STATION ACQUISITION AND STATION KEEPING WITH LOW-THRUST SYSTEMS

M. Eckstein

Deutsche Forschungs- und Versuchsanstalt für Luft- und Raumfahrt e. V. (DFVLR)
Wessling
Federal Republic of Germany

F. Hechler

European Space Operations Centre (ESOC)
Darmstadt
Federal Republic of Germany

ABSTRACT

Fuel and time minimum manoeuvres for a low-thrust station acquisition are determined by solving a special linear optimization problem. The proposed method also works if constraints are imposed on the manoeuvre times and directions.
Low thrust station keeping compensates the effects of natural perturbations and may be regarded as a series of station acquisition phases. It requires an optimal long term strategy defining the target orbits. Although this strategy is almost trivial for the secular perturbations of inclination and semi-major axis, the optimal compensation of long periodic effects of the eccentricity needs some analysis. Corresponding algorithms are derived by use of the "rope stretching method" and implemented in a station keeping simulation, the results of which are presented.

Keywords: Low-thrust, Station Acquisition, Station Keeping, Fuel and Time Minimum Control Problems, Longterm Strategy, Rope-stretching Method

1. INTRODUCTION

In future geostationary missions the application of highly accurate launchers (Ref. 1, page 2.17) and the replacement of the apogee boost motor by a restartable engine (Ref. 2) will lead to small ΔV-requirements for *station acquisition* which can even be generated by low-thrust systems in a reasonable short time interval. The use of low-thrust systems like electric propulsion systems instead of chemical systems will drastically reduce the fuel consumption in this phase of geostationary missions, and in the subsequent *station keeping* phase. The study will present some methods for the determination of optimum orbit correction sequences for station acquisition and for station keeping with low-thrust systems. The main optimality criterion will be a minimum fuel consumption, however, minimum time problems will be touched also.

The low-thrust systems considered here are characterized by the fact that the influence of a manoeuvre on the orbit cannot be described by a single impulsive variation of the satellite velocity. We consider systems with constant exhaust velocities ev. Hence the fuel consumption Δm during the controlled motion is proportional to the ΔV-requirement, i.e. the integral

$$\Delta m = \int \|\vec{u}(t)\| \, dt = \Delta V \qquad (1)$$

over the absolute values of the accelerations $\vec{u}(t)$ exerted by the control system on the satellite. $\|\ldots\|$ is the Euklidian norm in the $R^{(3)}$ and $\|u(t)\|$ is restricted by a sufficiently small and *constant* upper limit $\bar{u}$.

The station keeping phase can be split into cycles of duration T_i, i=1,..M. The T_i depend on the station-keeping tolerances, i.e. the permissible eccentricity and inclination of the orbit and the permissible longitude band. Each cycle comprises several orbits. In each of these cycles and in the station-acquisition phase one has to solve a typical rendez-vous problem : One is seeking for a sequence of orbit corrections which annihilate small deviations of a given departure orbit from a given target orbit during a fixed time T. We may call this a *short-term* problem since the influence of the natural perturbations (earth potential, sun, moon) on the orbit can be decoupled from the control problem.

The targets for these rendez-vous problems have to be defined by means of a *long-term strategy*, the *station-keeping strategy*. Only the combination of such a strategy with an optimum solution of the resulting short-term problems allows to correct the injection errors (station acquisition) and to compensate the influence of the natural perturbations on the orbit (station keeping) in a (fuel-) optimum way.

Operational constraints occurring mainly during station acquisition, complicate the rendez-vous problem. Periods during which the thrusters may not be fired (eclipse intervals, tracking periods) split the station acquisition phase into a sequence of disconnected intervals, and constraints on the firing directions cause similar problems as described in the paper "Midcourse Navigation for the European Comet Halley Mission" to be presented in this symposium. These constraints have to be taken into account in the short-term solution.

As is well known (Ref. 3, page 124 ff.), the equations of the controlled motion can be formulated in elegant way by means of the following variables. The control $\vec{u}(t)$ and the natural perturbations are represented by the 3-vector function $\vec{b}(t)$ with components

$T(t)$ in direction of the actual velocity
$N(t)$ in direction of the orbit pole
$R(t)$ normal to $N(t)$ and $T(t)$.

The orbit **state** is given by 6 nonsingular elements, for instance by a = semi-major axis; ex, ey = components of the eccentricity vector in the equatorial plane; $\Delta\lambda$ = mean off-station longitude in the rotating system **and** ix, iy = components of the orbit pole in the equatorial plane. With the following auxiliary notations

$$V = \text{mean velocity in synchronous orbit} \quad (2)$$

$$\omega_e = \text{mean angular velocity in synchronous orbit} \quad (3)$$

$$\lambda_o = \text{station mean longitude at epoch } t_o \quad (4)$$

$$\alpha = \alpha(t) = \Delta\lambda(t) + \omega_e(t-t_o) + \lambda_o = \text{satellite mean longitude} \quad (5)$$

$$\alpha_o = \Delta\lambda(t_o) + \lambda_o \quad (6)$$

and for *small eccentricities* and *inclinations* of all orbits in question, the (Gaussian-) equations of motion read (Ref. 3, page 134)

$$\dot{a} = \frac{2a}{V} T(t) \quad (7)$$

$$\dot{e}x = \frac{1}{V} [\sin\alpha\, R(t) + 2\cos\alpha\, T(t)] \quad (8)$$

$$\dot{e}y = \frac{1}{V} [-\cos\alpha\, R(t) + 2\sin\alpha\, T(t)] \quad (9)$$

$$\Delta\dot{\lambda} = \frac{1}{V} [-2R(t) + 3(\alpha_o-\alpha)T(t)] + \sqrt{\frac{\mu}{a_o^3}} - \omega_e \quad (10)$$

$$\dot{i}x = \sin\alpha\, N(t) \quad (11)$$

$$\dot{i}y = -\cos\alpha\, N(t) \quad (12)$$

The problem specific integration of these equations in the following chapters will be a *first order* integration. If necessary, its applicability must be justified by a comparison with a correct, possibly numerical integration of the nonlinear system (7) - (12). A first order integration is equivalent to an integration of the equations of motion linearized about a reference trajectory. Hence our problems fall into the class of linear control problems. This formulation of the problem has two advantages: It provides a closed solution of the linear system and permits to separate the influence of the natural perturbations on the orbit.

