellite from its initial state into a given final orbit. This is basically a two point boundary value problem (2PBVP). Specific solvers are available, however it is known that the problem is sensitive and difficult to solve unless a very good initial guess is provided. Rather than focusing on the initial guess generation, we have developed an iterative solving algorithm able to take advantage of the particularities of the problem. The algorithm is based on two main ideas, firstly, for any given values of initial co-states, the integrated trajectory is optimal but it will finish in a different orbit. Secondly, a delta-v cost can be defined between this orbit and the target one. The way to find the required trajectory is to drive this cost towards 0. To accomplish that, LOTTO uses a robust heuristic search method which finds iteratively the solution of any transfer and without initial guess from the user. In nearly all cases it converges and reaches the target orbit with sub-meter accuracy.

The usual mode to use LOTTO is to perform an intra-orbit averaging. Due to the low thrust, the change of the orbital elements during one revolution is small compared to the total required change and thrusting during many orbital arcs is required. Therefore, to obtain the long term evolution of the state vector and the co-states, it is possible to average them over one orbit. This intra-orbit averaging is performed by an integration over the true anomaly (if the Keplerian representation is selected) respectively the true longitude (if the equinoctial representation is selected), keeping the other five slow varying elements constant.

However it is also possible to run LOTTO in the non-averaging mode, which is slower, but more accurate and one can drop the constraint of low thrust.

3. Validation

A comprehensive validation has been performed. Benchmark scenarios have been taken from published papers. Tables 1 to 3 offer the comparison between published results and obtained results. All the scenarios have been run using a regular PC with an Intel Core i5 and 4 GB of RAM, without the need for iterations and providing a complete solution in a few seconds, as reported in the last column of the tables.

Table 1 shows a good matching for the simplest case, the minimum time problem without any constraint and pure Keplerian orbit propagation from GTO (a_0=24505.9 km, e_0=0.725, i_0=7°, Ω_0=0, ω_0=0°) to GEO with F=0.35 N, I_{sp}=2000 s, m_0=2000 kg. Orbit averaging is applied and the averaged state and co-states are integrated with a fourth-order Runge-Kutta (RK4) method.

Table 1. Validation: Simple Keplerian case (minimum time).

Case	*Mission*	*Reference*	*Final time (days)*		*Run time [s]*
			Reference	*LOTTO*	*LOTTO*
1	GTO-GEO	Geffroy, Epenoy [9]	137.5	137.40	1.6

The pitch and yaw steering angles in the orbital reference frame (radial-transverse-normal) are shown in figure 1. At the beginning (blue curves), both angles are small, the thrust is roughly tangential and increases the semimajor axis, while at the end (red curves) thrust is roughly inertial and reduces the eccentricity. The inclination is continuously reduced with a highest yaw steering angle of about 75° around day 50.

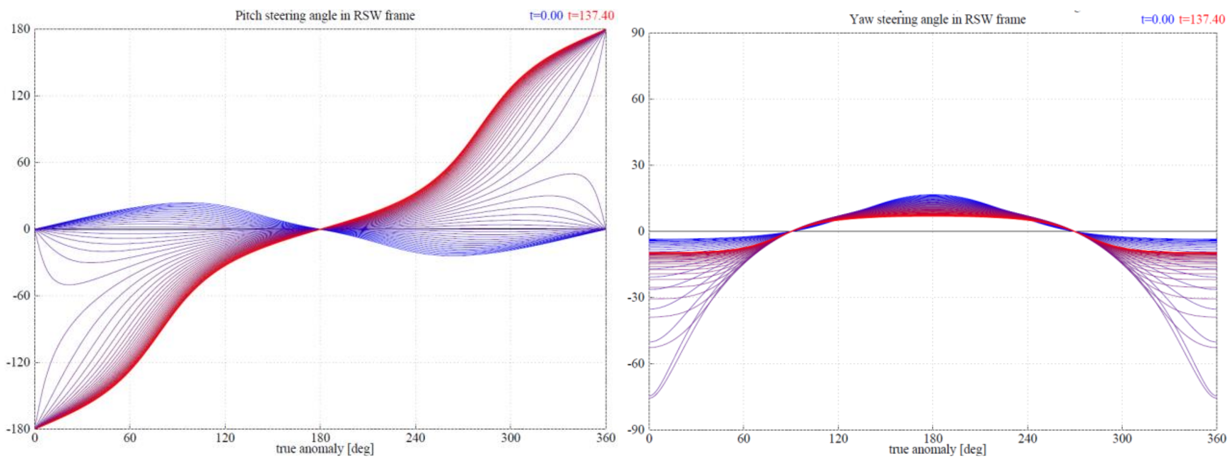

Figure 1. Left: in-plane (pitch) steering angle. Right: out-of-plane (yaw) steering angle for case 1. Blue is t=0, red is t=137.

