NEW METHODS FOR SPACECRAFT FORMATION DESIGN

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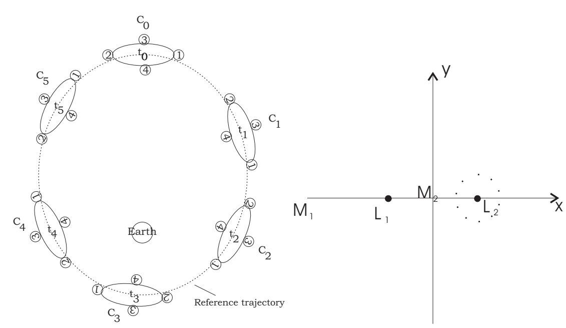
ABSTRACT This paper applies previous research on the solution of two-point boundary value problems to spacecraft formation dynamics and design. The underlying idea is to model the motion of a spacecraft formation as a Hamiltonian dynamical system in the vicinity of a reference solution. Then we can analytically describe the nonlinear phase flow using generating functions found by solving the Hamilton-Jacobi equation. Such an approach is very powerful and allows one to study any Hamiltonian dynamical system independent of the complexity of its vector field, and to solve any two-point boundary value problem using only simple function evaluations. We present the details of our approach through the study of a non-trivial example, the reconfiguration of a formation in Earth orbit and in the Hill three-body problem. Both continuous and impulsive thrusts are considered.

1. INTRODUCTION

Several missions and mission statements have identified formation flying as a means for reducing cost and adding flexibility to space-based programs. However, such missions raise a number of technical challenges. For example, the formation reconfiguration problem which consists of changing the geometry of the formation, requires one to solve a *large number* of two-point boundary value problems using an accurate dynamic models of the relative motion. To reconfigure a formation of N spacecraft using impulsive thrusts, there are $N!$ possibilities in general. Similarly, if continuous thrusts are used, this problem formulates as an optimal control problem and the necessary conditions for optimality yield a boundary value problem. Again, $N!$ boundary value problems need to be solved. Therefore, techniques and algorithms developed for solving boundary value problems (such as shooting and relaxation methods) are no longer appropriate as they require excessive computation and time. These observations have motivated the present work. Specifically, we present a novel approach for solving two-point boundary value problems. A fundamental difference with previous studies is that we are able to describe the relative motion, i.e., the phase space in the vicinity of a reference trajectory, as two-point boundary value problems whereas it is usually described as an initial value problem.

In this paper, to showcase the strength of our method, we have chosen to study two challenging reconfiguration problems. We first consider a spacecraft formation about an oblate Earth (the J_2 and J_3 gravity coefficients are taken into account) that must achieve 5 missions over a *one month* period. For each mission the formation must be in a given configuration C_i that has been specified beforehand, and we wish to minimize the overall fuel expenditure. The configurations C_i are specified as relative positions of the spacecraft with respect to a specified reference trajectory (Fig. 1(a)). Then, we consider the deployment problem. We assume several spacecraft at the Libration point L_2 in the Hill three-body problem and want to find the optimal control law that drive the formation to its final configuration (see Fig. 1(b)).



(a) At each t_i, spacecraft must be in the configuration C_i (b) Deployement problem

Figure 1: Representations of the two designs we study

2. SOLVING TWO-POINT BOUNDARY VALUE PROBLEM

In this section, we briefly review the method developed by Guibout and Scheeres [6, 4] for solving Hamiltonian two-point boundary value problems.

Definition 1 (Hamiltonian system). *A system is called Hamiltonian if there exists a smooth function $H(q,p,t)$ from $\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$ to $\mathbb{R}$ such that its dynamics can be described by equations of the form:*

$$\dot{q}_i = \frac{\partial H}{\partial p_i}\,,\ \dot{p}_i = -\frac{\partial H}{\partial q_i}\,. \tag{1}$$

H is called the Hamiltonian function and Eqns. (1) *are known as Hamilton's equations.*

Trajectories of Hamiltonian systems can also be characterized by the following variational principle:

Theorem 2 (Modified Hamilton's principle). *Critical points of $\int_{t_0}^{t_1}(\langle p, \dot{q}\rangle - H)dt$ in the class of paths $\gamma : \mathbb{R} \to \mathbb{R}^n \times \mathbb{R}^n$ whose ends lie in the n-dimensional subspaces $q = q_0$ at $t = t_0$ and $q = q_1$ at $t = t_1$ correspond to trajectories of the Hamiltonian system whose ends are q_0 at t_0 and q_1 at t_1.*

