THE NEWTON-RAPHSON METHOD IN ORBIT DETERMINATION

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ABSTRACT

In this paper, the applications of the Newton-Raphson iterative method to artificial satellite's orbit determination are discussed. The fundamental equations are derived. The solving process of the method is described. The convergency of the method is analysed. Some numerical examples are presented so as to demonstrate that the method is available. The future investigations are suggested.

LIST OF SYMBOLS

$\vec{r}$ - radius-vector from Earth's center to the satellite
$\vec{R}$ - radius-vector from the Earth's center to the tracking station
$\vec{\rho}$ - radius-vector from the tracking station to the satellite
r - distance from the Earth's center to the satellite
GM-the gravitational constant of the Earth
O- 3×1 zero-vector or zero matrix of 3×3
I - 3×3 unit matrix
The inertial rectangular coordinate system of 1950.0 is used in this paper
$x_1(t)$, $x_2(t)$, $x_3(t)$ - position components of satellite
$x_4(t)$, $x_5(t)$, $x_6(t)$ - velocity components of satellite
$X_1(t)=[x_1(t)\ x_2(t)\ x_3(t)]^T$ - position-vector
$X_2(t)=[x_4(t)\ x_5(t)\ x_6(t)]^T$ - velocity-vectOr
$X(t)=[X_1(t)\ X_2(t)]^T$ - state-vector
$R_1(t), R_2(t), R_3(t)$ - position components of the tracking station
$R_4(t), R_5(t), R_6(t)$ - velocity components of the tracking station
$R_1(t)=[R_1(t)\ R_2(t)\ R_3(t)]^T$ - position vector
$R_2(t)=[R_4(t)\ R_5(t)\ R_6(t)]^T$ - velocity-vector
$R(t)=[R_1(t)\ R_2(t)]^T$ -state-vector
$f_1(X,t), f_2(X,t), f_3(X,t)$ - acceleration components caused by acting forces
$f(X,t)=[f_1(X,t)\ f_2(X,t)\ f_3(X,t)]^T$ - acceleration-vector
$F(X,t)=[0\ \ f(X,t)]^T$ - function-vector
Superscripts:
' T ' - transpose of a vector or a matrix
' * ' - reference vector
' ~ ' - true vector

INTRODUCTION

A numercal technique in which the sixpoint boundary value problem of the differential equation system is solved by and is called 'Newton-Raphson method' can be used to determing the artificial satellite's orbit. On principle, this method can be used for any kind of the observation data obtained from a single tracking station and the complete dynamical equation containing perturbation forces. Therefore,it can be expected that the computing accuracy will be quite high.

The difficulty in the applications of the Newton-Raphson method to the orbit determination is that the convergency of the method isn't assured easily. This is caused by that on the one hand the observation-state relationships are the nonlinear algebraic equation system for the Ranging data or the Doppler data and on other hand the linearization of the state equation will results the errors which limits the convergency of the iterative process.

In this paper, the solving process of the nonlinear algebraic equation system is discussed and the convergency of the method is investigated. In the numerical examples, the equation which describes the motion of the satellite is the equation of the two-body motion and the simulation data are used. In fact, the method itself isn't limited by the two-body motion. The state equation can involves any acting forces. The future investigations will be suggested.

FUNDAMENTAL EQUATIONS

According to the Newton's law, the equation

*Some parts of calculation are finished by Wang-bin and Huang Wen-ling

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which describes the motion of the satellite can be written in general form as:

$$\frac{dX(t)}{dt} = F(X,t) \qquad (1)$$

where $F(X,t)$ may involves the acceleration caused by the central attraction of the Earth, nonspherical attraction of the Earth, the atmospheric drag, the attraction of the sun and moon, the solar radiation pressure.

It is assumed that the reference state-vector is denoted by $X^*(t)$, then

$$\delta X(t) = X(t) - X^*(t) \qquad (2)$$

is called the variation of the state-vector

$$\delta X(t) = \left[\delta X_1(t)\ \delta X_2(t)\right]^T \qquad (3)$$

where

$$\begin{cases} \delta X_1(t) = \left[\delta x_1(t)\ \delta x_2(t)\ \delta x_3(t)\right]^T \\ \delta X_2(t) = \left[\delta x_4(t)\ \delta x_5(t)\ \delta x_6(t)\right]^T \end{cases} \qquad (4)$$

and $\delta X_1(t)$ is the variation of the position-vector and $\delta X_2(t)$ is the variation of the velocity-vector

The partial differential matrix of $F(X,t)$ and $X^*(t)$ is denoted by the expression as follows:

$$A^*(t) = \left(\frac{\partial F(X,t)}{\partial X(t)}\right)_{X^*(t)} \qquad (5)$$

then the variation equation of the state-equation near the reference state-vector is as

$$\frac{d\delta X(t)}{dt} = A^*(t)\delta X(t) \qquad (6)$$

This is a linear differential equation about the $\delta X(t)$.

