AN ANALYTICAL SINGULARITY-FREE ORBIT PREDICTOR FOR NEAR-EARTH SATELLITES.

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Abstract

A completely analytical, first order satellite theory intended for low earth orbits is presented. Perturbations are unified under the non-singular Poincare-Similar element formulation. The theory includes short period, long period and secular effects of J_2 and all higher zonal harmonics; secular and quadratic effects of atmospheric drag; and the average mean motion considering all harmonics of the geopotential. Extensive use has been made of recursive equations instead of explicit formulas. In the drag theory, the density model accounts for not only changes in altitude but also the important effects of the sun and its location. The theory has been implemented in an operational computer program.

The fast execution times of the analytical methods make them very attractive as mission analysis and planning tools. But earlier analytical methods were difficult to apply because of the following problems:

1) The solutions were expressed by extremely lengthy formulas which required much more computer storage than numerical methods.
2) The solutions were based on simplified models of the perturbing forces and did not accurately represent the true orbital environment.
3) The analytical solution method did not provide enough accuracy.
4) The different perturbations were not unified under one non-singular formulation.

Introduction

Recent theoretical developments describing the orbital motion of a satellite using only analytical expressions have now been completed and implemented in the form of a computer program. The theory is intended to be used for computation of near-earth orbits including those of the Shuttle/Orbiter and its payloads. This paper gives an overview of the theory and discussion of the numerical comparisons.

Orbit computation methods can usually be given one of the following two classifications:

a) Numerical methods - The calculations are carried out in a step-by-step manner. High precision is possible, but computer runtime can be excessive.
b) Analytical methods - The calculations are carried out in one step regardless of the prediction interval. Therefore, these methods have extremely fast computation times.

The above mentioned problems have been overcome by the approaches described in this paper. All perturbations are treated by using the powerful tools provided by Hamiltonian mechanics. The geopotential is treated entirely using Von Zeipel's solution method.

The perturbations are unified under one non-singular formulation, namely the Poincaré-Similar elements (PSϕ). This is a canonical set of elements in an extended phase space with the true anomaly, not time, as the independent variable. The nature of these elements allows an important reduction in the number of formulas needed to express the solution.

Another important feature is the extensive use of recursive equations instead of explicit formulas. The recursive relations are well suited for computer applications and reduce considerably the overall computer storage. In addition, the recursive expressions enable the implementation of much more complex models of the perturbing forces. Thus, the

Proc. Int. Symp. Spacecraft Flight Dynamics, Darmstadt, FRG, 18-22 May 1981 (ESA SP-160, Aug. 1981)

theory includes the perturbations of all the harmonics of the geopotential and also the perturbations due to drag, in which the atmospheric density is strongly effected by the sun and its location. These are important features when the application is for near-earth satellite orbits.

True anomaly DS-elements

The DSϕ elements ref.(1),(2) listed below, are different from the classical Delaunay elements in that the unperturbed Jacobian equation is separated after the transformation of the independent variable rather than before.

The angular variables are

$\alpha_1 = \phi$ true anomaly
$\alpha_2 = g$ argument of pericenter
$\alpha_3 = h$ ascending node
$\alpha_4 = \ell$ time element

The action variables are

$\beta_1 = \Phi$ realted to two-body energy
$\beta_2 = G$ angular momentum magnitude
$\beta_3 = H$ Z component of angular momentum
$\beta_4 = L$ the total energy

These may be transformed to the canonical PSϕ elements:

$$\sigma_1 = \phi + g + h \qquad \rho_1 = \Phi$$
$$\sigma_2 = -\sqrt{2(\Phi - G)}\sin(g+h) \qquad \rho_2 = \sqrt{2(\Phi - G)}\cos(g+h)$$
$$\sigma_3 = -\sqrt{2(G-H)}\sin h \qquad \rho_3 = \sqrt{2(G-H)}\cos h$$
$$\sigma_4 = \ell \qquad \rho_4 = L$$