Proof. We proceed to the computation of the variation.

$$\delta \int_\gamma (\langle p, \dot{q}\rangle - H)dt =$$
$$\int_\gamma \left(\dot{q}_i \delta p_i + p_i \delta \dot{q}_i - \frac{\partial H}{\partial q_i}\delta q_i - \frac{\partial H}{\partial p_i}\delta p_i\right) dt = [p_i \delta q_i]_{t_0}^{t_1}$$
$$+ \int_\gamma \left[\left(\dot{q}_i - \frac{\partial H}{\partial p_i}\right)\delta p_i - \left(\dot{p}_i + \frac{\partial H}{\partial q_i}\right)\delta q_i\right] dt\,.$$

Therefore, since the variation vanishes at the end points, the integral curves of Hamilton's equations are the only extrema. □

We now introduce the concept of canonical transformation, a class of coordinate transformations that preserves the Hamiltonian structure of the system.

Definition 3. *A smooth map $f : \mathbb{R}^{2n} \times \mathbb{R} \to \mathbb{R}^{2n} \times \mathbb{R}$ is a canonical transformation from (q, p, t) to (Q, P, t) if and only if:*

1. *f is a diffeomorphism,*
2. *f preserves the time, i.e., there exists a function g_t such that $f(x, t) = (g_t(x), t)$,*
3. *Critical points of $\int_{t_0}^{t_1}\left(\langle P, \dot{Q}\rangle - K(Q, P, t)\right)dt$ correspond to trajectories of the Hamiltonian system, where $K(Q, P, t)$ is the Hamiltonian function expressed in the new set of coordinates.*

We consider a canonical transformation $f : (q, p, t) \mapsto (Q, P, t)$ and a Hamiltonian system defined by H. Along trajectories, we have by definition:

$$\delta \int_{t_0}^{t_1}\left(\sum_{i=1}^n p_i \dot{q}_i - H(q, p, t)\right) dt = 0\,, \qquad (2)$$

$$\delta \int_{t_0}^{t_1}\left(\sum_{i=1}^n P_i \dot{Q}_i - K(Q, P, t)\right) dt = 0\,. \qquad (3)$$

From Eqns. (2) - (3), we conclude that the integrands of the two integrals differ at most by a total time derivative of an arbitrary function F:

$$\sum_i p_i dq_i - H dt = \sum_j P_j dQ_j - K dt + dF\,. \qquad (4)$$

Such a function is called a generating function for the canonical transformation f and is, *a priori*, a function of both the "old" and the "new" variables and time. The two sets of coordinates being connected by the $2n$ equations, namely, $f(q, p, t) = (Q, P, t)$, F can be reduced to a function of $2n + 1$ variables among the $4n + 1$. Hence, we can define 4^n generating functions that have n "old" variables and n "new". Among these are the four kinds defined by Goldstein [3], $F_1(q, Q, t)$, $F_2(q, P, t)$, $F_3(p, Q, t)$ and $F_4(p, P, t)$. In the present work, we focus on the generating function of the first kind, F_1. In other words, we assume that (q, Q) are independent variables. Then, from $dF_1 = \frac{\partial F_1}{\partial q}dq + \frac{\partial F_1}{\partial Q}dQ + \frac{\partial F_1}{\partial t}$, Eqn. (4) simplifies to the following vector equation:

$$(p - \frac{\partial F_1}{\partial q})dq - H dt = (P + \frac{\partial F_1}{\partial Q})dQ - K dt + \frac{\partial F_1}{\partial t}$$

Hence, since (q, Q, t) are independent variables, we obtain:

$$\begin{aligned} q = \tfrac{\partial F_1}{\partial p}(q, Q, t)\,,\ Q = -\tfrac{\partial F_1}{\partial Q}(q, Q, t)\,, \\ \tfrac{\partial F_1}{\partial t} + H(q, \tfrac{\partial F_1}{\partial p}, t) = K(Q, -\tfrac{\partial F_1}{\partial Q}, t)\,. \end{aligned} \qquad (5)$$