It is assumed that the satellite is subjected only to Earth's central forces, the state equation which describes the motion of the satellite can be expressed in the form

$$\frac{dX(t)}{dt} = \begin{bmatrix} 0 & I \\ -\frac{GM}{r^3} I & 0 \end{bmatrix} \begin{bmatrix} X_1(t) \\ X_2(t) \end{bmatrix} \qquad (7)$$

The variation equation can be expressed in the form

$$\frac{d\delta X(t)}{dt} = \begin{bmatrix} 0 & I \\ \frac{GM}{r^{*3}}\left[\frac{3X_1^*(t)X_1^{*T}(t)}{r^{*2}} - I\right] & 0 \end{bmatrix} \begin{bmatrix} \delta X_1(t) \\ \delta X_2(t) \end{bmatrix} \qquad (8)$$

The geometrical relation between the satellite and the tracking station is drawn in Fig.1.

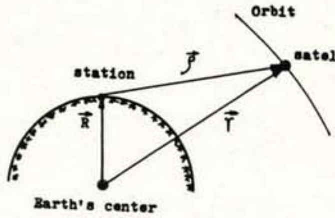


Fig.1 The geometrical relation between the satellite and the tracking station

The observation-state relationships can be written as follows:

$$q_j = \left[X_1(t_j) - R(t_j)\right]^T \left[X_1(t_j) - R(t_j)\right] - \rho_j^2 = 0 \qquad (9)$$

for the Ranging data where ρ_j are observation data and $t_j (j=1,2,\ldots,6)$ are observation times.

$$\rho_j = \left\{\left[X_1(t_j) - R_1(t_j)\right]^T \left[X_1(t_j) - R_1(t_j)\right]\right\}^{1/2} \qquad (10)$$

$$q_j = \left[X_1(t_j) - R_1(t_j)\right]^T \left[X_2(t_j) - R_2(t_j)\right] - \dot{\rho}_j \rho_j = 0 \qquad (11)$$

for the Doppler data where $\dot{\rho}_j$ are observation data.

SOLVING PROCESS

The problem investigated here is a sixpoint boundary value problem of the nonlinear differential equation system in which the boundary conditions are implicit type. The boundary value target technique can be used to solve this kind of the problem. The solving process can be outlined below:

1) To solve the state equation based on a guessed initial value $X^*(t_o)$ so as to obtain the reference state-vector and compute the transformation matrix $\Phi(t;t_o)$. The computed results are stored at the observation times $t_j (j=1,2,\ldots,6)$ and $\Phi(t_j;t_o)$ is resolved blocks as:

$$\Phi(t_j;t_o) = \begin{bmatrix} P_{11}(t_j) & P_{12}(t_j) \\ P_{21}(t_j) & P_{22}(t_j) \end{bmatrix} \qquad (12)$$

where $P_{ik}(i,k=1,2,3)$ are 3X3 matrixes.

The modified state-vector can be written in the form

$$\begin{cases} X_1(t_j)=X_1^*(t_j)+P_{11}(t_j)\delta X_1(to)+P_{12}(t_j)\delta X_2(to) \\ X_2(t_j)=X_1^*(t_j)+P_{21}(t_j)\delta X_1(to)+P_{22}(t_j)\delta X_2(to) \end{cases} \quad (13)$$

2) To compute the following functions:

$$Q_j^*=\left[X_1^*(t_j)-R_1(t_j)\right]^T\left[X_1^*(t_j)-R_1(t_j)\right]-\rho_j^2 \quad (14)$$

for the Ranging data.

$$\rho_j^*=\left\{\left[X_1^*(t_j)-R_1(t_j)\right]^T\left[X_1^*(t_j)-R_1(t_j)\right]\right\}^{1/2} \quad (15)$$

$$Q_j^*=\left[X_1^*(t_j)-R_1(t_j)\right]^T\left[X_2(t_j)-R_2(t_j)\right]-\dot{\rho}_j\,\rho_j^* \quad (16)$$

for the Doppler data.