2. 'FUEL AND TIME MINIMUM RENDEZ-VOUS'

2.1 Solution methods for the fuel minimum problem

Fuel or ΔV-minimum rendez-vous problems in a central force field are often solved by analysing the "primer". This method was introduced by Lawden in 1963 (Ref. 4). For our linear system, the primer is a 3-vector function of the time t depending on 6 *constant* Lagrange-multipliers λ_j, j=1,...,6. Due to technical reasons we can exclude intermediate thrust **arcs** from our solutions. Thus the fuel and time minimum solutions we are seeking for are always of the type "bang-bang". For details we refer to Marec (Ref. 3, chapter 7.2), fuel minimum problem) and Krabs (ref. 5, time minimum problem). In both cases the bang-bang principle normally leads to solution methods for the control problem which request the solution of a nonlinear system for the multipliers in parallel to a determination of the switch-on times ts and the switch-off times te of the thrust from a switching function. The number of switch times is a multiple of the number of orbits during the rendez-vous. Hence in our case a large number of unknowns is involved in this process. This number is increased by the constraints imposed on the manoeuvre times and on the thrust directions. Especially the latter constraints are not easily treatable by the primer method, because it implicitly assumes that $\vec{u}(t)$ is defined on a convex region in the $R^{(3)}$.

We therefore tried to solve the rendez-vous problem by means of the following discretisation method :

- First one selects in the rendez-vous interval $[0,T]$ a sufficiently dense set $\{t_i\}$ of *permissible* times or switch-points t_i, i=1,...,N.
- At each t_i one fixes a sufficiently dense set of permissible thrust directions $\vec{e}_{ij} = (eT_{ij}, eR_{ij}, eN_{ij})$, j=1,...,M.

Then the above control problem turns to the following *linear optimization problem :*

Determine N x M absolute values V_{ij} of impulsive velocity increments $\vec{V}_{ij} = V_{ij} \cdot \vec{e}_{ij}$ for which the cost function

$$\sum_{i=1}^{N} \sum_{j=1}^{M} g_i V_{ij} \quad (13)$$

assumes its minimum under the linear constraints

$$0 \le V_{ij} \le \bar{V}_i = \bar{u}(t_i - t_{i-1}), \quad i=1,\ldots,N \quad (14)$$

and under the linear rendez-vous conditions

$$\frac{V\,\Delta a}{2a_o} = \sum_{i=1}^{N} \sum_{j=1}^{M} V_{ij}\, eT_{ij} \quad (15)$$

$$V\,\Delta ex = \sum_{i=1}^{N} \sum_{j=1}^{M} (\sin\alpha_i\, eR_{ij} + 2\cos\alpha_i\, eT_{ij})\, V_{ij} \quad (16)$$

$$V\,\Delta ey = \sum_{i=1}^{N} \sum_{j=1}^{M} (-\cos\alpha_i\, eR_{ij} + 2\sin\alpha_i\, eT_{ij})\, V_{ij} \quad (17)$$

$$V\,\Delta\lambda = \sum_{i=1}^{N} \sum_{j=1}^{M} (-2eR_{ij} + 3(\alpha_o-\alpha_i)_i eT_{ij})\, V_{ij} + (\sqrt{\frac{\mu}{a_o^3}} - \omega_e)T \quad (18)$$

$$V\,\Delta ix = \sum_{i=1}^{N} \sum_{j=1}^{M} \sin\alpha_i\, eN_{ij}\, V_{ij} \quad (19)$$

$$V\,\Delta iy = \sum_{i=1}^{N} \sum_{j=1}^{M} -\cos\alpha_i\, eN_{ij}\, V_{ij} \quad (20)$$

emerging from the first order integrationof equations (7) - (12). The $\Delta a, \ldots, \Delta iy$ are the deviations to be annihilated. They enclose the effects of the natural perturbations on the orbit during $[0,T]$. The g_i are positive weights representing either the thruster efficiencies or the thrust mode.

This linear optimisation problem is an upper bounding problem. It *must be solved* under one *specific condition:* The solution must not contain more than one nonvanishing V_{ij} at each t_i. Otherwise the absolute value of a single velocity increment

$$|\vec{V}_i| = |\sum_{j=1}^{M} V_{ij}\, \vec{e}_{ij}|$$

at t_i will exceed the upper limit $\overline{V}_i$. These upper limits $\overline{V}_i$ in (13) are the maximum velocity increments producible by the thruster system within the given time intervals $[t_i, t_{i-1}]$, i=1,...,N between the switch points.

The linear optimization problem can be solved by a *modified version* of the *upper-bounding method*. The upper bounding method can be found in many text books on linear optimization like Lasdon (Ref. 6). For sufficiently dense discretisations the number of unknowns N x M may be rather large. However, the order of the problem is defined by only 6 rendez-vous conditions (7) - (12). Hence, the optimisation program is dealing only with 6x6 matrices and can be coded in such a way that only the coefficients $\cos\alpha_i$, $\sin\alpha_i$ and α_i of the Gaussian equations of motion at the N switch-times have to be stored.

One special case of this problem is of particular interest. In the station acquisition and in the station keeping phase almost all 3-axis stabilized satellites are kept in the following position with respect to the orbit: One axis points to the center of the earth and one axis is perpendicular to the orbital plane. This implies that the permissible optimum thrust directions are fixed along the orbit tangent and along the orbit normal. Hence the permissible directions at the switch time t_i are given by $\vec{e}_{i3} = (0,0,1)$ and (or) $\vec{e}_{i4} = (0,0,-1)$. The problem decouples into a linear 4-dimensional *in-plane* problem and into a linear 2-dimensional *out-off-plane* problem. Their solutions provide approximations for -say- K lower and upper bounds t_{1k} and t_{2k} respectively of the thrust-on intervals and they give the thrust uT_k and uN_k in those intervals. The 2K switch times can be improved by minimizing the cost-function

$$\sum_{k=1}^{K} g_k(t_{2_k} - t_{1_k}) \tag{21}$$

under the following conditions.

- The t_{1_k}, t_{2_k} remain in the permissible intervals.
- They must be in chronological order, i.e. (22)

$$0 \le t_{1_1} \le t_{2_2} \le \dots t_{1_k} \le t_{2_k} \le \dots T \quad ,$$

- and they fulfil the rendez-vous conditions (23)

$$\Delta a = \frac{2a}{V} \sum_{k=1}^{K} uT_k(t_{2_k} - t_{1_k}) \tag{24}$$

$$\Delta ex = \frac{2}{V\omega_e} \sum_{k=1}^{K} uT_k[\sin\alpha(t_{2_k}) - \sin\alpha(t_{1_k})] \tag{25}$$

$$\Delta ey = \frac{2}{V\omega_e} \sum_{k=1}^{K} -uT_k[\cos\alpha(t_{2_k}) - \cos\alpha(t_{1_k})] \tag{26}$$

$$\Delta\lambda = \frac{3}{V\omega_e} \sum_{k=1}^{K} uT_k[\alpha(t_{1_k}) - \alpha(t_{2_k})] + T\cdot\left(\sqrt{\frac{\mu}{a_0^3}} - \omega_e\right) \tag{27}$$

Similar relations hold for the out-off-plane problem, which will not be considered here. By virtue of the relation (4) between α and t, the equations (24) - (27) become nonlinear equations

$$h_e(t_{1_1}, \dots, t_{2_k}) = 0, \quad e=1,\dots,4 \tag{28}$$

in the 2K unknowns t_{1_k} and t_{2_k} respectively, whereas the conditions (22) and (23) are linear constraints in these unknowns.

