EFFECTIVE ALGORITHM FOR CALCULATING PROBABILITY OF SATISFACTION OF RESTRICTIONS ON SPACECRAFT MOVEMENT PARAMETERS

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Abstract

The algorithm is designed to perform the calculation of probability P of the satisfaction of the restrictions on spacecraft movement with respect to the celestial body (other spacecraft, orbital station, planet, satellite of a planet, etc.) at moments t from finite set T.

For each calculation of P it is necessary to compute the many-dimensional integral. Computing a many-dimensional integral is usually executed by the method of a separation of domain of integration into small parallelepipeds that is immediately followed from mathematical definition of a integral. In our case a calculation of integral by the methods of a separation of domain requires execution of a lot of operations even if it is done for one moment t only. In solving applied problems a set T that t belongs to being generally large, in computing it is impossible to limit oneself to above-mentioned method of a separation of a domain through a large time of its execution. In connection with this the more effective algorithm is offered for solving the problem.

Solving the problem is divided into two time stage:

1) preliminary stage and
2) stage of immediate computation of $P(t)$ for all t from set T.

A finite set J of integrals whose values are independent on t is computed in a preliminary stage when time to calculate is practically unlimited. For calculated mathematical expectations, covariance matrices and values of integrals from set J a computation of values $P(t)$ is realized in the stage 2 by using simple formula and does not require big time of execution.

Key words: Trajectory monitoring, Coming together of spacecraft and celestial body, Restrictions on spacecraft movement, Probability.

Introduction

A movement of mass centres of celestial bodies and spacecraft for used model of the acting forces (movement model) is defined by representation of the kinematic parameters of their trajectories in some initial moment and by the parameters of the movement model.

It is assumed that

* kinematic parameters of trajectories, parameters of movement model and their errors are known;

* restrictions on the parameters of spacecraft movement define the domain Ω which is not changed (in the course of time) in a Cartesian system of coordinates $OXYZ$;

* position of the axes of coordinate system $OXYZ$ in an inertial space is given function from time.

The algorithm for calculation of probability P utilizes practically justified assumptions about character of the errors in knowledge of initial movements for the celestial bodies and the parameters of movement model and about topology of the domain Ω:

* in initial moment (for the interval of prediction of movement) the errors are distributed in accordance with normal law (Gauss law) for which the parameters (mathematical expectation and covariance matrix) are known;

* in any moment (from the interval of prediction of movement) the dependence of deviations of kinematic parameters of the bodies' movement on deviations in initial moment of such parameters and parameters of movement model can be presented by linear part of Taylor series;

* domain Ω is bounded.

The algorithm was used successfully for

a) choosing the trajectories ensuring the autonomic closure between the vehicle and orbital station,

b) projection and implementation of the closure between artificial satellite of Mars and Fobos,

c) estimation of safe coming together of spacecraft and etc.

Description of the algorithm

Input information for algorithm are:

* set T;
* mathematical expectation and covariance matrices of
  a) initial conditions of movements of spacecraft and the celestial body;
  b) parameters of the manoeuvres to perform (jet thrust, orientation of the jet axis is space, etc.) and
  c) parameters of the movement model;
* parameters to define domain in which restrictions on spacecraft movement are satisfied.

Output information of algorithm are value of the probability

$$P = \frac{1}{(2\pi)^{q/2}(\det K)^{1/2}} \int_{\Omega} \exp(-(\bar{x}-\bar{a})^T K^{-1}(\bar{x}-\bar{a})/2)dx_1 \ldots dx_q \quad (1)$$

at every moment t from set T. Here $\bar{a}, K$ are mathematical expectation and covariance matrix for q kinematic parameters of spacecraft at the coordinate system $OXYZ$ in which the restrictions are imposed.

In common case the movement of celestial bodies is described by the system of differential equation. For calculation of the matrix K it is necessary to calculate the partial derivatives of current kinematic parameters of the spacecraft movement in coordinate system $OXYZ$ with respect to a) initial kinematic parameters of movement of the spacecraft and the celestial body, b) parameters of the manoeuvres and c) parameters of the movement model. These derivatives can be calculated by the method of the finite differences. In the algorithm the computation of their derivatives is executed by the numerical solution of Cauchy problem for the corresponding system of differential equations. The algorithm of such computation of the covariance matrix was presented[1,2] in detail.

After a computation of the covariance matrices $K(t)$ and the mathematical expectations $\bar{a}(t)$, $t \in T$, the values $P(t)$ are calculated. A computing of the probability P is highly laborious process even for the individual value of t. Our algorithm exploits effective the computation method for calculating probability P .

<u>Method for calculating probability.</u>

Without loss of generality the method is presented for the case when the Ω is three-dimensional 1-connected and bounded domain.