The accuracy control quantity is denoted by ε . If

$$\|Q_j^*\|\leqslant\varepsilon \quad (17)$$

then the computing process would be end. Else, the iterative process would be continued.

3) To compute corrections of the state-vector: Substituting (13) into (9) or (10) and (11), then the nonlinear algebraic equation system about $\delta X_1(to), \delta X_2(to)$ can be obtained. This equation system can be solved by means of whether the Newton iterative method or the gradient method. Solving $\delta X_1(to), \delta X_2(to)$, then the modified state-vector can be obtained:

$$\begin{cases} X_1(to)=X_1^*(to)+\delta X_1(to) \\ X_2(to)=X_2^*(to)+\delta X_2(to) \end{cases} \quad (18)$$

4) The $X_1(to), X_2(to)$ are served as new initial state-vector. It is necessary repeat the computing process mentioned above again until (17) is satisfied.

DISCUSSION ON CONVERGENCY

Whether the Newton iterative method solving nonlinear algebraic equation system where the zero point of Q_j^* is solved or the gradient method where the minimum point of $F=\sum_{j=1}^{6} Q_j^*$ is solved are all solving local zero point or local minimum point. Therefore, the discussion on the convergency of the method is necessary.

As computing time is not too long, for example, it is shorter than 300 second, then the transformation of the variation equation can be simplified below:

$$\Phi(t_j;to)=\begin{bmatrix} I & j\Delta t I \\ & \\ 0 & I \end{bmatrix} \quad (19)$$

where $J=0,1,\ldots5$ and Δt is the separate of the observation time which is taken $\Delta t=60$ sec.

The parameters are defined by

$$\begin{cases} C_1(t_j)=\left[\tilde{X}_2(t_j)-R_2(t_j)\right]^T-\dot{\rho}_j/\tilde{\rho}_j C_2(t_j) \\ C_2(t_j)=\left[\tilde{X}_1(t_j)-R_1(t_j)\right]^T \end{cases} \quad (20)$$

Then the expression of the $Q_j^*(j=1,2\ldots,6)$ can be simplified in the form:

$$\begin{aligned} Q_j^*=&2C_2(t_j)\delta X_1(to)+2j\Delta t C_2(t_j)\,\delta X_2(to) \\ &+\left[\delta X_1(to)\right]^T\delta X_1(to)+2j\Delta t\left[\delta X_1(to)\right]^T\delta X_2(to) \\ &+(2j\Delta t)^2\left[\delta X_2(to)\right]^T\delta X_2(to) \end{aligned} \quad (21)$$

for the Ranging data.

$$\begin{aligned} Q_j^*=&C_1(t_j)\delta X_1(to)+\left[C_2(t_j)+j\Delta t C_1(t_j)\right]\delta X_2(to) \\ &+\tfrac{1}{2}\,\dot{\rho}_j/\tilde{\rho}_j\left[\delta X_1(to)\right]^T\delta X_1(to) \\ &+(1-j\Delta t\,\dot{\rho}_j/\tilde{\rho}_j)\left[\delta X_1(to)\right]^T\delta X_2(to) \\ &+j\Delta t(1-\tfrac{1}{2}j\Delta t\,\dot{\rho}_j/\tilde{\rho}_j)\left[\delta X_2(to)\right]^T\delta X_2(to) \end{aligned} \quad (22)$$

for the Doppler data.

It can be seen that the Q_j^* are a quadratic type of the $\delta X_1(to)$, $\delta X_2(to)$ whether for Ranging data or the Doppler data.

As

$$\begin{cases} X_1^*(to)=\tilde{X}_1(to) \\ X_2^*(to)=\tilde{X}_2(to) \end{cases} \quad (23)$$

the value of the Q_j^* are zero which means that the true state-vector is a solution of the problem.