Abbreviations used in the text are:

$$p = \frac{1}{\mu}\left(G - \Phi + \frac{\mu}{\sqrt{2L}}\right)^2 \quad \text{(semi-latus rectum)}$$
$$q = G - \frac{1}{2}\left(\Phi - \frac{\mu}{\sqrt{2L}}\right)$$
$$e = \sqrt{1 - \frac{2L}{\mu}p} \quad \text{(eccentricity)}$$
$$s = \sin I = \sqrt{1 - \frac{H^2}{G^2}} \quad \text{(I is the inclination)}$$
$$c = \cos I = \frac{H}{G} \qquad r = \frac{p}{1+\zeta}$$
$$Q = \frac{\sqrt{\rho_4}}{\mu}\left(\frac{2\mu}{\sqrt{2\rho_4}} + G - \Phi\right)^{\frac{1}{2}} \qquad \zeta = e\cos\phi$$
$$P = \frac{\sqrt{2(G+H)}}{2G} \qquad \frac{dt}{d\tau} = \frac{r^2}{q}$$

(τ = independent variable)

The perturbing Hamiltonian

The geopotential perturbations are treated entirely by applying Von Zeipel's solution method to the DSϕ elements and carefully rewriting the solution in a nonsingular form using the PSϕ elements. The geopotential is expanded in recursion formulation to allow any order or degree model. Considerable simplifications are offered by the PSϕ elements. By including the true longitude as a canonical element, the zonal Hamiltonian becomes a finite fourier expansion in the canonical elements. Also, since the mean motion is related to the total energy, only the second order time dependent harmonics perturb this value. The result is errors in the down track which are of the same magnitude as the errors in the out of plane and radial directions.

The DSϕ Hamiltonian may be written in the form

$$F = F_o + \varepsilon F_1 + \varepsilon^2 F_2 \tag{1}$$

where

$$F_o = \Phi + \mu/\sqrt{2L} \qquad \text{two body} \tag{2}$$
$$\varepsilon F_1 = \frac{r}{q} C_{20}\left(\frac{R_e}{r}\right)^2 P_2^o\left(\frac{z}{r}\right) \qquad J_2 \tag{3}$$
$$\varepsilon^2 F_2 = \varepsilon^2 F_z + \varepsilon^2 F_T \qquad \text{(Zonals \& Time Dependent)} \tag{4}$$

and

$$\varepsilon^2 F_z = \frac{r}{q}\sum_{n=3}^{\infty} C_{no}\left(\frac{R_e}{r}\right)^n P_n^o\left(\frac{z}{r}\right) \tag{5}$$
$$\varepsilon^2 F_T = \frac{r}{q}\sum_{n=2}^{\infty}\sum_{m=1}^{n}\left(\frac{R_e}{r}\right)^n P_n^m\left(\frac{z}{r}\right)\left\{C_{nm}\cos m\lambda + S_{nm}\sin m\lambda\right\} \tag{6}$$

P_n^m are the associated Legendre polynomials; R_e is the mean equatorial radius; $C_{n,m}$, $S_{n,m}$ are the geopotential coefficients; λ is the longitude of the satellite with respect to the Greenwich meridian.

F_1, F_z and F_T may be written in the form of a fourier series in the DSϕ canonical angular elements, ref.(11).

$$\varepsilon F_1 + \varepsilon^2 F_z = \frac{p}{q}\sum_{n=2}^{\infty}\sum_{p=0}^{n}\sum_{k=1-n}^{n-1}\left(\frac{R_e}{p}\right)^n F_{nop} G_{n-1,ok} \times \left\{A_{no}\cos\psi_{nopk} + B_{no}\sin\psi_{nopk}\right\}, \tag{7}$$

$$\varepsilon^2 F_T = \frac{p}{q}\sum_{n=2}^{\infty}\sum_{m=1}^{n}\sum_{p=0}^{n}\sum_{k=-\infty}^{\infty}\left(\frac{R_e}{p}\right)^n F_{nmp} G_{n-1,mk} \left\{A_{nm}\cos\psi_{nmpk} + B_{nm}\sin\psi_{nmpk}\right\} \tag{8}$$

where

$$A_{nm} = \begin{cases} C_{nm} & n-m \quad \text{even} \\ -S_{nm} & n-m \quad \text{odd} \end{cases}$$

$$B_{nm} = \begin{cases} S_{nm} & n-m \quad \text{even} \\ C_{nm} & n-m \quad \text{odd} \end{cases}$$

