OPTIMAL SPACE MANEUVERS IN A KEPLERIAN FORCE FIELD IN THREE DIMENSIONS

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Abstract

Problems related to orbital transfers are of considerable importance[1,2]. Considering non-coplanar orbits, it is particularly important to minimize the fuel expenditure necessary for a specific plane change, since this kind of maneuver is the main fuel consumer. The method developed and presented here allows the choice of the orbital regions where the impulses can be applied. So, the contribution of this paper is to develop, implement and test a new set of equations to solve the problem of the minimum fuel bi-impulsive maneuver where it is possible to include constraints considerations, such as impulses positions restrictions, e.g., maneuvers been performed only in visible regions from a given groundstation.

Key words: Orbital maneuvers, fuel optimization, Keplerian field.

Nomenclature

e	= eccentricity
H	= angular momentum
i	= inclination
r	= magnitude of the position vector
$\boldsymbol{r}$	= position vector
$\boldsymbol{S}$	= unit vector normal to the orbital plane
t	= unit vector normal to the transfer plane
V	= velocity
x, y, z	= reference system unit vectors
ϕ	= true anomaly
λ	= angle between the transfer and the final orbits
μ	= gravitational constant (398600.64 km^3/sec^2 for Earth)
σ	= transfer angle
ω	= perigee argument
Ω	= longitude of the ascending node
ζ	= angle between the transfer and the initial orbits

Subscripts

A	= initial orbit
B	= final orbit
T	= transfer orbit
k	= normal component
r	= radial component
θ	= transversal component
1	= first impulse
2	= second impulse

Introduction

The solution of the spacecraft bi-impulsive transfer between two elliptical and non-coplanar given orbits problem, with minimum fuel consumption under a Keplerian dynamics was found. A numerical algorithm was developed for fast practical use to obtain the minimum velocity increment needed to perform this kind of maneuver. It was supposed that the unique forces considered by the system dynamics are the spacecraft propulsion forces (instantaneous) and the Earth's gravitational attraction (assumed as a point mass).

The problem is to obtain a Keplerian transfer orbit between the given initial and final non-coplanar orbits. This maneuver should be performed in a such way that the addition of the two magnitudes of the impulses applied be a minimum.

Problem Formulation

As the initial and the final orbits are given, five orbital elements of each one are known. They are the semi-major axis, eccentricity, inclination, perigee argument and longitude of the ascending node. The first task is to obtain the two unit normal vectors of each orbit. This can be done by arbitrarily choosing two values for the true anomaly (called points 1 and 2) and obtaining the unit vectors S_A and S_B from the following equations:

$$\boldsymbol{S}_A = \frac{\boldsymbol{r}_{1A} \times \boldsymbol{r}_{2A}}{\left|\boldsymbol{r}_{1A} \times \boldsymbol{r}_{2A}\right|}, \quad \boldsymbol{S}_B = \frac{\boldsymbol{r}_{1B} \times \boldsymbol{r}_{2B}}{\left|\boldsymbol{r}_{1B} \times \boldsymbol{r}_{2B}\right|} \qquad (1)$$

To define the transfer orbital plane it is necessary to specify the start and the end transfer position vectors. This could be done by assuming the start and the end true anomalies. It is important to note that these true anomalies refer to their respective orbital planes and, so, they are angles on different orbital planes. These values can be varied on each complete orbit (0^o to 360^o) by chosen steps to obtain the couple of true anomalies that minimize the fuel consumption establishing the orbital places to apply the impulses. It could be chosen one or more subintervals on the region 0^o to 360^o of each orbit, to represent the possible visibility maneuvers constraints.

In other words, first the problem of minimum fuel transfer between two fixed points (one at the initial and the other on the final orbit) will be formulated and solved and then these two points will be circulated by the orbits involved to obtain the minimum consumption transfer.

With the initial and final position vectors defined, it is possible to obtain the transfer angle between these vectors:

$$\cos(\sigma) = \frac{\boldsymbol{r}_1 \cdot \boldsymbol{r}_2}{\left|\boldsymbol{r}_1\right| \cdot \left|\boldsymbol{r}_2\right|} \qquad (2)$$

The unit vector normal to the transfer plane can be calculated by:

$$\boldsymbol{t} = \frac{\boldsymbol{r}_1 \times \boldsymbol{r}_2}{\left|\boldsymbol{r}_1 \times \boldsymbol{r}_2\right|} \qquad (3)$$

It is important to note that if in this step the vectors r_1 and r_2 are collinear, there is a singularity. For this reason, vectors very close to this condition are not satisfactory for the method developed here. This restriction imposes constraints to the method, that is not able to solve special geometry problems (Hohmann transfer, bi-elliptical, etc.). But, it should be take into account that this class of problems presents immediate solutions and doesn't need to be solved by the proposed method.