The above problem is equivalent to the problem emerging from the primer method. Provided there are given appropriate initial values for the unknowns, it can be solved for instance by a penalty method, in which the linear cost function (21) and the nonlinear conditions (24) - (27) are tied together by means of positive penalty parameters $p_1, \dots p_4$ to a new cost function

$$\sum_{k=1}^{K} g_k\cdot(t_{2_k} - t_{1_k}) + \sum_{e=1}^{4} p_e h_e^2(t_{1_1}, \dots, t_{2_k}) \quad . \tag{29}$$

For suitably selected p_e, the values t_{1k}, t_{2k} rendering (29) a minimum under the linear constraints belonging to (22) and (23) provide an improved solution for the in-plane rendez-vous problem. For suitable gradient methods we refer to Avriel (Ref. 7). The cost function (29) was used in Ref. 8 for the determination of optimal switching times in combination with an approximate analytical solution providing the initial values.

2.2 Solution of time minimum problem

Since the *fuel available* for station acquisition is usually computed at an X%-probability level of NSO-dispersions it will exceed in X% of all cases the *fuel* really *needed* for a rendez-vous . The *excess fuel* could be used for speeding up the station acquisition which turns the fuel minimum problem into a time minimum problem.

Because of the non-convexity of the domain on which $\vec{u}(t)$ is defined, again some classical solution methods for time minimum problems fail, like the method of Eaton (Ref. 2) or Fujisawa-Yasuda (Ref. 9). One way out of this dilemma is the following approach :

One selects monotonically decreasing rendez-vous times $T_i < T_{i-1}$, i=1,... and solves for each T_i the corresponding fuel minimum problem. The minimum T_i, for which there still exists a solution of the fuel minimum problem is at least a permissible approximation for the minimum time T^* in which the rendez-vous can be completed. One can show indeed, that this procedure provides a solution method for time minimum problems if the linear control problem is normal. Unfortunately, our problem is not normal if constraints are imposed on the manoeuvre times, so one must live with a solution method not necessarily yielding an unique solution.

2.3 Test examples

The figure 1 shows the deviations of a low-thrust controlled, near synchronous orbit from the synchronous target orbit during a 10 days in-plan station acquisition. It is the solution of a *fuel minimum* problem. The following longitude intervals $[\lambda e, \lambda u]$ were forbidden for manoeuvres

Table 1

orbit No.	1	2	3	4	5	6	7	8	9	10
λe(deg)	166	167	168	169	170	171	173	175	---	---
λu(deg)	204	203	202	201	200	199	197	195	---	---

These intervals contained the perigees of all intermediate orbits. Hence one of the thrust-on intervals in each orbit was cut into two pieces. Such a set of forbidden intervals is for operations during "eclipse seasons" around the equinoxes.

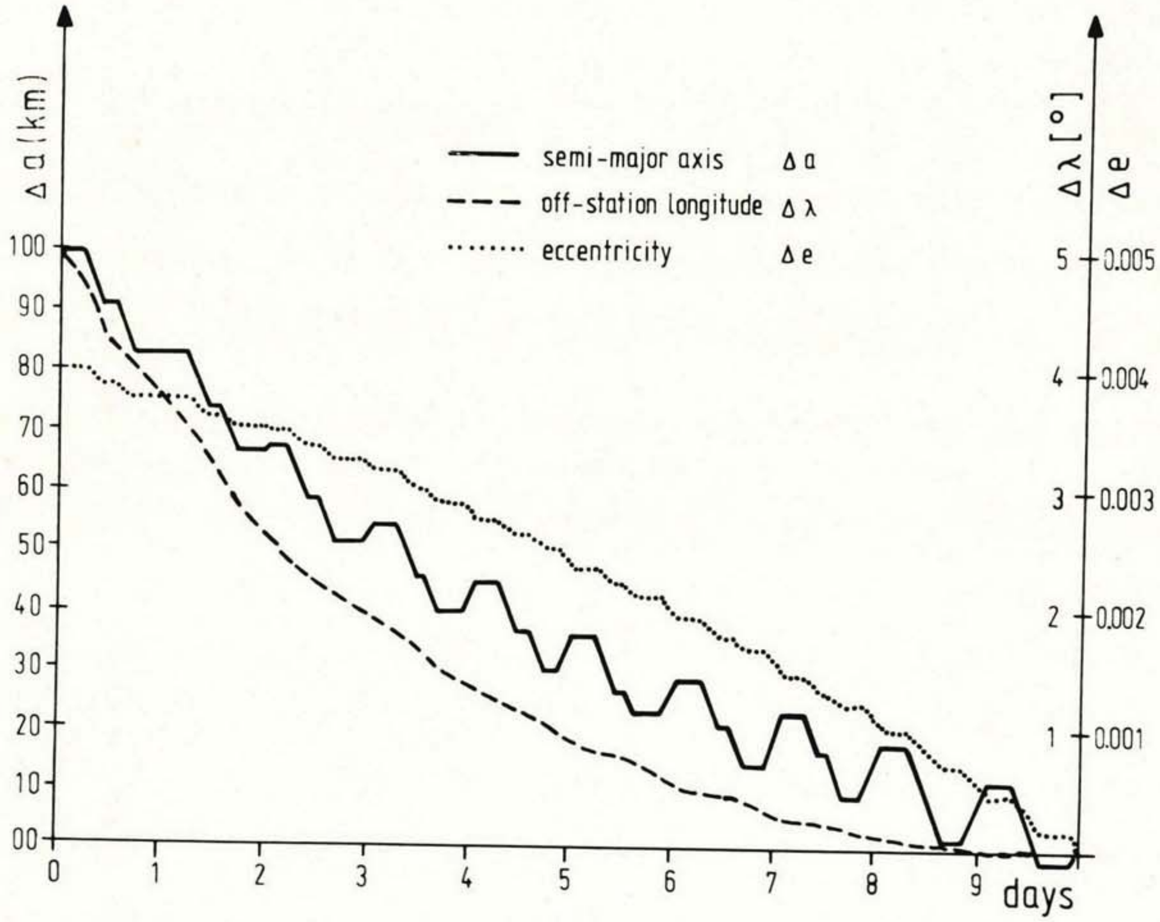

Figure 1. Deviations NSO -SO during station aquisition

The discretisation with 300 switch-points during 10 days resulted in a solution with a total ΔV -requirement of 8.050 m/s. The gradient search reduced this value to 8.001 m/s. For a 500 kg satellite and an exhaust velocity ev = 20.000 m/s, the fuel consumption would amount to 0.2 kg. The chosen thrust level of 0.01 Newton made it necessary to fire the thrusters over a total angular interval of 1686 degrees. This corresponds to a duty-ratio r = 50.2% between switch-on time and available time.