$$\psi_{nmpk} = (n-2p+k)\phi + (n-2p)g + m(h-\omega_{\oplus}\ell+\theta_o)$$

$$F_{nmp} = F_{nmp}(c) \equiv \text{Inclination Function}$$

$$G_{nmk} = G_{nmk}(e,m\nu) \equiv \text{Eccentricity Function}$$

ν = ratio of earth rotation rate and mean motion

Recursive expressions for F_{nmp} may be found in Giacaglia ref.(3). Unlike the expansions in classical elements, the true, not the mean anomaly appears in the angular argument. *Note that the zonal perturbation is a finite series and does not contain the time element* ℓ. The eccentricity function G_{nmk} is similar to the Hansen Coefficients ref.(4) in classical theory except that an additional small argument ν appears in the series expressing the function. Since ν for low earth satellites is about 1/16, the series expression for G_{nmk} tends to converge faster than Hansen Coefficients.

These expressions may be rewritten in the non-singular PSϕ elements to remove the singularities. The expansions then become

$$\epsilon F_1 + \epsilon^2 F_z = \tag{10}$$

$$\frac{\mu}{q} \sum_{n=2}^{\infty} \sum_{p=0}^{n} \sum_{k=1-n}^{n-1} \left(\frac{R_e}{p}\right)^n \overline{F}_{nok}\overline{G}_{n-1,ok} \times \left\{ \overline{A}_{nopk} \cos\Theta_{nopk} + \overline{B}_{nopk} \sin\Theta_{nopk} \right\}$$

$$\epsilon^2 F_T = \tag{11}$$

$$\frac{\mu}{q} \sum_{n=z}^{\infty} \sum_{m=0}^{n} \sum_{p=0}^{n} \sum_{k=-\infty}^{\infty} \left(\frac{R_e}{p}\right)^n \overline{F}_{nmk}\overline{G}_{n-1,mk} \times \left\{ \overline{A}_{nmpk} \cos\Theta_{nmpk} + \overline{B}_{nmpk} \sin\Theta_{nmpk} \right\}$$

where

$$\overline{A}_{nmpk} = A_{nm} R_{kq} - B_{nm} I_{kq}$$

$$\overline{B}_{nmpk} = A_{nm} I_{kq} + B_{nm} R_{kq}$$

$$\overline{F}_{nmp} = \frac{F_{nmp}}{s^{|q|}} \quad q = m - n + 2p$$

$$\overline{G}_{nmk} = \frac{G_{nmk}}{e^{|k|}}$$

$$\Theta_{nmpk} = (n-2p+k)\,\sigma_1 - m(\omega_{\oplus}\ell+\theta_o)$$

$$R_{kq} = e^{|k|} s^{|q|} \cos(k(g+h) + qh)$$

$$I_{kq} = e^{|k|} s^{|q|} \sin(k(g+h) + qh)$$

Nonsingular recursion relations exist for $\overline{F}_{nmp}$ R_{kq} and I_{kq} which are all polynomial of $\sigma_1, \sigma_2, \sigma_3, \rho_2$ and ρ_3. G_{nmk} may be obtained from a series expression similar to those of Hansen Coefficients.

Von Zeipels Solution Method

The objective of the Von Zeipel method is to transform the system so that the angular variables $(\phi, g, h, \omega_{\oplus}\ell)$ are removed from the DS Hamiltonian and, therefore, admit a solvable system of differential equations.

To eliminate the short and intermediate periodicities one assumes a generating function of the form

$$S = S_0 + \epsilon S_1 + \epsilon^2 S_2$$

$$S_2 = S_z + S_T \tag{12}$$

where S_o gives the identity transformation, S_T is a periodic function of ϕ and $\omega_{\oplus}\ell$ and S_1 and S_z are periodic functions of ϕ only. The transformation in the DSϕ space is given by

$$\alpha' = \alpha + \epsilon\frac{\partial S_1}{\partial \beta'} + \epsilon^2\frac{\partial S_2}{\partial \beta'} \tag{13}$$

$$\beta = \beta' + \epsilon\frac{\partial S_1}{\partial \alpha} + \epsilon^2\frac{\partial S_2}{\partial \alpha}$$