The expression (1) can be rewritten in the form:

$$P(t) = Q\iiint_{Q} \exp(-\bar{x}^T K^{-1}\bar{x}/2 + \bar{a}^T K^{-1}\bar{x})dxdydz \quad (2)$$

where

$\bar{x} = (x, y, z)^T$,

$Q(t) = (2\pi)^{3/2}(\det K)^{-1/2}\exp(-\bar{a}^T K^{-1}\bar{a}/2)$

The formula (2) is replaced by the approximate formula

$$P(t) = Q\sum_{j=0}^{N}(1/j!)\iiint_{\Omega}(\bar{x}^T A\bar{x} + \bar{b}^T\bar{x})^j dxdydz \quad (3)$$

where

$A(t) = -K^{-1}(t)/2$,

$\bar{b}(t) = K^{-1}(t)\bar{a}(t)$

(the limit of (3) as N becomes infinite is equal to (2)).

It is obvious that calculation of $P(t)$ by the formula (3) is reduced to calculation of the set J of the integrals

$$J(k,l,m) = \iiint_{\Omega} x^k y^l z^m dxdydz. \quad (4)$$

Solving the problem is divided into two time stage:

1. preliminary stage,
2. stage of immediate computation of $P(t), t \in T$.

<u>Stage 1.</u>

Being independent on t the values of integrals (4) are calculated in stage 1. The calculation of integrals (4) are realized for all nonnegative integers k,l,m that satisfy the inequality $k+l+m \le 2N$.

In addition centre coordinates and radii of two spheres (S_M, S_m) are calculated. The spheres S_M, S_m satisfy relations $S_M \subset \Omega, S_m \supset \Omega$ and they have maximal and minimal radii respectively.

<u>Stage 2.</u>

In stage 2 a computation of $P(t)$ is executed by using linear dependence of (3) on the integrals $J(k,l,m)$ with coefficients that are relatively simple functions of components of the vector $\bar{a}(t)$ and the covariance matrix $K(t)$.

The stage is executed in the following succession.

1. Let E_U be ellipsoid which is similar to the concentration ellipsoid of the stochastic vector with covariance matrix K. A centre and directions of principal axes of E_U is the same as those of ellipsoid of concentration but form of E_U is determined by matrix U^2K where U is a given scalar. If by researching it has been found that either

$$E_U \cap S_M = E_U \qquad (5)$$

or

$$E_U \cap S_m = \varnothing \qquad (6)$$

then P is set equal to 1 in case (5) or 0 in case (6).

It is perfectly permissible for practical calculations if $U = 5$.

2. For known values of the integrals (4) the following values

$$V_j = \frac{1}{j!}\iiint\limits_{\Omega} (\bar{x}^T A\bar{x} + \bar{b}^T \bar{x})^j \, dxdydz \qquad (7)$$

are calculated in succession for $j = 0,1,\ldots,N$ by using simple relations. The calculation are executed until

$$\left| Q \cdot V_j \right| \le \delta \qquad (8)$$

where δ is a given small value.

If the inequality (8) is satisfied for $j^* \le N$, the probability P is set equal

$$P = Q\sum_{j=0}^{j^*} V_j \, . \qquad (9)$$

It should be noted that the number N is empirical found with regard for restrictions on execution time and on size of computer storage for coefficients of polynomials $(\bar{x}^T A\bar{x} + \bar{b}^T \bar{x})^j$, $j = 1,2,\ldots,N$. Taking in into account we have recommended $N = 14$ for three-dimensional domain (N is set equal 11, 9, 7 for four-, five- and six-dimensional domain Ω respectively). In addition it should be noted that generally for this value of N a number j^* satisfying the condition (8) exists in most practical tasks and, consequently, calculation of the probability P is completed by the formula (9).

Characteristic of the algorithm

The algorithm was put in the base of the software (software A) to calculate probability of satisfaction of the restrictions on spacecraft movement parameters. The basic characteristics of the algorithm are showed in connection with problem of coming together of two spacecrafts (S_1 and S_2).

The mass centres (points O_1 and O_2) of spacecrafts move along their trajectories in a Cartesian system of coordinates $OXYZ$. The point O_2 is centre of the sphere with radius R (sphere of safety). For each moment $t \in T$ it is necessary to calculate the probability P of the event when point O_1 hits into sphere of safety.