Next we computed an approximate solution of the *time minimum* problem. Table 2 contains the ΔV-requirements and the duty ratios for different station acquisition times T.

Table 2

T(days)	10	8.7	8.0	7.8	7.7
ΔV(m/s)	8.05	8.84	10.70	11.19	no solution of fuel minimum problem
r(%)	50.2	63.87	84.56	91.10	

The minimum station acquisition time will be between 7.7 and 7.8 days. It is interesting to notice that the maximum duty ratio stays below 92%. This is due to the fact that the given time minimum problem is not normal.

3. LONG TERM STRATEGY FOR LOW THRUST STATION KEEPING

In addition to the injection errors the natural perturbations acting on the satellite cause increasing deviations from its desired geostationary position. Hence the orbit has to be corrected from time to time so as to compensate for the perturbation effects. The frequency of corrections and the target orbits have to be defined by a long term strategy which is designed to minimize the fuel under the constraints imposed by the low thrust system.

The natural perturbations may be split into secular, long periodic and short periodic contributions to the non-singular elements, a, ex, ey, Δλ, ix, iy. Since the secular effects vary linear with time, they will eventually violate the tolerance window and have to be compensated by corrections in regular time intervals (correction cycles). Some long periodic effects show amplitudes of some 10^{-2} degrees and have to be - at least partially- compensated if the tolerance window is small, for instance, ±0°1. Most of the periodic perturbations are, however, very small and may be considered globally by reducing the tolerance window by some 10^{-3} degrees.

3.1 Corrections of secular perturbations

The conventional long term strategy for correcting the secular effects with high thrust systems makes use of the space for the tolerance window so as to minimize the frequency of corrections. The orbit is corrected whenever one of the elements reaches the boundary of its admissible range. Hence the time interval between corrections is dictated by the tolerance window.

In case of *low thrust station keeping*, the magnitude of corrections is limited by the maximum variation ΔE_i^{max} of the elements that can effectively be achieved by a single burn. This leads to very small corrections ΔE_i which are carried out rather frequently, up to twice a day. However, the total velocity increment required for the entire mission does not depend on the frequency and magnitude of corrections except for the efficiency loss associated with non-impulsive burns.

Hence the long term strategy for correcting the secular effect of an element E_i simply consists of corrections

$$\Delta E_i = E_i^* - E_i^o - \dot{E}_i \, \Delta t$$
$$i = 1,2,\ldots,6 \tag{30}$$

Where E_i^* and E_i^o are the desired and actual values of the element E_i, $\dot{E}_i$ its secular rate and Δt the time interval between corrections. Since $E_i^o \rightarrow E_i^*$ for perfect station keeping, Δt must be chosen sufficiently small in order to guarantee feasible corrections $|\Delta E_i| < |\Delta E_i^{max}|$.

This strategy applies to corrections of the semi-major axis and the inclination where all periodic effects are sufficiently small for tolerance windows of about ±0°1 in longitude and latitude.

3.2 Corrections of long periodic perturbations

The optimal long term strategy for correcting the long periodic perturbations shall be demonstrated for the case of eccentricity corrections. Solar pressure causes the eccentricity vector $\vec{e} = (ex,ey)$ to approximately describe a drift circle with radius R in the ex, ey plane during one year (Fig. 2). Complete compensation of this motion would require yearly corrections amounting to

$$\sum_n |\Delta \vec{e}_n| = 2\Pi R \tag{31}$$

However, this amount can be reduced by choosing a strategy which takes advantage of the space offered by the tolerance window.

If one requires $|\vec{e}| \leq e_{max}$ throughout the mission, the corrections Δe_n have to be chosen so as to minimize the cost function

$$F = \sum_{n=1}^{N} |\Delta \vec{e}_n| \tag{32}$$

under the constraint that the point $\vec{e} = (ex,ey)$ stays within a tolerance circle of radius $r = e_{max}$ around the origin. The number N of corrections during the entire mission may be very large for low thrust station keeping.
Inspite of the constraint $|\vec{e}| \leq r$, the long term strategy has to include the case $|\vec{e}| > r$ because of execution errors and approximations in the simplified algorithms. If the initial eccentricity $\vec{e}_o$ is inside the tolerance circle, one has to solve a fuel-optimal problem. Otherwise, if $|\vec{e}_o| > r$, the tolerance circle should be reached as soon as possible, i.e., a time-optimal problem also arises.

Hence, depending on the radio of the drift and tolerance circles, the 4 cases shown in Table 3 may occur:

Table 3

	$r \leq R$	$r>R$	control
$\lvert\vec{e}_o\rvert \leq r$	I	II	fuel optimal
$\lvert\vec{e}_o\rvert > r$	III	IV	time-optimal

The algorithms defining the eccentricity corrections for the 4 cases will briefly be outlined below.

3.2.1. Case I ($|\vec{e}_o| \leq r < R$) In Fig. 2 the eccentricity is marked by the point E within the tolerance circle K_r. It moves along the drift circle K_r with a given angular velocity and would exceed the tolerance circle after some time if no corrections were applied. Since the tolerance region is convex and the fuel is proportional to the length of the correction vector $\Delta\vec{e}$, the latter can be found by means of the so-called "rope stretching method" (Ref. 10). A rope is thought to be fixed at the centre 0 of the tolerance circle and drawn through a series of other circles K_n, n=1,2,...,N with radius r and their centres located on the drift circle K_R. Each of the circles K_n is associated with the time where the uncorrected eccentricity would pass through its centre. The other end of the rope is freely movable inside the last circle K_N which may be several revolutions apart from the first one and corresponds to the end of the mission.