We desire to transform the elements such that the new Hamiltonian is no longer a function of the angular variables. The necessary Von Zeipel equations to derive S_1 and S_2 are given by ref.(2).

$$\epsilon^o \quad F_o'(\Phi', L') = F_o(\Phi', L') \tag{14}$$

$$\epsilon^1 \quad \frac{\partial F_o}{\partial \Phi}\frac{\partial S_1}{\partial \phi} = -F_1(\beta', \phi, g) + F_1'(\beta', g) \tag{15}$$

$$\epsilon^2 \quad \frac{\partial F_o}{\partial \Phi'}\frac{\partial S_T}{\partial \phi} + \frac{\partial F_o}{\partial L'}\frac{\partial S_T}{\partial \ell} = -F_T(\beta', \phi, g, h, \omega\,\ell) \tag{16}$$

$$S_z = 0$$

where

$$F_1'(\beta', g) = \frac{1}{2\pi}\int_{-\pi}^{\pi} F_1 d\phi$$

$$F_2'(\beta', g) = \frac{1}{2\pi}\int_{-\pi}^{\pi} F_z - \frac{\partial S_1}{\partial\phi\partial\Phi'}\frac{\partial S_1}{\partial\phi} + \frac{\partial^2 S_1}{\partial\phi\partial G'}\frac{\partial S_1}{\partial g}\, d\phi$$

The first order long period perturbations may be found by assuming another generating function of the form

$$S^* = S_0^* + \epsilon S_1^* \tag{17}$$

to eliminate the appearance of the g in the Hamiltonian. The necessary equations are

$$\frac{\partial F_1'}{\partial G''}\frac{\partial S_1^*}{\partial g'} = -F_2'(\beta', g') + F_z''(\beta') \tag{18}$$

$$F_2'' = \frac{1}{2\pi}\int_{-\pi}^{\pi} F_2'(\beta', g')dg'$$

Finally, one must rewrite the solutions for S^* and S entirely in the PSϕ elements removing any singularities for small inclinations and eccentricities. The short, intermediate and long periodic elimination in the PSϕ elements, neglecting second order terms becomes

$$\sigma'' = \sigma + \varepsilon \frac{\partial S_1}{\partial \rho} + \varepsilon \frac{\partial S_1^*}{\partial \rho} + \varepsilon^2 \frac{\partial S_T}{\partial \rho}$$

$$\rho'' = \rho - \varepsilon \frac{\partial S_1}{\partial \sigma} - \varepsilon \frac{\partial S_1^*}{\partial \sigma} - \varepsilon^2 \frac{\partial S_T}{\partial \sigma} \qquad (19)$$

Applying the Von Zeipel method to the geopotential expansion one finds

$$\varepsilon S_1 = \frac{p}{q} \sum_{p=0}^{2} \sum_{\substack{k=-1 \\ k \neq 0}}^{1} \left(\frac{R_e}{p}\right)^2 \bar{F}_{2op} \bar{G}_{2op}/k \times \qquad (20)$$

$$\left\{ \bar{A}_{2opk} \sin\Theta_{2opk} - \bar{B}_{2opk} \cos\Theta_{2opk} \right\}$$

$$\varepsilon S_1^* = \frac{1}{q}\left[\varepsilon \frac{\partial F_1'}{\partial G}\right]^{-1} \varepsilon^2 \hat{S}$$

$$\varepsilon^2 \hat{S} = p \sum_{n=3}^{\infty} \sum_{\rho=1}^{3} \left(\frac{R_e}{p}\right)^n \bar{F}_{nop} \bar{G}_{n-1,0,q}/q$$

$$\left\{ \begin{matrix} -A_{no} I_{q,-q} & n=\text{even} \\ B_{no} R_{q,-q} & n=\text{odd} \end{matrix} \right\}_{\substack{q = n - 2p \\ q \neq 0}} \qquad (21)$$

$$+ \frac{9}{8} R_e^4 C_{20}^2 \, (3s^2 - 2 + 12;c/H) I_{2,-2}$$

$$\varepsilon^2 S_T = -\frac{p}{q} \sum_{n=z}^{\infty} \sum_{m=1}^{n} \sum_{p=0}^{n} \sum_{k=-\infty}^{\infty} \left(\frac{R_e}{p}\right)^n \frac{\bar{F}_{nmp} \bar{G}_{n-1,mk}}{(n-2p+k-\nu m)} \times$$

$$\left\{ \bar{A}_{nmpk} \sin\Theta_{nmpk} - \bar{B}_{nmpk} \cos\Theta_{nmpk} \right\} \qquad (22)$$

If one is interested in corrections for the mean motion, all terms in S_T should be considered. Otherwise, only terms where $n-2p+k=0$ should be computed.