The next step is to calculate the angle between the initial and the transfer orbits and the angle between the transfer and the final orbits:

$$\cos(\zeta) = \boldsymbol{S}_A \cdot \boldsymbol{t}, \quad \cos(\lambda) = \boldsymbol{S}_B \cdot \boldsymbol{t} \qquad (4)$$

With the two vectors r_1 and r_2, it is possible to calculate the transfer orbit inclination, longitude of the ascending node and the angle that represents the addition of the perigee argument and the true anomaly. The following equations are used:

$$\cos(i_T) = \boldsymbol{t} \cdot \boldsymbol{z}, \quad \cos(\Omega_T) = \boldsymbol{M} \cdot \boldsymbol{x} \qquad (5)$$

$$\cos(\omega_T + \phi_T) = \frac{\boldsymbol{r}_1 \cdot \boldsymbol{M}}{\left|\boldsymbol{r}_1\right| \cdot \left|\boldsymbol{M}\right|}, \quad \boldsymbol{M} = \left(\frac{\boldsymbol{z} \times \boldsymbol{t}}{\left|\boldsymbol{z} \times \boldsymbol{t}\right|}\right) \qquad (6)$$

The other transfer orbit elements could only be determined when the fuel minimization problem is solved.

As the model used here consider bi-impulsive transfers, the velocity variation must be split in two parts: one refers to the first impulse applied on the initial orbit and the other one refers to the second impulse applied on the transfer orbit to achieve the final one.

The radial, transversal and normal velocity components at the initial orbits projected on the transfer plane are:

$$\begin{aligned} V_{r1T} &= \frac{\mu}{H_1} e_1 \sin(\phi_1) \\ V_{\theta 1T} &= \frac{\mu}{H_1} e_1 \left[1 + \cos(\phi_1)\right] \cdot \cos(\zeta) \\ V_{k1T} &= \frac{\mu}{H_1} e_1 \left[1 + \cos(\phi_1)\right] \cdot \sin(\zeta) \end{aligned} \qquad (7)$$

The velocity components after the first impulse on the transfer orbit are:

$$\begin{aligned} V_{rT1} &= \frac{\mu}{H_T} e_T \sin(\phi_{T1}) \\ V_{\theta T1} &= \frac{\mu}{H_T} [1 + e_T \cos(\phi_{T1})] \\ V_{kT1} &= 0 \end{aligned} \qquad (8)$$

So, the first impulse can be determined as a function of three unknown: transfer orbit angular momentum, eccentricity and the true anomaly of the first impulse. The equation is:

$$\Delta V_1^2 = \left(V_{\mathrm{rT1}} - V_{\mathrm{r1}}\right)^2 + \\ + \left(V_{\theta\mathrm{T1}} - V_{\theta 1}\cdot\cos(\zeta)\right)^2 + \left(V_{\theta 1}\cdot\sin(\zeta)\right)^2 \qquad (9)$$

The same approach can be developed to the second impulse. So, this impulse can also be determined as a function of the same three unknown variables. The final equation is:

$$\Delta V_2^2 = \left(V_{\mathrm{r2}} - V_{\mathrm{rT2}}\right)^2 + \\ + \left(V_{\theta 2}\cdot\cos(\lambda) - V_{\theta\mathrm{T2}}\right)^2 + \left(V_{\theta 2}\cdot\sin(\lambda)\right)^2 \qquad (10)$$

The problem now can be reduced to the minimization of the total impulse:

$$\Delta V(H_T, e_T, \phi_{T1}) = \left|\Delta V_1\right| + \left|\Delta V_2\right| \qquad (11)$$

subject to two constraints, that express the fact that the two vectors r_1 and r_2 have the same values, independent if they are calculate on the transfer orbit or on the initial and final ones. These constraints can be written as:

$$\frac{1}{r_1} = \frac{\mu}{H_1^2}[1 + e_1\cos(\phi_1)] = \frac{\mu}{H_{\mathrm{T}}^2}[1 + e_{\mathrm{T}}\cos(\phi_{\mathrm{T1}})] \qquad (12)$$

$$\frac{1}{r_2} = \frac{\mu}{H_2^2}[1 + e_2\cos(\phi_2)] = \frac{\mu}{H_{\mathrm{T}}^2}[1 + e_{\mathrm{T}}\cos(\phi_{\mathrm{T1}} + \sigma)] \qquad (13)$$