Results in Table 2 aim to validate the use of non-Keplerian effects (here J2 only) together with thrust off during eclipse. Test cases are taken from Reference 8: case 2 is an Ariane LEO (a_0=6780 km, e_0=0, i_0=5.2°, Ω_0=0°) to GEO transfer with F=1.86948 N, I_{sp}=1800 s, m_0=5500 kg. Case 3 is a Taurus LEO (a_0=6926.657 km e_0=0, i_0=28.5°, Ω_0=0°) to GEO with F=0.40171 N, I_{sp}=3300 s, m_0=1200 kg. Case 4 is a Taurus GTO (a_0=24364.3 km, e_0=0.731, i_0=27°, Ω_0=99°, ω_0=0°) to GEO transfer with F=0.20085 N, I_{sp}=3300 s and m_0=450 kg.

Table 2. Validation: Minimum time, including J2 and eclipse shutoff.

Case	*Mission*	*Reference*	*Final time (days)*		*Run time [s]*
			Reference	*LOTTO*	*LOTTO*
2	LEO-GEO	Kluever&Oleson [8] case 1	167.8	167.71	11.1
3	LEO-GEO	Kluever&Oleson [8] case 2	198.8	198.87	22.8
4	GTO-GEO	Kluever&Oleson [8] case 4	66.6	66.71	5.1

Figure 2 shows the time evolution of the Case 4 intra-orbit averaged Keplerian elements. The averaged state and co-states were integrated with an Adams-Bashfort-Moulton method. The match with the results from [8] obtained with SEPSPOT [13] is very good.

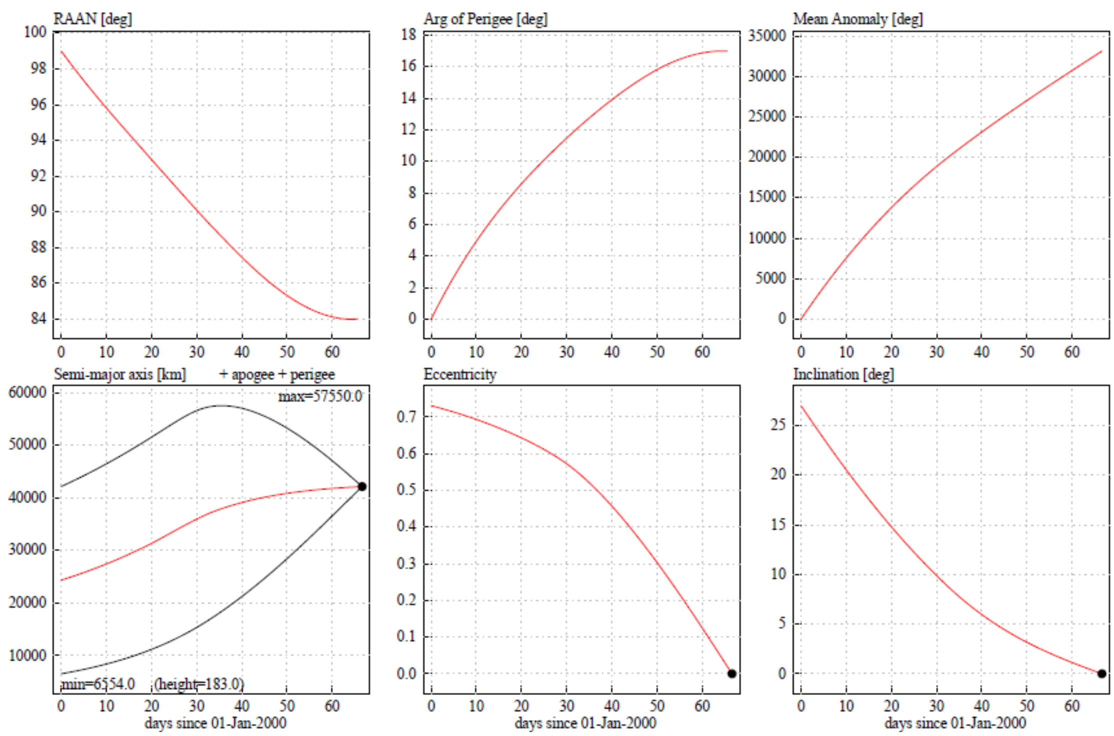

Figure 2. Keplerian elements for Case 4 (GTO to GEO)

Table 3 presents results for the more challenging minimum fuel problem. Here the final time is given and apart from the optimum steering law, the software needs to provide optimum coast periods to maximize mass. The optimization is performed without any constraint or non-Keplerian effects and uses intra-orbit averaging.