The new DSϕ Hamiltonian becomes

$$F'' = F_o''(\beta'') + \varepsilon F_1''(\beta'') + \varepsilon^2 F_2''(\beta'') \qquad (23)$$

where

$$F_o'' = \Phi'' + \frac{\mu}{\sqrt{2L''}} \qquad (24)$$

$$\varepsilon F_1'' = \frac{R_e^2 C_{20}}{p q} \bar{F}_{201} \qquad (25)$$

$$\varepsilon^2 F_2'' = \frac{p}{q} \sum_{\substack{n=2 \\ n \neq \text{odd}}}^{\infty} \left(\frac{R_e}{p}\right)^n \bar{F}_{n0,n/2} \bar{G}_{n-1,00} C_{no} +$$

$$\frac{C_{20}^2 R_e^4 \delta}{128 p^2 q^2} \qquad (26)$$

$$\delta = \frac{e^2}{q}(-3s^4 + 24s^2 - 8) + \frac{18s^4}{q}$$

$$- \sqrt{\frac{p}{\mu}} \left(\frac{e^2}{p} + \frac{L}{\mu}\right)(60s^4 - 96s^2 + 32)$$

$$- \frac{2c^2 s^2}{G} (24c^2 + 36)$$

Observe, since F'' is only a function of the DSϕ momenta, one is able to solve for secular changes in the angular DSϕ conjugates.

$$\alpha'' = \frac{\partial F''}{\partial \beta''}\tau + \alpha_o'' \qquad (27)$$

$$\beta'' = \beta_o'' \qquad (28)$$

True Anomaly PS Elements

For vanishing eccentricities and inclinations the above set of DSϕ elements is not suited because their perturbing differential equations have singularities in these cases. It is then necessary to introduce 8 canonical elements ρ_k, σ_k in a similar way as it is customary in classical theory where the corresponding elements are called Poincaré-elements:

$$\rho_1 = \Phi \quad , \qquad \sigma_1 = \phi + g + h$$

$$\rho_2 = C \cos(g+h) \quad , \qquad \sigma_2 = -C \sin(g+h)$$

$$\rho_3 = D \cos h \quad , \qquad \sigma_3 = -D \sin h$$

$$\rho_4 = L \qquad\qquad \sigma_4 = \ell$$

where

$$C = \sqrt{2(\Phi - G)} \text{ and } D = \sqrt{2(G-H)} \; .$$

The expressions may then be rewritten in the PSϕ space. For example,

$$\sigma_1'' = \alpha_1'' + \alpha_2'' + \alpha_3'' \qquad (29)$$

Replacing the above expressions and grouping

$$\sigma_1'' = \left(\frac{\partial F''}{\partial \beta_1''} + \frac{\partial F''}{\partial \beta_2''} + \frac{\partial F''}{\partial \beta_3''}\right)\tau + \sigma_{10}'' \qquad (30)$$

Similar expressions can be found for the rest of the PSϕ elements. The reverse transformation neglecting $O(\varepsilon^2)$ terms may be found by reversing the signs in equation (19).

Observe that although considerable use has been made of the singular DSϕ elements to derive the solution, in the computational algorithm this is quite transparent since all calculation are made using the well-defined PSϕ elements.

Geopotential Numerical Comparisons.

All the theory involving drag except for the small daily periodic effects have been implemented in an operational computer program - ASOP, ref.(10).