If $K_i, \bar{r}_i$ are covariance matrix and mathematical expectation of position of the point O_i, $i = 1,2.$ then expression (1) may be written in form:

$$P = \frac{1}{(2\pi)^{3/2}(\det K)^{1/2}} \iiint\limits_{x^2+y^2+z^2 \le 1} \exp(-(\bar{x}-\bar{a})^T K^{-1}(\bar{x}-\bar{a})/2)dxdydz \qquad (10)$$

(the domain Ω is unit sphere; the centre of sphere is point O_2). Here, $K, \bar{a}$ are covariance matrix and mathematical expectation of position of spacecraft S_1 with respect to spacecraft S_2,

$$K = (K_1 + K_2)/R^2,$$
$$\bar{a} = (\bar{r}_1 - \bar{r}_2)/R,$$
$$\bar{x} = (x, y, z)^T.$$

The basic characteristics of the algorithm are represented in Tables 1, 2, 3 for the series of matrices K and vectors $\bar{a}$.

The values ME, $\sigma_x, \sigma_y, \sigma_z$ are modulus of the mathematical expectation $\bar{a}$ and square roots of eigenvalues of the covariance matrix K. The values P_M and P_S are results of calculations of the probability P by using software A (with $\delta = 0.001$ in inequality (8)) and the algorithm of the separation of domain Ω into parallelepipeds (software S) respectively. The parallelepiped edges are equal

$$\min(\frac{\sigma_x}{N},\frac{1}{N}),\quad \min(\frac{\sigma_y}{N},\frac{1}{N}),\quad \min(\frac{\sigma_z}{N},\frac{1}{N}),$$

where $N = 40$. This value of N makes possible to compute probability with accuracy 0.0005.

Table 1: Characteristics of algorithm. Domain contains mass centre of spacecraft.

ME $\sigma_x, \sigma_y, \sigma_z$	P_M	P_S	T_{rel}
0.0 0.8, 0.8, 0.80	0.3321	0.3323	292
0.0 0.6, 0.6, 0.60	0.5729	0.5729	371
0.0 0.8, 0.6, 0.40	0.5671	0.5667	163
0.0 0.9, 0.6, 0.33	0.5468	0.5453	53
0.5 0.8, 0.8, 0.80	0.2887	0.2888	350
0.5 0.6, 0.6, 0.60	0.4742	0.4742	151
0.5 0.8, 0.6, 0.40	0.4957	0.4955	168
0.5 0.9, 0.6, 0.33	0.4877	0.4866	55

Table 2: Characteristics of algorithm. Boundary of domain contains mass centre of spacecraft.

ME $\sigma_x, \sigma_y, \sigma_z$	P_M	P_S	T_{rel}
1.0 0.8, 0.8, 0.80	0.1887	0.1888	159
1.0 0.6, 0.6, 0.60	0.2612	0.2612	75
1.0 0.8, 0.6, 0.40	0.3292	0.3293	163
1.0 0.8, 0.6, 0.33	0.3449	0.3451	52

Table 3: Characteristics of algorithm. Domain do not contains mass centre spacecraft.

ME $\sigma_x, \sigma_y, \sigma_z$	P_M	P_S	T_{rel}
1.5 0.8, 0.8, 0.80	0.0918	0.0918	103
1.5 0.6, 0.6, 0.60	0.0898	0.0896	82
1.5 0.8, 0.6, 0.40	0.1637	0.1638	163
1.5 0.9, 0.6, 0.33	0.1924	0.1925	89
2.0 0.8, 0.8, 0.80	0.0328	0.0327	110
2.0 0.6, 0.6, 0.60	0.0180	0.0179	47
2.0 0.8, 0.6, 0.40	0.0599	0.0599	168
2.0 0.9, 0.6, 0.33	0.0837	0.0842	99
2.5 0.8, 0.8, 0.80	0.0085	0.0084	110
2.5 0.6, 0.6, 0.60	0.0021	0.0020	62
2.5 0.8, 0.6, 0.40	0.0159	0.0158	163
2.5 0.9, 0.6, 0.33	0.0282	0.0284	99
3.0 0.8, 0.8, 0.80	0.0017	0.0015	>1752
3.0 0.6, 0.6, 0.60	0.0000	0.0001	>2049
3.0 0.8, 0.6, 0.40	0.0032	0.0030	414
3.0 0.9, 0.6, 0.33	0.0074	0.0073	181

An algorithm effectiveness is characterized by value

$$T_{rel} = \frac{t_S}{t_M} \qquad (11)$$

where t_S, t_M are times of the computations by using the softwares S and A respectively.

The problem of coming together of two spacecrafts (Earth satellites) was solved by software A and software S repeatedly. The movement model was sufficiently precise[3] . The calculations showed effectiveness of software A. The calculation time of probability P by using software A is approximately by two order less that this time by using software S. The value T_{rel} (see (11)) increases if dimension of domain is increased. Consequently the effectiveness of algorithm increases.

References

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