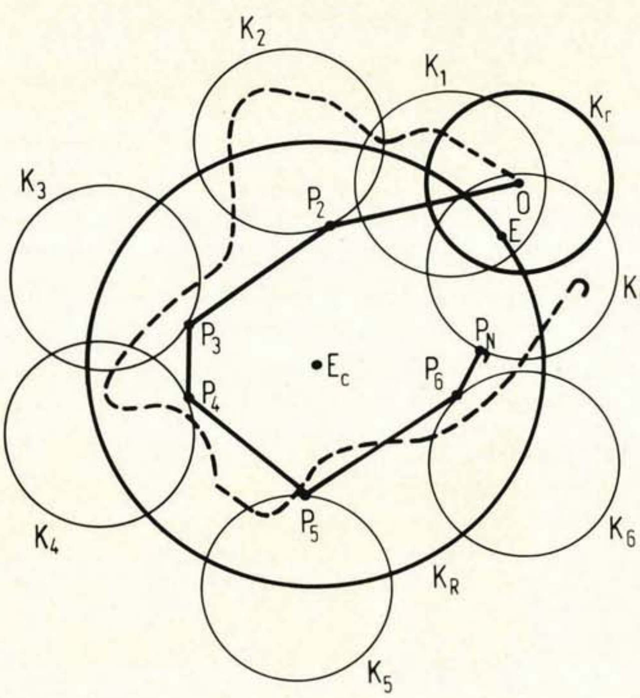

Figure 2. Application of the "rope stretching method" in Case I ($|\vec{e}_o| < r < R$).

The point E representing the eccentricity is inside the tolerance circle K_r and moves around the drift circle K_R during 1 year. The centres of the circles K_n located on the drift circle K_R correspond to the correction times. A rope (broken line) is thought to be fixed in O and drawn through the circles K_n, the other end being freely movable within K_N. Stretching the rope in O leads to an open polygon O,P2,...,PN which gives the optimal eccentricity corrections. Note that the circle K_1 does not influence the result.

The optimal corrections can then be found by stretching the rope in the point O, which results in a polygon whose lines represent the correction vectors to be carried out between the times associated with the circles.

Inspite of the simplicity of this method, the at - tempt to develop analytic formulas for the computation of the corrections is almost hopeless if N>2. Fortunately, typical in-plane correction cycles of 1 or 2 weeks correspond to polygons with 25 to 50 corners per year which is a good approximation to the limiting case where the polygon approaches a circle K_ρ with radius ρ = R-r around the centre of the drift circle and a tangent $\overrightarrow{OP}$ from the origin to K_ρ (Fig. 3). Note that $\overrightarrow{OP}$ is also parallel to a tangent E_rE_φ common to both the drift- and the tolerance circles.

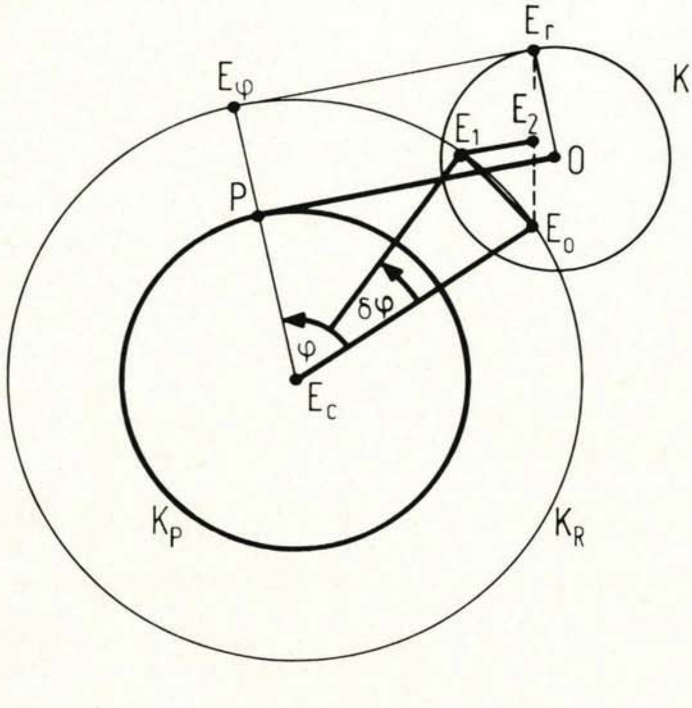

Figure 3. Optimal strategy for frequent eccentricity corrections ($N\to\infty$) in case I. For the very frequent corrections enforced by the low thrust system the polygonal of Fig. 1 approaches a circle $K\rho$ and the tangent $\overrightarrow{OP} = \overrightarrow{E_rE_\rho}$. The correction $E_\varphi E_r$ should be carried out while the natural perturbations move the eccentricity from E_0 to E_φ. Hence the combined effect is a motion from E_0 to E_r. The contribution of each correction cycle can be chosen such that this combined motion follows roughly a straight line (dashed line from E_0 to E_r).

This means that one should first correct along this tangent so as to complete the vector $\overrightarrow{E_\varphi E_r} = \overrightarrow{PO}$ by the time T_φ when natural perturbations would have moved the eccentricity from E_o to E_φ. Afterwards the corrections should continue along the circle K_ρ with radius R-r, which obviously requires less fuel than the complete compensation according to Eq. (31).

The correction $\overrightarrow{PO}$ may consist of contributions by several successive correction cycles. Since the "rope stretching method" does not specify the contribution of each correction cycle, additional considerations are neccessary.

It can be seen from Fig. 3 that the combined effects of natural perturbations $\overrightarrow{E_0E_\varphi}$ and corrections $\overrightarrow{E_\varphi E_r}$ should move the eccentricity from E_o to E_r by the prescribed time T_φ. Furthermore, the tolerance circle should not be exceeded during this motion. This can simply be guaranteed by forcing the eccentricity to move - at least approximately - along the straight line from E_o to E_r. Since the correction of each cycle has to be parallel to $\overrightarrow{PO}$, the magnitude is chosen such that the combined effects of natural perturbation $\delta\vec{e} = \overrightarrow{E_0E_1}$ plus correction $\Delta e = \overrightarrow{E_1E_2}$ puts the eccentricity into the point E_2 on the straight line $\overline{E_0E_r}$.

Defining position vectors $\vec{e}_o = \overrightarrow{OE_o}$, $\vec{e}_r = \overrightarrow{OE_r}$, $\vec{e}_\varphi = \overrightarrow{OE_\varphi}$, $\vec{e}_c = \overrightarrow{OE_c}$ and the angle $\delta\varphi$ associated with the natural perturbation $\delta\vec{e}$ during one cycle, one obtains

$$\delta\vec{e} + \Delta\vec{e} + y(\vec{e}_o - \vec{e}_r) = 0\ , \quad 0 \le y \le 1 \qquad (33)$$

Scalar multiplication by $\vec{e}_r$ results in

$$\delta\vec{e}\cdot\vec{e}_r = y(\vec{e}_r - \vec{e}\,)\vec{e}_r \qquad (34)$$

because $\Delta\vec{e}$ is perpendicular to $\vec{e}_r$. The perturbation vector $\delta\vec{e}$ is given by

$$\delta\vec{e} = \begin{pmatrix} 1-\cos\delta\varphi & \sin\delta\varphi \\ -\sin\delta\varphi & 1-\cos\delta\varphi \end{pmatrix} (\vec{e}_o - \vec{e}_c) \qquad (35)$$

and $\vec{e}_r$ can be obtained from the conditions

$$\vec{e}_r^{\,2} = r^2,\ (\vec{e}_r - \vec{e}_\varphi)(\vec{e}_\varphi - \vec{e}_c) = 0,,$$

$$\vec{e}_\varphi - \vec{e}_c = \frac{R}{2}\,\vec{e}_r \qquad (36)$$

with the result

$$\vec{e}_r = \frac{r}{|\vec{e}_c|^2}\begin{pmatrix} r-R & \pm\sqrt{\vec{e}_c^{\,2}-(R-r)^2} \\ \mp\sqrt{\vec{e}_c^{\,2}-(R-r)^2} & r-R \end{pmatrix}\vec{e}_c \qquad (37)$$

Since $\vec{e}_o$ is known, y can be obtained by (34), (35) and (37) and hence $\Delta\vec{e}$ from (33). The two signs in (37) correspond to the two possible tangents common to the circles K_r and K_R. For increasing φ the upper sign has to be chosen.