From (21) or (22), it can be drived that

$$\begin{cases} \left(\dfrac{\partial Q_j^*}{\partial X_1(to)}\right)=2C_2(t_j)+2\left[\delta X_1(to)\right]^T+2j\Delta t\left[\delta X_2(to)\right]^T \\ \left(\dfrac{\partial Q^*}{\partial X_2(to)}\right)=2j\Delta t C_2(to)+2j\Delta t\left[\delta X_1(to)\right]^T \\ \qquad +2(j\Delta t)^2\left[\delta X_2(to)\right]^T \end{cases} \quad (24)$$

for the Ranging data.

$$\begin{cases} \left(\dfrac{\partial Q_j^*}{\partial X_1(to)}\right) = c_1(t_j) + \dot{\rho}_j/\tilde{\rho}_j \left[\delta X_1(to)\right]^T \\ \qquad + (1 - j\Delta t\, \dot{\rho}_j/\hat{\rho}_j)\left[\delta X_2(to)\right]^T \\ \left(\dfrac{\partial Q_j^*}{\partial X_2(to)}\right) = c_2(t_j) + j\Delta t c_1(t_j) + (1 - j\Delta t\, \dot{\rho}_j/\hat{\rho}_j)\left[\delta X_1(to)\right]^T \\ \qquad + 2j\, t(1 - \tfrac{1}{2} j\Delta t\, \dot{\rho}_j/\hat{\rho}_j)\left[\delta X_2(to)\right]^T \end{cases} \tag{25}$$

for the Doppler data.

It can be derived that the extremity of the are as follows:

$$\begin{cases} X_1^*(to) = R_1(to) \\ X_2^*(to) = R_2(to) \end{cases} \tag{26}$$

This means that when the guessed initial state-vector is equal to state-vector of the tracking station, then the Q_j^* will take extremity.

The $X_1^*(to)$, $X_2^*(to)$ which satisfy below formula are other zero point of the Q_j^*

$$\begin{cases} X_1^*(to) = R_1(to) - \left[X_1(to) - R_1(to)\right] \\ X_2^*(to) = R_2(to) - \left[X_2(to) - R_2(to)\right] \end{cases} \tag{27}$$

which indecates the wrong solution of the problem.

From results obtained above, the convergency of the method can be summarized as follows:

If
1) $\tilde{x}_k(to) > R_k(to)$

and the guessed initial state-vector satisfy

$$x_k^*(to) > R_k(to) \quad (k=1,2,\dots,6)$$

or
2) $\tilde{x}_k(to) < R_k(to)$

and the guessed initial state-vector satisfy

$$x_k^*(to) < R_k(to)$$

then it can be predicated that all Q_j^* $(j=1,2,\dots,6)$ are monotone functions of the $\delta X_1(to)$, $\delta X_2(to)$, the zero point corresponding to true state-vector is a unique local compressed point, the true solution can be obtained by means of the Newton-Raphson method.

If
3) $\tilde{x}_k(to) > R_k(to)$

and the guessed initial state-vector satisfy

$$x_k^*(to) < R_k(to)$$

or
4) $\tilde{x}_k(to) < R_k(to)$

and the guessed initial state-vector satisfy

$$x_k^*(to) > R_k(to)$$

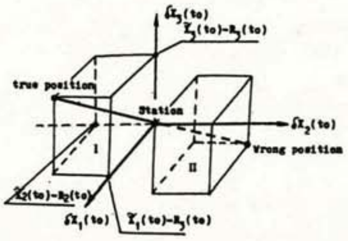
then it can be predicated that all Q_j^* $(j=1,2,\dots,6)$ are monotone functions of the $\delta X_1(to)$, $\delta X_2(to)$, but the zero point corresponding to the wrong solution is a unique local compressed point, the wrong solution can be obtained.

If the guessed initial state-vector neither satisfy the condition in 1) and 2) nor satisfy the condition in 3) and 4), then it is not predicated that all $Q_j^*(j=1,2,\dots,6)$ are uniform monotone functions of the $\delta X_1(to)$ and $\delta X_2(to)$, the iterative process may diverges.

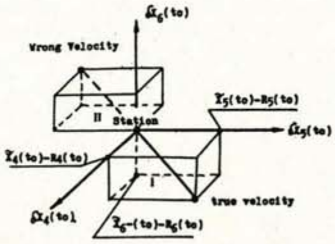
The illustration diagram of the convergency of the method are drawn in Fig.2. Where it is assumed that the true state-vector take values as follows:

$$\tilde{x}_1(to) > R_1(to),\ \tilde{x}_2(to) < R_2(to),\ \tilde{x}_3(to) > R_3(to)$$

$$\tilde{x}_4(to) > R_4(to), \tilde{x}_5(to) > R_5(to), \tilde{x}_6(to) < R_6(to)$$



a) condition of convergency for position



b) condition of convergency for velocity

Fig. 2 illustration diagram for convergency

If the guessed initial state-vector are in I region indecated in Fig.2, then the process will converges towards the true solution; If the guessed initial state-vector are in II region indecate in Fig.2, then the iterative process will converges towards the wrong solution; otherwise, the iterative process will diverges.