Table 3. Validation: Minimum fuel

Case	*Mission*	*Reference*	*Consumed mass [kg]*		*Run time [s]*
			Ref	*LOTTO*	*LOTTO*
5	GTO-GEO	Verlet et al. [12]	171	169.50	3.9
6	MEO-GEO	Verlet et al. [12]	263	259.01	5.3
7	SSTO-GEO	Verlet et al. [12]	329	329.61	3.4
8	SMART1	Cano et al. [10]	22.1	22.13	3.8

Cases 5 to 7 describe transfers from GTO (h_p=250 km, h_a=35786 km, i=5°), from MEO (h_p=5000 km, h_a=10000 km, i_0=13°) and from SSTO (h_p=11497 km, h_a=65000 km, i=13.8°) to a GEO orbit. The thrust equals 0.58 N and the specific impulse equals 1800 s The duration is 180 days for case 5&6 and 160 days for case 7. The initial mass for case 5&6 is 2000 kg and for case 7 it is 4600 kg. The results of case 7 present a good matching while for 5 and 6 LOTTO finds slightly better consumptions. Figure 3 shows the on/off periods for all three scenarios as function of time and true anomaly. The GTO transfer consists mainly of apogee burns (true anomaly=180°) except at the very end of the transfer where perigee burns appear. The MEO transfer starts with perigee burns and ends with apogee burns. The SSTO transfer starts with apogee burns, but perigee burns appear after a few days.

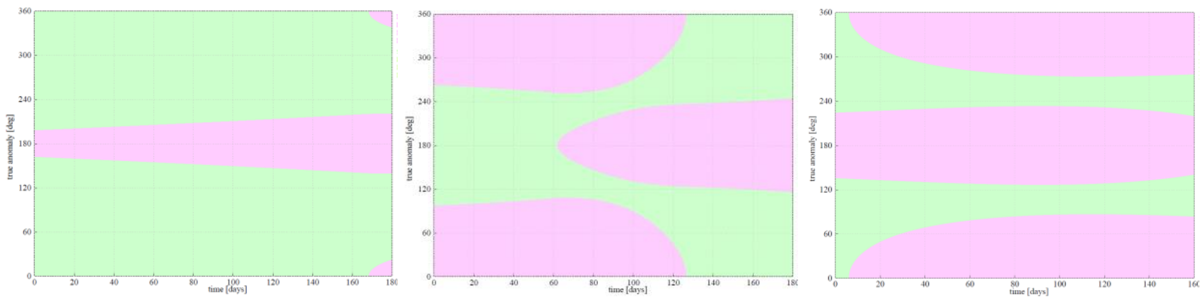

Figure 3. Optimum on (magenta) / off (green) periods for the solutions of Case 5 = GTO→GEO, Case 6 = MEO→GEO and Case 7 = SSTO→GEO (left to right)

Case 8 is a SMART-1 trajectory optimization without any constraint and pure Keplerian orbit propagation. Initial orbital parameters are r_p=20000 km, r_a=58068 km, i=6.655°, Ω=244.21° and ω=200.23°. Final orbit parameters are r_a=219400 km, i=5.49°, Ω=0.25° and ω=79.66°. The final perigee radius is free. Initial spacecraft mass is 325.966 kg, the transfer duration is fixed to 284 days. Thrust is 45.82 mN and I_{sp} is 1496.3 s. Because the final perigee altitude r_p was free, multiple runs with different r_p's were done and the minimum consumption was found to be 22.128 kg for a perigee radius of 26075 km. This is close to the results carried out in [10] and cited in [11] which obtained a consumption of 22.1 kg and a final perigee radius of 26500 km.