3.2.2 Case II ($|\vec{e}_o| < r$, $r > R$) If the drift circle initially lies completely inside the tolerance circle, obviously no corrections are necessary. Otherwise the "rope stretching method" (Fig. 4) shows that one has to apply corrections parallel to $-\vec{e}_c = \overrightarrow{E_cO}$ so as to shift the drift-circle K_R into the tolerance circle K_r (Fig. 5). Again, the time history of this shift is not defined by the "rope stretching method". It can be obtained as follows.

If S is the intersection of the two circles, the angle $\psi = \widehat{OE_cS}$ increases from $\psi_o > 0$ to π as the two circles approach each other.

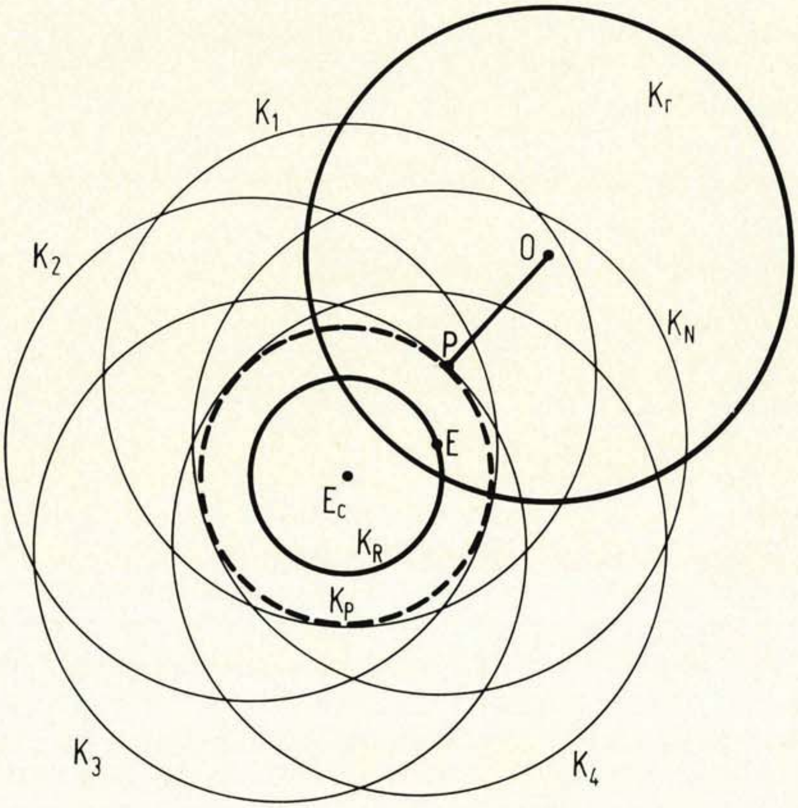

Figure 4. Application of "rope stretching method" in Case II ($|\vec{e}_o| < r$, $R < r$). As in Fig. 2 the centres of the circles K_n are fixed on K_R and a rope is drawn from O through the circles, ending somewhere inside K_N. As the number of circles $N \to \infty$, an enveloping circle K_ρ (broken curve) is formed. Stretching the rope in O moves the end into P on K_ρ and yields the correction $\overrightarrow{PO}$, which shifts the drift circle K_R on the shortest way into the tolerance circle K_r.

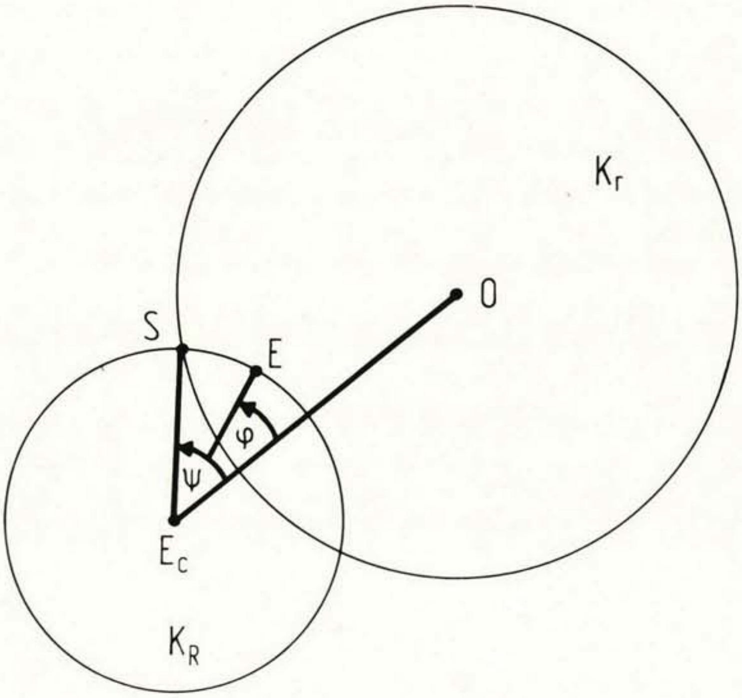

Figure 5. Optimal strategy for frequent corrections in Case II ($|\vec{e}_o| < r$, $r > R$). The eccentricity marked by E stays within K_r as long as $\varphi < \psi$ during the shift of K_R into K_r.