Let us particularize Eqns. (5) for the canonical transformation induced by the inverse of the phase flow (a proof that this transformation is canonical can be found in [1]). Such a transformation maps the state of the system at time t to its state at the initial time while preserving the time. Thus, it maps the system to a trivial one with constant Hamiltonian function that can be chosen to be 0. The associated generating function F_1 verifies Eqns. (5) where (Q, P) now denotes the initial state (q_0, p_0) and $K = 0$:

$$q = \frac{\partial F_1}{\partial p}(q, q_0, t)\,,\ p_0 = -\frac{\partial F_1}{\partial q_0}(q, q_0, t)\,, \qquad (6)$$

$$\frac{\partial F_1}{\partial t} + H(q, \frac{\partial F_1}{\partial p}, t) = 0\,. \qquad (7)$$

Given two positions q_0 and q, and a transfer time T, we immediately notice that Eqns. (6) solves the two-point boundary value problem that consists of going from q_0 to q in T units of time. This remark is of prime importance since it provides us with a very general technique for solving Hamiltonian position to position boundary value problems. However, this approach relies on knowledge of F_1. In the next section, we develop an algorithm for computing this function.

3. COMPUTING THE GENERATING FUNCTIONS

The Hamilton-Jacobi theory provides us with a direct approach for computing the generating functions. Indeed, it tells us that they are solutions of the Hamilton-Jacobi equation (Eqn. (7)). This is a partial differential equation that is difficult to solve in general. However, the Hamiltonian function for describing the relative motion has a particular structure that enables us to solve this differential equation.

3.1 Relative motion

Consider a Hamiltonian system with Hamiltonian function $H(q,p,t)$. Let (q_0^0, p_0^0) and (q_0^1, p_0^1) be two points in phase space such that $q_0^1 = q_0^0 + \Delta q_0\,,\ p_0^1 = p_0^0 + \Delta p_0$, where $(\Delta q_0, \Delta p_0)$ is small enough to guaranty the convergence of the Taylor series in Eqn. (12). We denote by (q^i, p^i) the trajectory with initial conditions (q_0^i, p_0^i), i.e.,

$$q^1 = q(q_0^1, p_0^1, t)\,,\, p^1 = p(q_0^1, p_0^1, t)\,, \qquad (8)$$

$$q^0 = q(q_0^0, p_0^0, t)\,,\, p^0 = p(q_0^0, p_0^0, t)\,. \qquad (9)$$

and we define $X^h = \begin{pmatrix} \Delta q \\ \Delta p \end{pmatrix}$ the relative state vector by:

$$X^1 = X^0 + X^h\,, \qquad (10)$$

where $X^i = \begin{pmatrix} q^i \\ p^i \end{pmatrix}$. For convenience we shall call (q^0, p^0) the reference trajectory and (q^1, p^1) the displaced trajectory .

Using our previous notation, Hamilton's equations for the displaced trajectory reads:

$$\dot{X}^0 + \dot{X}^h = J\nabla H^1\,. \qquad (11)$$

We expand the right hand side of Eqn. (11) about the reference trajectory X^0, assuming $(\Delta q, \Delta p)$ small enough for convergence of the series:

$$\nabla H(q^1, p^1, t) = \nabla H(q^0, p^0, t) + \left(\begin{array}{c} \frac{\partial^2 H}{\partial q^2}(q^0, p^0, t)\Delta q + \frac{\partial^2 H}{\partial q \partial p}(q^0, p^0, t)\Delta p \\ \frac{\partial^2 H}{\partial q \partial p}(q^0, p^0, t)\Delta q + \frac{\partial^2 H}{\partial p^2}(q^0, p^0, t)\Delta p \end{array} \right) + \cdots$$

Substituting this into Eqn. (11) yields $\dot{X}^h = J\nabla H^h$, where

$$H^h(X^h, t) = \sum_{p=2}^{\infty} \sum_{\substack{i_1, \cdots, i_{2n}=0 \\ i_1+\cdots+i_{2n}=p}}^{p} \frac{1}{i_1! \cdots i_{2n}!} \frac{\partial^p H}{\partial q_1^{i_1} \cdots \partial q_n^{i_n} \partial p_1^{i_{n+1}} \cdots \partial p_n^{i_{2n}}}(q^0, p^0, t) {X_1^h}^{i_1} \ldots {X_{2n}^h}^{i_{2n}}$$

Thus, the dynamics of a particle relative to a known trajectory is Hamiltonian with a Hamiltonian function $H^h(X^h, t) = H^h(\Delta q, \Delta p, t)$. The coefficients of the Taylor series $\frac{1}{i!j!}\frac{\partial^{i+j} H}{\partial q^i \partial p^j}(q^0, p^0, t)$ are time varying quantities and are easily evaluated for any Hamiltonian once the reference trajectory is known.