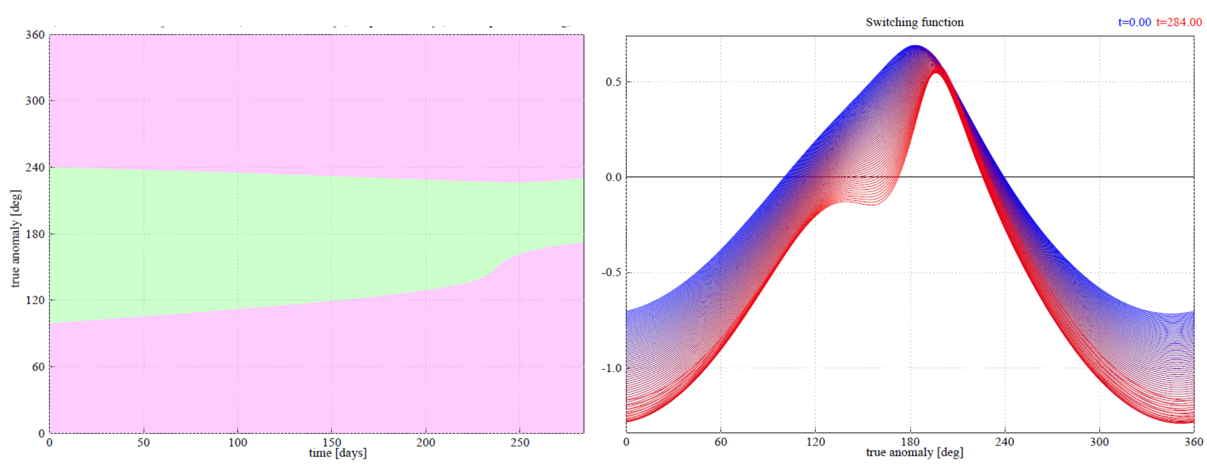

Figure 4. Optimum on (magenta) / off (green) periods for the solutions of case 8. The corresponding switching function is show on the right side.

4. Study cases

In this section we present several study cases together with findings and recommendations.

4.1. Comparison of averaged and non-averaged trajectories.

In this example we want to assess the accuracy of the intra-orbit averaging method. To be able to visualize the differences to the non-averaged method, we selected a SSTO transfer with relative high acceleration such that the target orbit is reached in only 68 revolutions.
At the same time we selected some more difficult options (non-Keplerian perturbations, longitude targeting, slew rate limitations, eclipse shutdown).

The initial osculating orbital parameters are h_p=350 km, h_a=70000 km, i=28.5°, Ω=265°, ω=170° and M=0°. The satellite has an initial mass of 1500 kg, the thrust is 0.5 N and the specific impulse is 1800 s. The target orbit is a perfect GEO orbit for which the subsatellite longitude at 50° East is targeted. The initial epoch is 01-Jan-2016 00:00:00 and the non Keplerian perturbations considered are J2 effect, Sun & Moon third body perturbations and solar radiation pressure with C_rA=90 m^2. During eclipse, the thrusters are shut off. The angular rate of the thrust direction in inertial frame is constrained to below 100°/hour. For the averaged method, the osculating initial and target elements have be replaced by the mean elements because only the secular effects of J2, Sun& Moon etc. are considered in the orbital propagation.

Figure 5 compares the time evolution of equinoctial elements for the averaged and non-averaged case. The matching is pretty good and will be even better for longer transfers with lower thrust.