The coordinate system is composed which three axes parallel with three axes of the inertial system and the origin is placed at the tracking station. This rectangular coordinate system divites space around the tracking station into eight regions. As the guessed initial state-vector is in same region with the true state-vector, then the true solution can be obtained. As the guessed initial state-vector is in the symmetrical region with the true state-vector about the origin, then the wrong solution can be obtained. As the guessed initial state-vector is in other regions, then the iterative process will diverges.

NUMERICAL EXAMPLES

It is assumed that the initial time is 6:00, March 21, 1981. The simulation data are as following:

Table 1 The initial value of true state-vector

$\tilde{x}_1(to)$ (Km)	$\tilde{x}_2(to)$ (Km)	$\tilde{x}_3(to)$ (Km)	$\tilde{x}_4(to)$ (Km/Sec)	$\tilde{x}_5(to)$ (Km/Sec)	$\tilde{x}_6(to)$ (Km/Sec)
5836.89070	1265.61600	3411.49600	0.31460	6.94010	-3.11250

Table 2 Location of the tracking station

geography latitude	geography longitude	height
23° 10' N	113° 15' E	0.15 Km

Table 3 State-vector of the station

$R_1(to)$ (Km)	$R_2(to)$ (Km)	$R_3(to)$ (Km)	$R_4(to)$ (Km/sec)	$R_5(to)$ (Km/sec)	$R_6(to)$ (Km/sec)
5463.14996	2139.12637	2493.77850	-0.15551	0.39717	0.0

Table 4 Observation data

t_j(sec)	0	60	120	180	240	300
ρ_j(Km)	1321.1040	953.7183	647.7150	526.5433	695.2631	1018.4470
$\dot{\rho}_j$(Km/sec)	-6.35643	-5.79697	-4.07108	0.50565	4.55832	5.95061

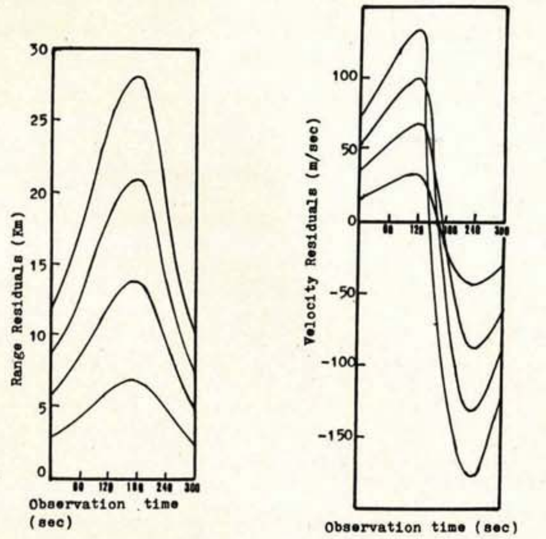
It is assumed that the values of $\delta x_k(to)$ $(k=2,3,\ldots,6)$ take zero and the values of the $\delta x_1(to)$ as below Table.

Table 5 four initial deviation of $x_1(to)$

Number	1	2	3	4
$\delta x_1(to)$ (km)	10	20	30	40

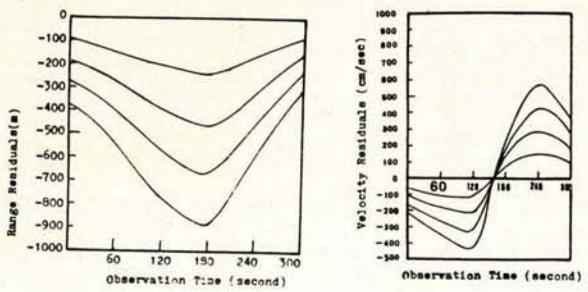
The initial residuals are drawn in Fig.3.