The analytical geopotential solutions have been compared to precise numerically integrated solutions to determine the accuracies. The results are shown in Figures 1, 2 and 3. A typical shuttle type orbit has been integrated numerically for 100 revolutions using an 8th order, 8th degree (8x8) model. This has been compared to the analytically 8x8 model labeled #1, an 8th order zonal model (#2), and a 2nd order zonal model (#3). In Figure 1, the combined out of plane and radial position differences are shown for each solution. Note that there is no discernable difference between #1 and #2 and that both remain small and periodic. However, by neglecting the higher order zonals in #3, one sees a small secular error growth. The small periodic error in #1 and #2 is due mainly to the daily periodic effects which have not been implemented. In Figure 2 the in track position differences are shown. Note that only solution #1 (which corrects for the mean motion due to the time dependent terms) remains small and periodic. This periodic error in #1 remains always less than 600 meters. In Figure 3 we present the in track and out of track differences for #1 over one day. The inclusion of the daily periodic effects into the program could reduce this error to about 10 m. The error growth exhibited by #1 in Figure 2, is smaller than the modelling error expected from the insufficient precision knowledge of the gravitational constant: μ .

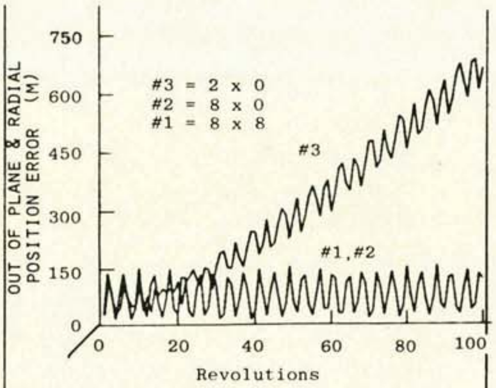

FIGURE 1

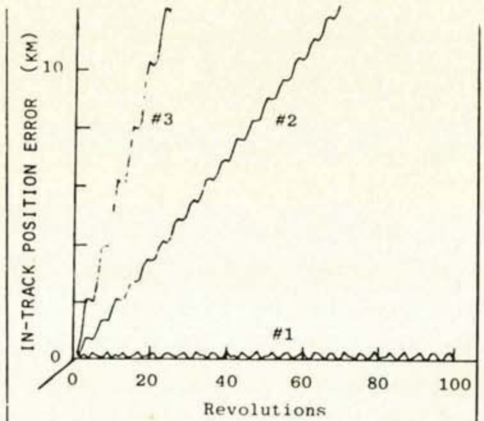

FIGURE 2

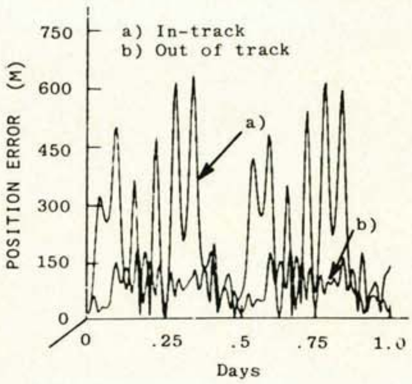

FIGURE 3

Atmospheric Drag Perturbation

The goals of the drag analytical satellite theory were 1) to base the theory completely on an canonical formalism whereby one can use the powerful tools provided by Hamiltonian mechanics; 2) to not simplify the model used to describe the forces acting on a satellite such that the theory becomes only a mathematical exercise; and, 3) to result in a concise theory so that the accuracy gained outweighs the extra computer costs to reach that accuracy. We feel all these goals have been satisfied.

The assumption in the theory is that the drag force is proportional to the square of the velocity magnitude relative to the inertial atmospheric velocity and acting along this relative velocity.

Mueller ref.(6) has developed an accurate density model which takes into account recent investigations (7) and (8), showing that the density of the upper atmosphere is extremely effected by the sun activity and its position. Complicated models are available but prove to be too unwieldy for application in satellite theories. A completely new model had to be developed in which it is able to simulate the more complicated models yet can still be written in the form of a fourier series.

The approach taken was to construct a model which simulates the Jacchia density model along a particular orbit. The value of the coefficients in the new model are determined by a procedure called "calibration". A simple formulation allows the model to be inverted, i.e., given the density at different points along the orbit (as determined from Jacchia), one can compute the coefficients of the model. The coefficients are implicit functions only of long period effects and can be considered constants in the analytical theory. The new model was then implemented into the theory using a careful balance of explicit equations and recursive relations to minimize core requirements.