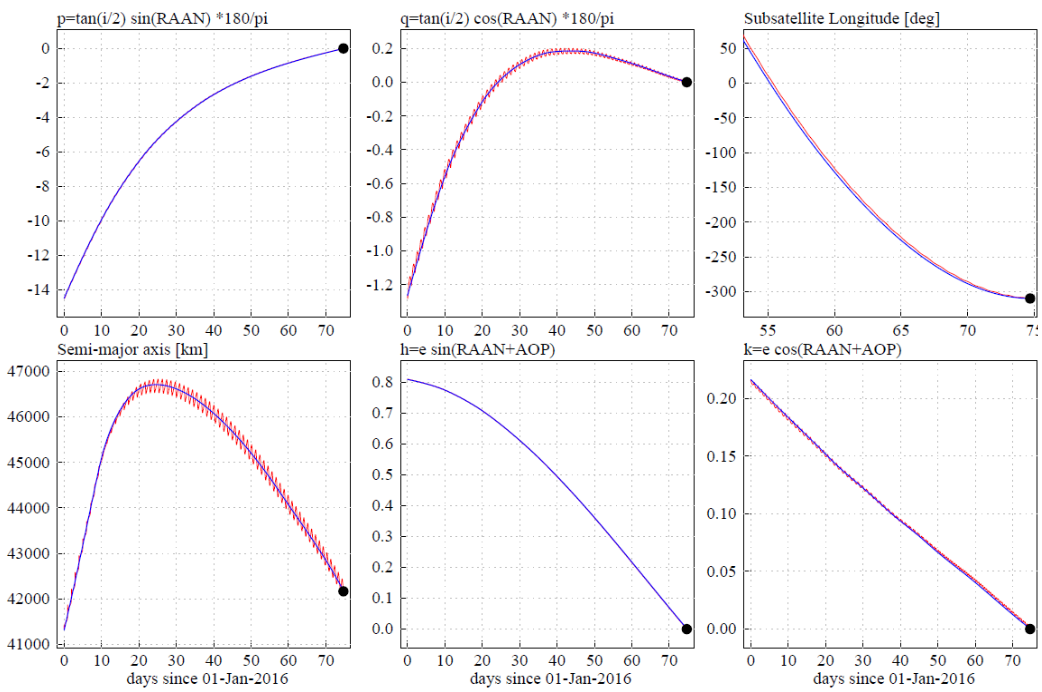

Figure 5. Time evolution of equinoctial elements for averaged (blue) and non-averaged (red) method. Note that only the last 20 days are displayed for the subsatellite longitude.

The delta-v determined with the averaging method integrated with RK4 and a step size of one day is 2272.2 m/s. It is 0.1 percent higher than the delta-v of 2269.3 m/s from a non-averaged integration with RK4 and a step size of 10 minutes. The CPU time is 15 s respectively 200 s.
Figure 6 compares the time evolution of the co-states for the averaged and non-averaged case. From that plot it is clear that the converged averaged co-states λ_0 are very good initial guesses for the non-averaged co-states, thus accelerating the convergence for the relative CPU intensive non-averaged case.

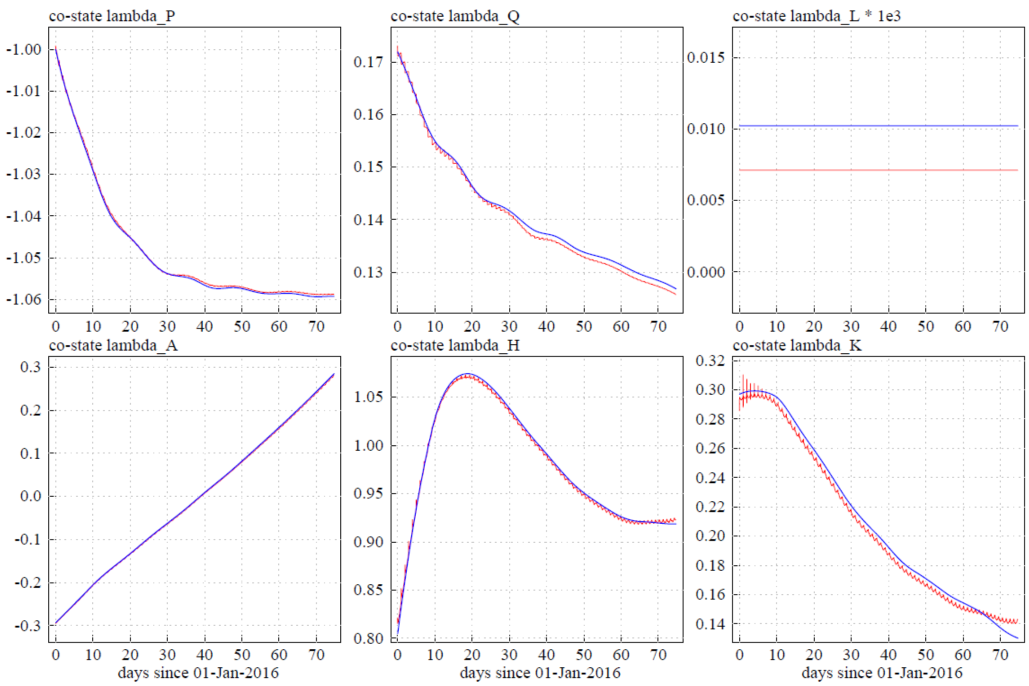

Figure 6. Time evolution of co-states averaged (blue) and non-averaged (red)

With the non-averaged method it is also possible to abandon the low thrust assumption and LOTTO can even optimize a sequence of two high thrust chemical LAE burns, but this will not be presented here.