3.2 Algorithm

We found that the Hamiltonian describing the dynamics of two particles relative to each other is a power series in its spatial variables, with time-dependent coefficients. At first glance, the associated Hamilton-Jacobi equation may appear impractical. However, if we truncate H^h, a closed-form solution for the generating functions can be found. In this section we briefly review the solution procedure. We refer to [5, 4] for additional details and a study of the convergence properties of our algorithm.

We assume that F_1 can be expressed as a Taylor series about the reference trajectory in its spatial variables.

$$F_1(y, t) = \sum_{q=2}^{\infty} \sum_{\substack{i_1, \cdots, i_{2n}=0 \\ i_1+\cdots+i_{2n}=q}}^{q} \frac{1}{i_1! \cdots i_{2n}!} f_{q, i_1, \cdots, i_{2n}}(t) y_1^{i_1} \cdots y_{2n}^{i_{2n}}\,, \qquad (12)$$

where $y = (\Delta q, \Delta q_0)$. We substitute this expression in the Hamilton-Jacobi equation (Eqn. (7), with $H = H^h$). The resulting equation is an ordinary differential equation that has the following structure:

$$P(y, f_{q, i_1, \cdots, i_{2n}}^{p,r}(t), \dot{f}_{q, i_1, \cdots, i_{2n}}^{p,r}(t)) = 0\,, \qquad (13)$$

where P is a series in y with time dependent coefficients that are functions of $f_{q, i_1, \cdots, i_{2n}}(t)$ and $\dot{f}_{q, i_1, \cdots, i_{2n}}(t)$. Eqn. (13) holds for all y if and only if all the coefficients of P are zero. In this manner, we transform the ordinary differential equation (13) into a set of ordinary differential equations whose solutions are the coefficients of the generating function F_1.

This approach provides us with a closed form approximation of the generating functions. However, there are inherent difficulties as generating functions may develop singularities which prevent the integration from going further (see [1, 6] for more details on singularities). Techniques that rely on the Legendre transformation have been developed [5] to bypass this problem but have a cost in terms of computation.

An alternative approach for computing F_1 has been explored in [5, 4]. We present the main ideas of this approach as in the following we combine both methods to increase performance. We suppose that $\Delta q(\Delta q_0, \Delta p_0, t)$ and $\Delta p(\Delta q_0, \Delta p_0, t)$ can be expressed as series in the initial conditions $(\Delta q_0, \Delta p_0)$ with time dependent coefficients. We truncate the series to order N and insert these into Eqn. (1). Hamilton's equations reduce to a series in $(\Delta q_0, \Delta p_0)$ whose coefficients depend on the coefficients of the series $\Delta q(\Delta q_0, \Delta p_0, t)$ and $\Delta p(\Delta q_0, \Delta p_0, t)$ and their time derivatives. By balancing terms of the same order, we transform Hamilton's equations into a set of ordinary differential equations whose variables are the coefficients defining Δq and Δp as a series in Δq_0 and Δp_0.

For linear systems, this approach recovers the state transition matrix. Then, a series inversion of the phase flow provides us with the gradient of the generating functions that can be integrated to find the generating functions.

The main advantage of this approach is that the phase flow is never singular, therefore the ordinary differential equations are always well-defined. However, this method requires us to solve more equations than the previous method and provides us with the value of F_1 at a given time only (the time at which we perform the series inversion).

In the following, we use a "combined" algorithm. The alternative approach is used to compute the phase flow over a long time span. Then, we compute the value of F_1 at a time of interest, t_1, and solve the Hamilton-Jacobi equation around t_1. For both examples, we compute the first four terms in the series expansion of F_1. We will see that they provide an accurate picture of the nonlinear dynamics about the reference trajectory.