It is assumed that the observation data are the Doppler data. The technique of the solution is the boundary value target method where the solving process of the implicit boundary condition is the Newton iterative technique. The residuals curve after the first time iterative are drawn in Fig.4. The Range residuals curve after second time iterative are drawn in Fig.5 and the velocity residuals are given in Table 6. The residuals after third and fouth time iterative are given in Table 7, Table 8, 9, 10 respectively. The final value of the components x(t) and its errors are given in Table 11.



a) Range residuals b) Velocity residuals

Fig.3. residuals for initial state-vector



a) Range residuals b) Velocity residuals

Fig.4. The residuals after first time iterative

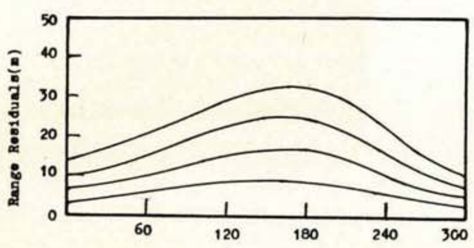


Fig.5. Range residuals after second time iterative

Table 6 Velocity residuals after second time iterative

δx_1(to) \ $\delta\dot{f}$(cm/sec) \ t_1(sec)	0	60	120	180	240	300
10(Km)	2.22	3.39	4.37	-2.42	-5.73	-3.61
20(Km)	4.27	6.51	8.39	-4.64	-11.02	-6.94
30(Km)	6.26	9.56	12.32	-6.81	-16.17	-10.18
40(Km)	8.32	12.70	16.37	-9.06	-21.49	-13.53

Table 7 Range residuals after third iterative

δx_1(to) \ δf(cm) \ t_1(sec)	0	60	120	180	240	300
10(Km)	-13.64	-19.79	-28.86	-33.07	-21.57	-11.17
20(Km)	-26.11	-37.89	-55.25	-63.33	-41.30	-21.39
30(Km)	-38.27	-55.54	-80.99	-92.83	-60.53	-31.35
40(Km)	-50.82	-73.75	-107.55	-123.27	-80.38	-41.63

Table 8 Velocity residuals after third iterative

δx_1(to) \ $\delta\dot{f}$(cm/sec) \ t_1(sec)	0	60	120	180	240	300
10(Km)	-0.08	-0.13	-0.17	0.09	0.21	0.13
20(Km)	-0.16	-0.25	-0.32	0.17	0.41	0.26
30(Km)	-0.23	-0.36	-0.46	0.25	0.60	0.38
40(Km)	-0.31	0.47	-0.61	0.33	0.79	0.50

Table 9 Range residuals after fourth iterative

δx_1(to) \ δf(cm) \ t_1(sec)	0	60	120	180	240	300
10(Km)	0.40	0.58	0.86	0.98	0.63	0.33
20(Km)	0.86	1.25	1.83	2.10	1.36	0.71
30(Km)	1.31	1.91	2.78	3.19	2.08	1.08
40(Km)	1.78	2.58	3.77	4.32	2.81	1.46

Table 10 Velocity residuals after fourth iterative

δx_1(to) \ $\delta\dot{f}$(cm/sec) \ t_1(sec)	0	60	120	180	240	300
10(Km)	0.01	0.00	0.00	0.00	-0.01	0.00
20(Km)	0.01	0.00	0.01	-0.01	-0.02	-0.01
30(Km)	0.01	0.01	0.01	-0.01	-0.02	-0.01
40(Km)	0.01	0.01	0.02	-0.01	-0.03	-0.02

Table 11 State-vector and errors as convergency

δx_1(to) (Km)	10	20	30	40
x_1(to) (Km)	5836.89071	5836.89073	5836.89075	5836.89076
Δx_1(to) (cm)	1.00	3.00	5.00	6.00

CONCLUSTION AND DISCUSSION

The computing accuracy of the method can commensurate with the current observation data and computing program is simple which can be operated on a personal computer.

The condition of the convergency of the method is not too serious which is satisfied easily in practice.

The dynamical equation of the two-body motion is used in numerical examples and the observation data are computed based on the relationships of the two-body motion. The model errors and the observation errors did not contained in the results. In practice, the model and observation errors can not be avoid. Therefore, it would be studied what type of the perturbation is suitable and how the observation errors are eliminated in future investigations.

From the discussion on the convergency above, the method is available for the orbit improvement where the deviation of the initial state-vector are smaller in general case. In order to determine the preliminary orbit, the convergency of the method must be improve in future investigations.

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