4.2. Minimum fuel vs minimum time transfers to GEO and lifetime savings.

In some regions of the orbit it is less efficient to change orbital elements than in others, for instance inclination corrections at the perigee are much more “expensive” than at the apogee because of the very high velocity at perigee. Therefore it is obvious that the total delta-v needed to reach the target orbit can be reduced if the thruster can be shut off in the “inefficient” portions of the orbit. This section shows how much fuel can be saved by not firing continuously and extending the transfer time. For simplicity, it uses pure Keplerian motion without any other perturbations, no constraints and applies intra-orbit averaging. The initial orbit is a typical GTO orbit: h_p=300 km, h_a=35786 km, i=6°, Ω=270° and ω=180°. The satellite has an initial mass of 5000 kg, the thrust is 0.8 N and the specific impulse 1800 s.

The lowest possible delta-v is 1486.0 m/s and corresponds to the Simplified Nodal Transfer (SNT) solution [14] that can be used because the apses are located at the nodes (ω=180°). It can

be reached asymptotically for an infinite transfer duration, in which case the thrust will be off during the entire orbit, except at in infinite short time at the apses.
The minimum duration solution has 149.42 days, with a delta-v of 2196.9 m/s and a consumption of 585.11 kg. Figure 7 shows the time evolution of the orbital elements for this case and also for a duration of 153 days with a delta-v of 2091.76 m/s and a consumption of 558.74 kg. Coasting is mainly around perigee and extends from day 32 to day 103 (see figure 8 right, top/right).

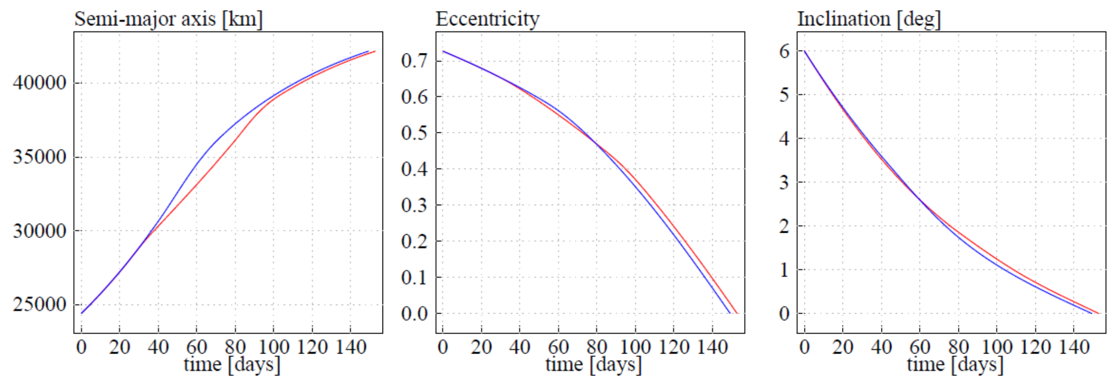

Figure 7. Time evolution of a, e, i for min duration (blue) and min fuel with 153 days (red).

Figure 8 left shows how the delta-v and the fuel consumption decrease with increasing transfer duration. The duration is extended by at most 25 days in which the required delta-v to reach GEO is reduced by 333 m/s compared to the continuous firing case.

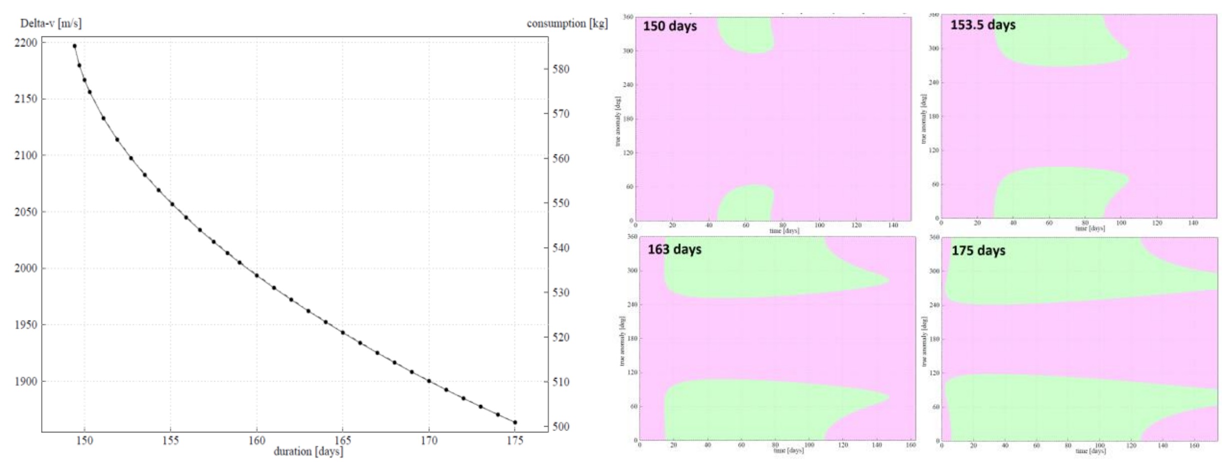

Figure 8. Left: delta-v and consumption as function of transfer duration. Right: optimum on (magenta)/ off (green) periods for 4 selected durations.