4. A MULTI-TASK MISSION ABOUT THE EARTH

4.1 Problem settings

The motion of a satellite under the influence of the Earth modeled by an oblate sphere (J_2 and J_3 gravity coefficients are taken into account) in the fixed coordinate system (x, y, z) whose origin is the Earth center of mass is described by the following Hamiltonian:

$$H = \frac{1}{2}(p_x^2 + p_y^2 + p_z^2) - \frac{1}{\sqrt{x^2+y^2+z^2}} \Big[1 - \frac{R^2}{2r_0^2(x^2+y^2+z^2)}\left(3\frac{z^2}{x^2+y^2+z^2} - 1\right) J_2 - \frac{R^3}{2r_0^3(x^2+y^2+z^2)^2}\left(5\frac{z^3}{x^2+y^2+z^2} - 3z\right) J_3\Big] ,$$

where

$$GM = 398600.4405\ km^3 s^{-2}\,,\ R = 6378.137\ km\,,$$
$$J_2 = 1.082626675 \cdot 10^{-3}\,,\ J_3 = 2.532436 \cdot 10^{-6}\,,$$

and all the variables are normalized. Distances are normalized by r_0, the radius of the trajectory at the initial time, and the time is normalized by $\sqrt{r_0^3/GM}$.

We consider a "reference" trajectory whose state is designated by (q^0, p^0) and study the relative motion of spacecraft with respect to it. The reference trajectory is chosen to be highly eccentric and inclined, but any other choice could have been considered. At the initial time its state is:

$$\begin{aligned}
q_x^0 &= r_p\,, & p_x^0 &= 0\ kms^{-1}\,,\\
q_y^0 &= 0\ km\,, & p_y^0 &= \sqrt{\frac{GM}{\frac{1}{2}(r_a+r_p)}}\sqrt{\frac{r_a}{r_p}}\cos(\alpha)\ km\,s^{-1}\,,\\
q_z^0 &= 0\ km\,, & p_z^0 &= \sqrt{\frac{GM}{\frac{1}{2}(r_a+r_p)}}\sqrt{\frac{r_a}{r_p}}\sin(\alpha)\ km\,s^{-1}\,,\\
\alpha &= \frac{\pi}{3} rad\,, & r_p &= 7{,}000\ km\,,\ r_a = 13{,}000\ km\,.
\end{aligned}$$

Without the J_2 and J_3 gravity coefficients the reference trajectory would be an elliptic orbit with eccentricity $e = 0.3$, inclination $i = \pi/3\ rad$, argument of perigee $\omega = 0$, longitude of the ascending node $\Omega = 0$, semi-minor axis $r_p = 7,000\ km$, semi-major axis $r_a = 13,000\ km$ and of period $t_p = 2\pi\sqrt{\frac{1}{2^3}\frac{(r_a+r_p)^3}{r_p^3}}\ sec \approx 2\ hours\ 45\ min$. The Earth oblateness perturbation causes (see Chobotov [2] for more details) secular drifts and short terms oscillations in the orbital elements. In Fig. 2, we plot the orbital elements for this trajectory as a function of time over a month period (about 300 revolutions about the Earth). The symplectic implicit Runge-Kutta integrator built in $Mathematica^{©}$ is used for integration of Hamilton's equations.

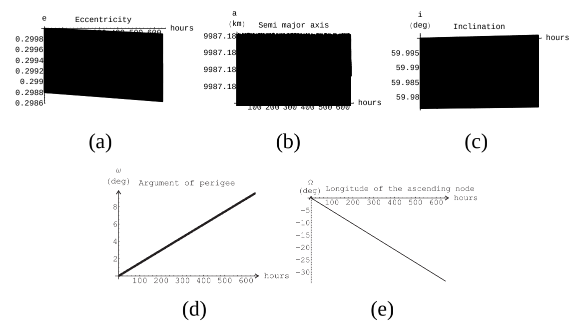


Figure 2: Time history of the orbital elements of the reference trajectory over a month period

4.2 Multi-task mission

We consider four imaging satellites flying in formation about the reference trajectory. We want to plan spacecraft maneuvers over the next month knowing that they must observe the Earth, i.e., must be in a given configuration C_i at the following instants (chosen arbitrarily for our study):

$$\begin{aligned}
t_0 &= 0\,, & t_1 &= 5\ d\ 22\ h\,, & t_2 &= 10\ d\ 20\ h\,,\\
t_3 &= 16\ d\ 2\ h\,, & t_4 &= 21\ d\ 14\ h\,, & t_5 &= 26\ d\ 20\ h\,.
\end{aligned}$$