In order to guarantee that E stays always inside K_r, one has to require $\psi \geq \varphi$ until K_R is completely inside K_r. This can be acchieved, for instance, by requiring for each cycle

$$\delta\psi = \frac{\pi-\psi o}{\pi-\varphi o}\,\delta\varphi \qquad (38)$$

where the subscript 0 refers to the initial state and $\delta\varphi$ is the known angular motion of E on K_R during one cycle. Since the correction during one cycle is orientated along $-\vec{e}_c$, one obtains ψ and the vector $\Delta\vec{e}$ from the relations

$$(|\vec{e}_c|-R\cos\psi o)^2 + (R\sin\psi o)^2 =$$

$$(|\vec{e}_c + \Delta\vec{e}| - R\cos\psi)^2 + (R\sin\psi)^2 = r^2 \qquad (39)$$

$$\psi = \psi o + \delta\psi \qquad (40)$$

with the result

$$\Delta\vec{e} = -\vec{e}_c[1-(R\cos\psi + \sqrt{r^2 - R^2\sin^2\psi})/|\vec{e}_c|] \qquad (41)$$

3.2.3 Case III ($|\vec{e}_o| > r$, $r < R$) Since the eccentricity is initially outside the tolerance circle (Fig. 6), it should be moved as fast as possible to a position Eη on that circle such that fuel optimal corrections are possible afterwards. While the eccentricity completes the angle $\varphi\eta$ from E_o to Eη on the drift circle, the two circles should approach each other such that the tolerance circle touches the drift circle from inside in the point Eη. Then the correction vector $\Delta\vec{e}$ for a single cycle is obtained form the relation $\overrightarrow{O^*O} = \overrightarrow{O^*E_c} + \overrightarrow{E_cO}$ or, using the previous definitions

$$n\Delta\vec{e} = -\vec{e}_c - \frac{R-r}{R}\begin{pmatrix}\cos\varphi_n & -\sin\varphi_n\\ \sin\varphi_n & \cos\varphi_n\end{pmatrix}(\vec{e}_o - \vec{e}_c) \qquad (42)$$

$$\varphi\eta = \eta\delta\varphi \qquad (43)$$

where η is the number of cycles necessary to acchieve the total correction. If Δe_{max} is the largest possible correction per cycle this number is obtained by iteration form the inequality

$$n\Delta e_{max} \geq |n\Delta\vec{e}| \geq (n-1)\, e_{max} \qquad (44)$$

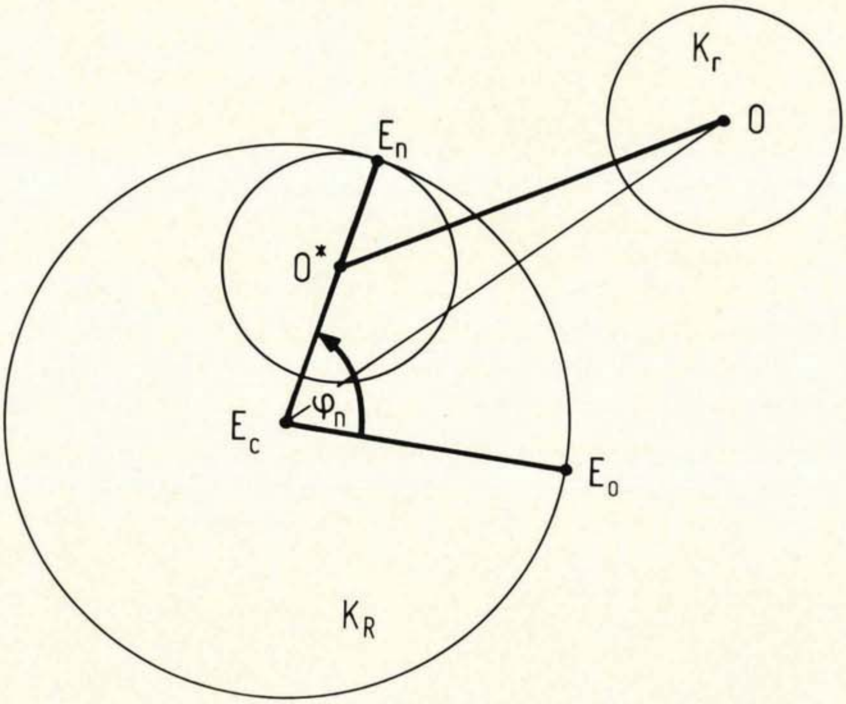

Figure 6. Optimal correction in Case III ($|\vec{e}_o| > r$, $R > r$).
The total correction is a relative shift of the circles K_r and K_R such that K_r touches K_R from inside in a point $E\eta$ which is reached by the eccentricity after η cycles. The number η is to be chosen as small as possible, observing that the magnitude of the correction per cycle is limited because of the low thrust.

3.2.4 Case IV ($|\vec{e}_o| > r > R$) In this case the drift circle should be shifted into the tolerance circle as fast as possible, i.e., in a direction opposite to $\vec{e}_c = \overrightarrow{OE_c}$ (Fig. 7). If η cycles are necessary, the shift $\Delta\vec{e}$ is obtained from

$$n\Delta\vec{e} = \overrightarrow{E_cE_c} = -\vec{e}_c + \frac{\vec{e}_c}{|\vec{e}_c|}(r-R) , \qquad (45)$$

where η is again defined by (44).

3.3 Station keeping simulation

The algorithms developed in the foregoing section were applied to simulate 181 days of low-thrust station keeping of the eccentricity for the following example:

Start of station keeping:	01.01.1983, 0 h U.T.
Station longitude:	19^o West
Initial eccentricity:	$10^6\vec{e}_o = (-8, +31)$
Area/Mass ratio:	0.0501 m^2/kg
Radius of tolerance circle:	r = 0.0004
Radius of drift circle:	R ~ 0.00055
Correction cycle duration:	10 days
Station keeping period:	181 days

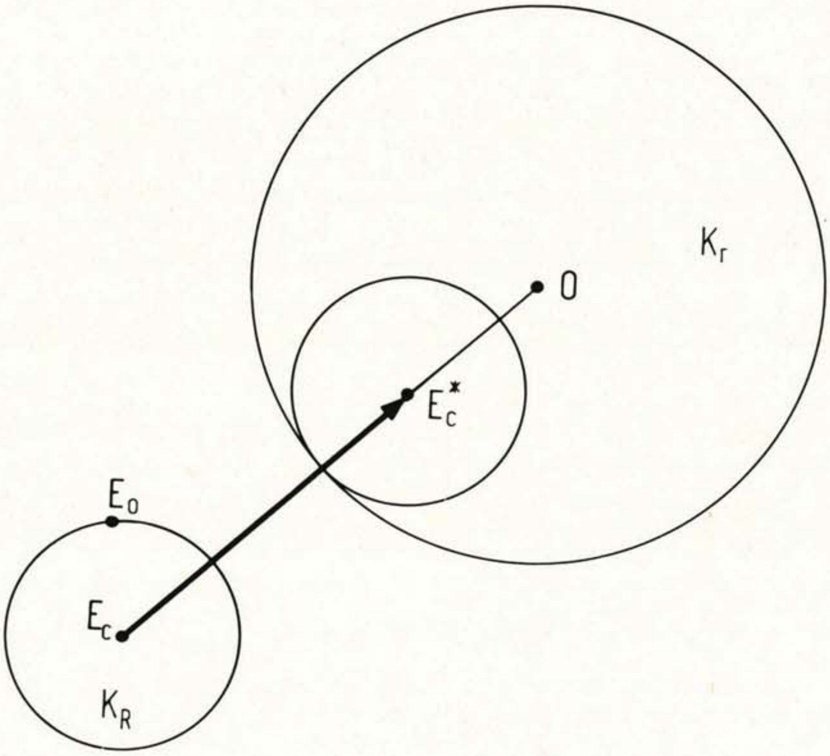

Figure 7. Optimal correction in Case IV ($|\vec{e}_o| > r > R$).
The drift circle K_R should be moved into the tolerance circle K_r as fast as possible, which results in a total correction $\overrightarrow{E_c E_c^*}$.