For a geostationary satellite for which the yearly delta-v is about 50 m/s for inclination control, this corresponds to more than six years of propellant lifetime gain (!), assuming that it uses the same propulsion system with identical I_{sp} for both stationkeeping and transfer orbit.
We have also observed that the coasting periods correspond to those regions in which the slew rates are the highest in the continuous thrust solution. When entering regions of very low efficiency, solutions with continuous thrust present high rates which are risky and inefficient manoeuvers for the spacecraft, adding little gain. By adding just one or two days to the minimum

duration, these regions are normally avoided. On top of that, due to the initial high slope of the curve (Figure 8), the gain in lifetime by adding just a few days to the transfer is very interesting: we can basically trade days by years of lifetime (adding just 2 days gives around 2 years of lifetime increase). Another advantage of this approach is that it is well suited for a fixed mission completion date because unexpected underperformances in the early phase of the transfer can be compensated by less coasting in the remaining part of the transfer.
Each fixed duration run used as initial guess the converged solution of the previous duration and the CPU for a single fixed duration run was about 6 seconds.

4.3. In-orbit low thrust relocations.

In this example, we apply the minimum fuel, fixed duration optimization without intra-orbit averaging. A 15° relocation from a GEO orbit at 65° East to 50° East is performed in a fixed time of 10 days. The satellite has an initial mass of 5000 kg, the thrust is 0.2 N and the specific impulse 3000 s. All non-Keplerian perturbations are neglected.

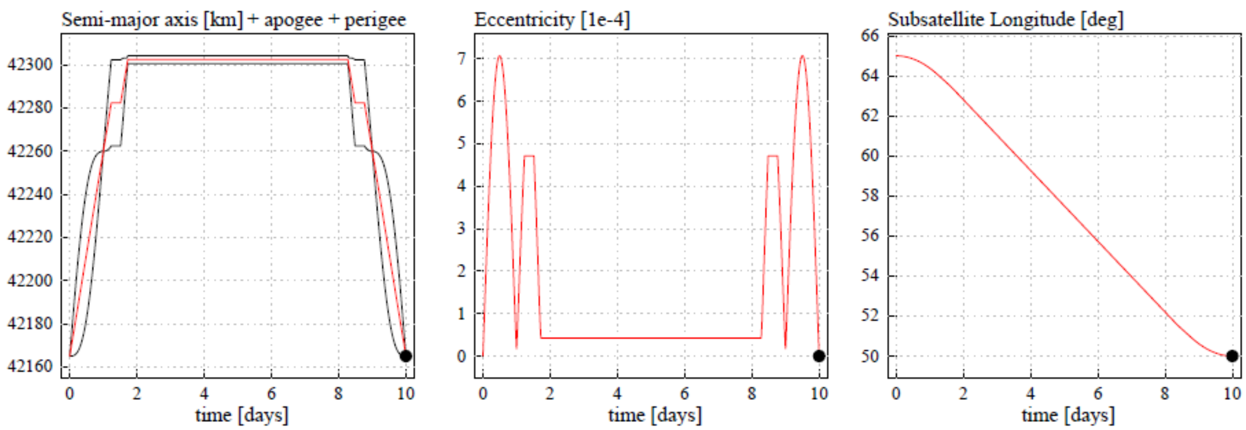

Figure 9. Time evolution of Keplerian elements for GEO relocation.

The minimum delta-v is 9.996 m/s and the minimum consumption 1.6985 kg. The start sequence and the stop sequence are nearly perfectly symmetric and both are split by the switching function (fig. 10) into 2 burns: b_1=[0:1.2357], b_2=[1.5159:1.7262], b_3=[8.2739:8.4841], b_4=[8.7644:10].