These constraints equations are manipulated to be transformed into equations for the eccentricity and angular momentum as a function of the true anomaly of the first impulse. The equations are:

$$e_T(\phi_{T1}) = \frac{Q - 1}{\cos(\phi_{T1}) - Q\cdot\cos(\phi_{T1} + \sigma)} \qquad (14)$$

$$Q = \frac{H_2^2}{H_1^2}\frac{\left[1 + e_1\cdot\cos(\phi_1)\right]}{\left[1 + e_2\cdot\cos(\phi_2)\right]} \qquad (15)$$

$$H_T(\phi_{T1}) = \sqrt{\frac{1 - \dfrac{\cos(\phi_{T1})}{\cos(\phi_{T1} + \sigma)}}{Q1 - \dfrac{\cos(\phi_{T1})}{\cos(\phi_{T1} + \sigma)}\cdot Q2}} \qquad (16)$$

$$Q1 = \frac{\left[1 + e_1\cdot\cos(\phi_1)\right]}{H_1^2} \qquad (17)$$

$$Q2 = \frac{\left[1 + e_2\cdot\cos(\phi_2)\right]}{H_2^2} \qquad (18)$$

Now the velocity variation is a function only of the true anomaly of the first impulse. Several methods can be employed to find the minimum value of the function of a single variable. In this work, it was used a function minimization routine[3] that found the value of the true anomaly of the first impulse which makes the velocity variation a minimum. With this value, it is possible to calculate the others transfer orbit elements, such as, semi-major axis, eccentricity and perigee argument. So the transfer orbit and the two impulses magnitude, direction and location are determined.

Results

A transfer between two elliptical non-coplanar orbits with a slight variation in all orbital elements is presented, as an example of the method described. The initial orbit was: semi-major axis = 12030.0 km; eccentricity = 0.02000; inclination = 0.00873 rd; perigee argument = 3.17649 rd; longitude of the ascending node = 0.00000 rd. The final orbit was: semi-major axis = 11994.7 km; eccentricity = 0.01600; inclination = 0.00602 rd; perigee argument = 3.05171 rd; longitude of the ascending node = 0.15568 rd. The minimum fuel consumption transfer orbit was found to be: semi-major axis = 12038.1 km; eccentricity = 0.01945; inclination = 0.00865 rd; perigee argument = 3.16620 rd; longitude of the ascending node = 0.01215 rd; true anomaly of the first impulse = 4,03754 rd; true anomaly of the second impulse = 5,91049 rd. The first impulse velocity variation was 0.00226 km/sec and the second one was 0.01997 km/sec. The total velocity variation was 0.02223 km/sec.

If one consider restrictions on the application impulses points there could be an increase on the fuel consumption. The same orbital transfer of the previous example with this kind of constraint has a different result. Considering that the true anomaly of the first impulse at the initial orbit can vary only between 0 and 1.5 rd, and the true anomaly of the second impulse on the final orbit between 2.0 and 3,2 rd, the results for the velocity variation were: first impulse velocity variation was 0.01414 km/sec and the second one was 0.00874 km/sec. The total velocity variation was 0.02288 km/sec. It represents a slight increase in the fuel expenditure, that can be larger in different situations.

Conclusions

An analytical formulation was derived and implemented, based on a method derived by Prado and Broucke[4] to coplanar orbits, to solve the problem of bi-impulsive orbital transfers between elliptical non-coplanar orbits in a Keplerian dynamics problem with minimum fuel consumption. The results to numerous tests here performed are in agreement with the results obtained by a method derived from the work of Altman[5], implemented and tested by Paulo[6] at INPE.

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References

[1]Gobetz, F.W.; Doll, J.R., A survey of impulsive trajectories, *AIAA Journal*, Vol. 7, No. 5, 1969, pp. 801-834.

[2]Edelbaum, T.N., How many impulses ?, *Astronautics and Aeronautics*, Vol. 5, No. 11, 1967, pp. 64-69.

[3]Press, W.H., Teukolsky, S.A., Vetterling, W.T., Flannery, B.P, Two Point Boundary Value Problem *Numerical Recipes in FORTRAN (the art of scientific computing).* 2nd ed., Cambridge University Press, 1992, pp. 745-778.

[4]Prado, A.F.B.A.; Broucke, R.A., The Minimum Delta-V Lambert's Problem, *Controle e Automação*, Vol. 7, No. 2, 1996, pp. 84-90.

[5]Altman, S.P.; Pistiner, J.S., Analysis of the Orbital Transfer Problem in Three-Dimensional Space, *Proceedings of the Astrodynamics Conference*, AIAA, New Haven, Connecticut. 1963, pp. 627-654.

[6]Paulo, M.S., private communication, National Institute of Space Research (INPE), São José dos Campos, SP, Brazil, 1996.

[7]Schulz, W., Transferências Bi-Impulsivas entre Órbitas Elípticas não Coplanares com Consumo Mínimo de Combustível, *M.S. Thesis*, Space Mechanics and Control Division, National Institute of Space Research (INPE), São José dos Campos, SP, Brazil, Mar. 1997.