The results is shown in Fig. 8. Curve A represents the eccentricity variation due to natural perturbations without corrections, according to an approximate analytical orbit model obtained from (Ref. 11) neglecting short periodic perturbations. The combined effects of perturbations and corrections result in curve B. Since the example corresponds to Case I, the corrections are applied such that the eccentricity first increases until it reaches the tolerance circle and then continues to move along that circle.

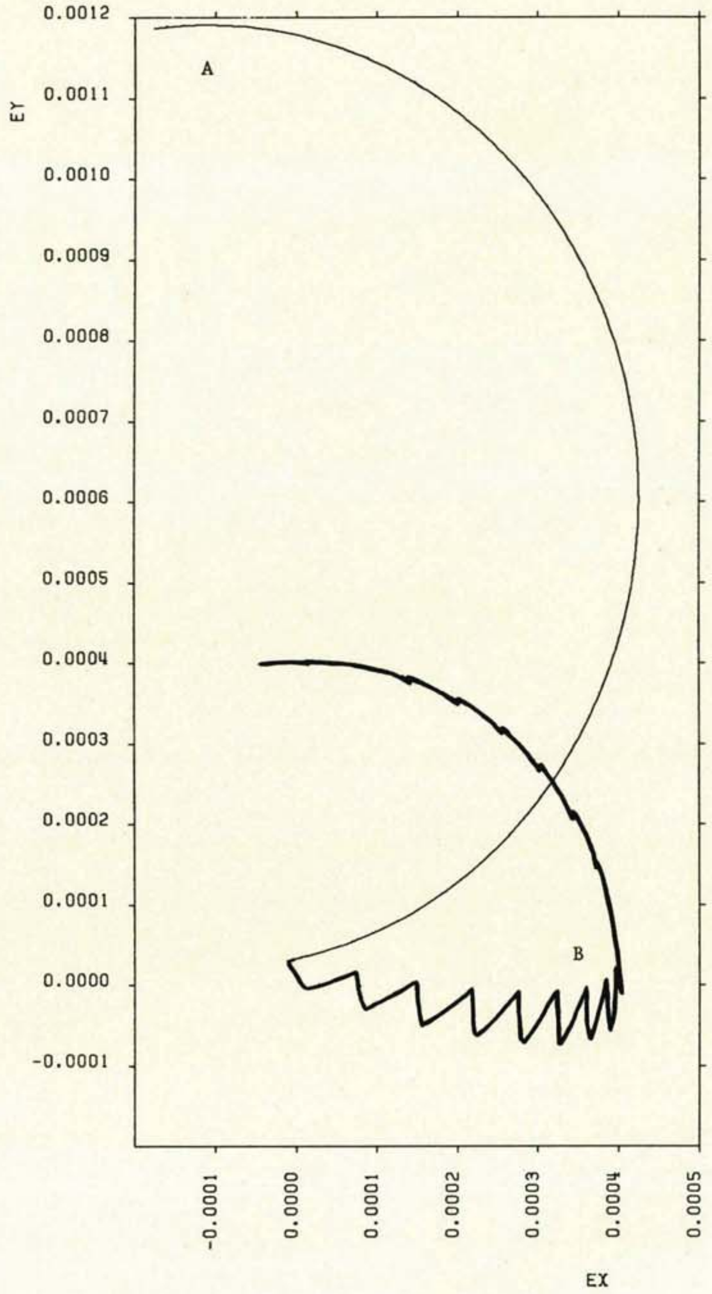

Figure 8. Controlled (curve B) and uncontrolled (curve A) long term eccentricity variation during 181 days.

4. CONCLUSIONS

It was shown that the optimum control problems with low thrust station acquisition and station keeping during geostationary missions can be linearized and solved by discretisation of the thrust times and the thrust directions. This technique has considerable advantages in comparison to the classical methods especially in the presence of constraints. Furthermore, a procedure applying the discretization method to a sequence of fuel-optimal minimum problems may be used to solve certain time minimum problems even in cases where some classical methods fail.

The simple concept of the "rope stretching method" turns out to be a useful tool in deriving algorithms for the long-term strategy of correcting eccentricity perturbations. Some examples of both short-term and long-term optimal control problems associated with low thrust systems demonstrate the capability of the described methods.

5. REFERENCES

1. Ariane, Users Manual, ESA-CNES, 1980

2. Eurosatellite GmbH, Eurosatellite-Industrial Proposal TV-SAT-TDF1, *Mission Description and Rationale* Doc. DF50-TN-00-212-00-03.

3. Marec J P 1979, *Optimal space trajectories*, Amsterdam-Oxford-New York, Elsevier scientific pupl. comp.

4. Lawden D F 1963, *Optimal trajectories for space navigation*, London, Butterworths.

5. Krabs W 1978, *Einführung in die Kontrolltheorie*, Darmstadt, Wissenschaftliche Buchgesellschaft.

6. Lasdon L S 1970, *Optimisation theory for large systems*, London, MacMillan Series in Operations Research.

7. Avriel M 1976, *Nonlinear Programming, Analysis and Methods*, Englewood Cliffs, New Jersey, Prentice Hall.

8. Eckstein M C 1980, *Positionshaltung geostationärer Satelliten mit elektrischen Triebwerken*, DGLR Walter-Hohmann-Symposium Raumflugmechanik, Köln, DGLR 80-009.

9. Fujisawa T & Yasuda Y 1967, *An iterative procedure for solving the time optimal regulator problem*, SIAM J. of Control, Vol. 5, No. 4, 501-512.

10. Hechler F, Soop M & Schäfer M 1977, *Solution of a fuel minimum long term attitude control problem by a numerical technique and an ISEE-B dedicated low-cost hardware*, ESOC, OAD Working Paper No. 100.

11. Eckstein M C 1979, *Ein Verfahren zur Herstellung analytischer Bahnmodelle für geostationäre Satelliten*, DFVLR-FB 79-16.