The coupling of drag and J_2 is neglected except in the critical effects of J_2 on the radius and thus the density. A surprising result is that the true longitude (which is so important in computing J_2 short period effects) is not affected by the in plane drag perturbation. This is a critical decoupling of J_2 and drag and reflects the fact that the geometry of the motion is fully separated from the dynamics within the orbit, typical of the PSϕ formulation.

Canonical Drag Equations of Motion.

The generalized PSϕ differential equations are

$$\frac{d\sigma_k}{d\tau} = \frac{\partial F}{\partial \rho_k} - T_k \qquad (31)$$

$$k = 1,2,3,4$$

$$\frac{d\rho_k}{d\tau} = -\frac{\partial F}{\partial \sigma_k} + U_k$$

where F is the geopotential Hamiltonian described earlier, and T_K and U_K are the canonical drag forces. The derivation of the canonical forces has been described in the extended phase space in references (2) and (5). The canonical in plane forces reduce to

$$T_k = \frac{r^2}{q} v\rho C \left\{ (1-\kappa)\left(u \frac{dr}{d\rho_k} - \begin{bmatrix} 0 \\ 0 \\ \frac{1}{2}\sigma_3 \\ 0 \end{bmatrix} \right) - (1-2\kappa) v^2 \frac{dt}{d\rho_k} \right\}$$
$$U_k = \frac{r^2}{q} v\rho C \left\{ (1-\kappa)\left(1 \frac{dr}{d\sigma_k} + \begin{bmatrix} G \\ 0 \\ \frac{1}{2}\rho_3 \\ 0 \end{bmatrix} \right) - (1-2\kappa) v^2 \frac{dt}{d\sigma_k} \right\} \tag{32}$$

The out of plane forces may be expressed in the canonical forces as

$$T_k = \frac{r^2}{q} C\rho v \omega_{\oplus} \left\{ y\frac{dx}{d\rho_k} - x\frac{dy}{d\rho_k} \right\}$$
$$U_k = \frac{r^2}{q} C\rho v \omega_{\oplus} \left\{ y\frac{dx}{d\sigma_k} - x\frac{dy}{d\sigma_k} \right\} \tag{33}$$

where $\kappa = \dfrac{\omega_{\oplus} H}{2L}$

$$u = \frac{QZ_2(1-\eta^2)}{\beta^2 P}$$

$$Z_2 = \rho_2 \sin\sigma_1 + \sigma_2 \cos\sigma_1$$

$$\beta^2 = \frac{\sqrt{2L}}{\mu}$$

$$\eta^2 = \frac{\beta^2}{2} (\sigma_2^2 + \rho_2^2)$$

C = Area/Mass

Analytical Density Model.

In a manner similar to Santora ref.(9), we define a density along a mean reference orbit above a sphere which has a radius equal to that of the average the satellite observes in its orbit. Just as the radius may be expanded in a power series in ζ we write the reference density as

$$\rho_o = \sum_{k=0}^{n} (a_k + db_k)\zeta^k \tag{34}$$

where a_k and b_k are the coefficients to be found through calibration and d reflects the magnitude of the diurnal effect. The expression for d is

$$d = \left(\frac{1 + \cos\psi}{2}\right)^2 \tag{35}$$

where

$$\cos\psi = \frac{1}{r}\left\{ z \sin\delta_s + \cos\delta_s (x\cos\gamma + y\sin\gamma) \right\}$$

$\gamma = \alpha_s + \phi$; $(\alpha_s, \delta_s) \equiv$ right ascension and declination of sun

$\phi \equiv$ defines the lag of diurnal bulge behind the sun ($\phi = 37^{\circ}$)

The total density model then may be expressed as

$$\rho = \rho_o e^{\alpha \Delta h} \approx \rho_o (1+\alpha\Delta h) \tag{36}$$

where α is a constant defining the change in density with small changes in height and Δh is the small changes in the altitude about the reference orbit and sphere. This includes the variations in height due to J_2 periodicities, the oblate figure of the earth and when the satellite is low, the drag effect itself.

Drag Numerical Comparisons.