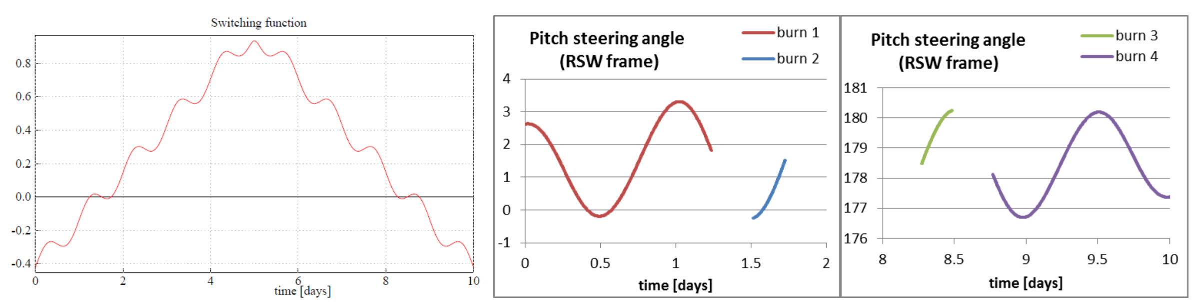

Figure 10. Switching and pitch steering angle during a GEO relocation.

The interruptions between burns 1 and burn 2 and between burns 3 and burn 4 are required to target eccentricity and longitude simultaneously (see figure 9).

The thrust vector lies in the orbital plane, but is not perfectly aligned with the velocity vector: the optimal pitch steering angle varies between -0.2° and 3.3° during the two start burns (see figure 10) and between 176.7° and 180.2° at the two stop burns.

5. Conclusions

A new low thrust trajectory optimization tool LOTTO based on an indirect method has been developed and successfully validated. It does not need an initial guess and in most cases converges without problems. It is fast, typical CPU times lie in the order of seconds. It has the capability to solve different kind of problems like longitude targeting, non-averaged high thrust scenarios or transfers where coasting phases help to reduce the total propellant consumption compared to a minimum time solution. It can handle constraints like eclipse shutoffs or slew rate limitation.

6. References

[1] Lawden, Derek F. "*Optimal trajectories for space navigation*." Butterworths, 1963.
[2] Pontryagin L.S, Boltyanskii V.G, Gamkrelidze R. V, Mishchenko E. F, "*The Mathematical Theory of Optimal Processes* (Russian)", English translation: Interscience 1962.
[3] Broucke, Roger A., and Paul J. Cefola. "*On the equinoctial orbit elements.*" Celestial Mechanics 5.3 (1972): 303-310
[4] Battin R. H., 1999, "*An Introduction to the Mathematics and Methods of Astrodynamics*." AIAA Education Series, Washington DC.
[5] Conway, Bruce A., ed. "*Spacecraft trajectory optimization*". Vol. 32. Cambridge, UK: Cambridge University Press, 2010.
[6] Dickmanns, Ernst D., and Klaus H. Well. "Approximate solution of optimal control problems using third order Hermite polynomial functions." IFIP-TC7, Springer 1975.
[7] Vallado, D. A., "*Fundamentals of Astrodynamics and Applications*". Fouth Edition. Space Technology Library, Microcosm Press, 2013
[8] Kluever, C. A., & Oleson, S. R. (1998). "*Direct approach for computing near-optimal low-thrust earth-orbit transfers*", Journal of Spacecraft and Rockets, 35(4), 509-515
[9] S. Geffroy and R. Epenoy, "*Optimal Low-Thrust Transfers with Constraints. Generalization of Averaging Techniques*", Astronautica Acta, vol. 41, no. 3,pp. 133–149, 1997.
[10] J.L. Cano, J. Schoenmaekers, R. Jehn, M. Hechler. "SMART-1 Mission Analysis: Collection of Notes on the Moon Mission." S1-ESC-RP-5004, 1999
[11] Petukhov, V. "*Optimal Multirevolutional Transfers between Non-Coplanar Elliptical Orbits*" *Proceedings of 2nd International Symposium on Low-Thrust Trajectories LOTUS-2, Toulouse, France*. 2002.
[12] Verlet, M., et al. "*Coupled optimization of launcher and all-electric satellite trajectories*". 24th International Symposium on Space Flight Dynamics. 2014
[13] Sackett, L. L., Malcow, H. L., & Edelbaum, T. N. (1975). "*Solar electric geocentric transfer with attitude constraints: analysis*." Charles Stark Draper Laboratory.
[14] Vinh, N.X., Kuo, S.H., Marchal, C., "*Optimal Time-Free Nodal Transfers between Elliptical Orbits*", Acta Astronautica, Vol. 17, No. 8, 1988, pp. 875-880
[15] Tarzi Z.; Speyer J.L.; Wirz R.E.. "*Fuel optimum low-thrust elliptic transfer using numerical averaging*". Acta Astronautica, Volume 86, p. 95-118