As in the geopotential solution comparisons, the drag analytical theory is compared to a precise numerical solution. In all the cases, the numerical solution includes the J_2 and drag forces using the Jacchia 71 density model. This is compared to the analytical J_2 (first order secular and short period terms only) and drag solution where the analytical density model is calibrated to the Jacchia model used in the numerical solution. In both numerical and analytical solutions the solar flux $F_{10.7}$ and geomagnetic index K_p were set equal to their average values. The total position difference between solutions over a 5 day period is shown in each of the figures 4, 5 and 6. Also shown is the position difference with drag turned off. In each case the initial conditions are the same. We chose a polar orbit with an h_p=300 km and h_a=556 km and a perigee which lies above the equator. The node is positioned such that the orbit lies in the diurnal bulge when the sun is at the vernal equinox. Figure 4 represents the case where the sun is at the vernal equinox and is in a period of high solar activity $F_{10.7}$ =250. Figure 5 represents the case in which the sun is at the summer soltice and $F_{10.7}$= 250 again. Figure 6 is the case in which the sun is at the vernal equinox but the solar activity is low, $F_{10.7}$ = 75.

Table 1 gives some typical numbers on the differences of analytical versus numerical integration for a realistic shuttle orbit after 20 revolutions.

Model	e=0	e=.015	e=.1
	position difference in km		
Neglect Drag	1481	1506	1920
With Drag	.97	1.01	2.18

TABLE 1

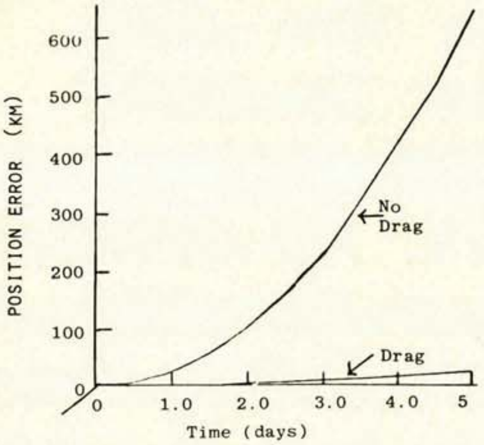

FIGURE 4

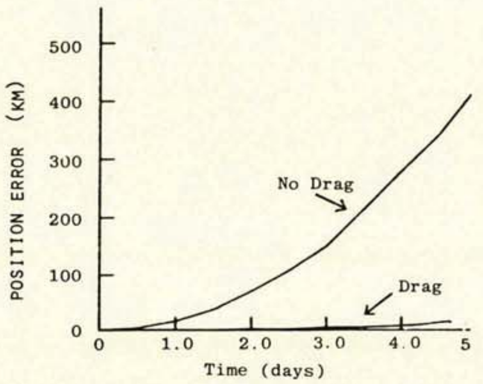

FIGURE 5

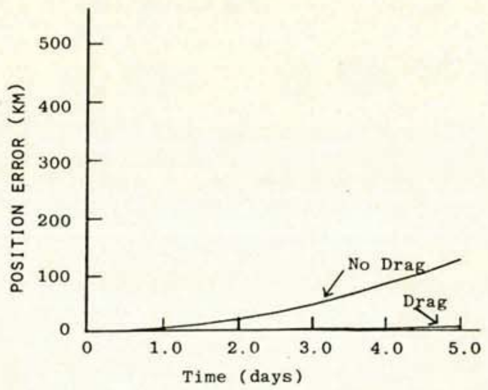

FIGURE 6

Computer Program.

The theory has been implemented into a structured modular designed program called Analytical Satellite Orbit Predictor (ASOP). All the different perturbations are separated into modules, so that a user may select only the modules he needs. The execution times vary with orbit and size of model but are on the order of 25ms to initialize and 5ms to take a step (Univac 1110). The program storage requirements can also vary with the size of model but ranges in the neighborhood of 18 k, 36 bit words (all coding in double precision).

Conclusion

Inaccuracies of the computer program based on the methods described are given by the physical limitations of the force models rather than the neglections made while carrying out the analytical solution itself. - For near earth orbiters we may have reached the point where numerical integration of orbits becomes obsolete.

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