Define the local horizontal by the unit vectors $(\hat{e}_1, \hat{e}_2)$ such that $\hat{e}_2$ is along $r^0 \times v^0$ and $\hat{e}_1$ is along $\hat{e}_2 \times r^0$. At every t_i, the configuration C_i is defined by the four following relative positions (or slots):

$$\begin{aligned}
q^1 &= 700\ m\ \hat{e}_1\,,\ q^2 = -700\ m\ \hat{e}_1\,,\\
q^3 &= 700\ m\ \hat{e}_2\,,\ q^4 = -700\ m\ \hat{e}_2\,.
\end{aligned} \qquad (14)$$

Note that at t_i, q^1 is in front of the reference state (in the local horizontal plane), q^2 is behind, q^3 is on the left and q^4 is on the right (see Fig. 1(a)). At each t_i, there must be one spacecraft per slot and we want to determine the sequence of reconfigurations that minimizes the total fuel expenditure (other cost functions such as equal fuel consumption for each spacecraft may be considered as well).

For the first mission, there are $4!$ configurations (number of permutation of the set $\{1, 2, 3, 4\}$). For the second mission, for each of the previous $4!$ configurations, there are again $4!$ configurations, that is a total of $4!^2$ possibilities. Thus for 5 missions, there are $4!^5 = 7,962,624$ possible configurations.

In this example, we assume impulsive controls that consist of impulsive thrusts applied at $t_{i\in[0,5]}$. For each of the four spacecraft, we need to compute the velocity at t_i so that the spacecraft moves to its position specified at t_{i+1} under gravitational forces only. As a result, we must solve $5 \cdot 4! = 120$ position to position boundary value problems (given two positions at t_i and t_{i+1}, we need to compute the associated velocity). Using the generating functions, this problem can be handled at the cost of only 120 function evaluations. Then, we need to evaluate the fuel expenditure (sum of the norm of all the required impulses, assuming zero relative velocities at the initial and final times) for all the permutations (there are $7,962,624$ combinations) to find the sequence that minimizes the cost function. Fig. 3 represents the number of configurations as a function of the values of the cost function. We notice that most of the configurations require at least three times more fuel than the best configuration, and less than 6% yield values of the cost function that are less than twice the value associated with the best configuration. The cost function for the optimal sequence of reconfigurations is $0.00644\ km \cdot s^{-1}$ whereas it is $0.0396\ km \cdot s^{-1}$ in the least optimal design.

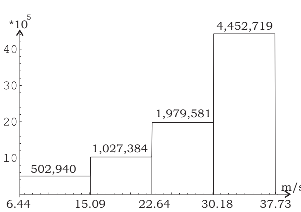
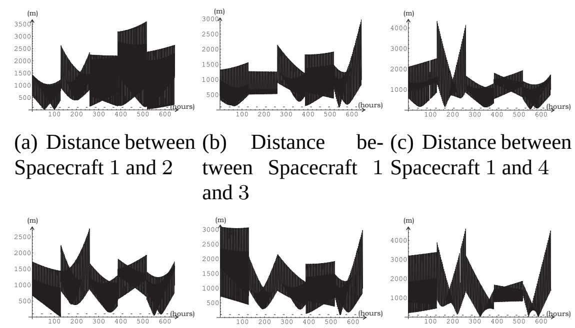


Figure 3: Number of configurations as a function of the value of the cost function

We may verify, *a posteriori*, if the solutions found meet the mission goals, i.e., if the order 4 approximation of the dynamics is sufficient to simulate the true dynamics. Explicitly comparing the analytical solution with numerically integrated results shows that the spacecraft are at the desired positions at every t_i with a maximum error of $1.5 \cdot 10^{-8}\ km$.

4.3 Considerations on collision management

Our algorithm does not consider the risk of collision in the design. Integrating Hamilton's equations shows that the best scenario yields collisions. Therefore, it cannot be used for this problem and we need to find another design. It can be proven that for this specific mission, the minimum relative distance between the spacecraft is at best about $15\ m$, and is achieved in $3,360$ different designs. Among these $3,360$ possibilities, we represent in Fig. 4 the time history of the relative distance between the spacecraft for the design that achieves minimum fuel expenditure (the total fuel expenditure is $60\ \%$ larger than in the best case). For times at which the spacecraft are



(a) Distance between Spacecraft 1 and 2

(b) Distance between Spacecraft 1 and 3

(c) Distance between Spacecraft 1 and 4

(d) Distance between Spacecraft 2 and 3

(e) Distance between Spacecraft 2 and 3

(f) Distance between Spacecraft 3 and 4

Figure 4: Distance between the spacecraft as a function of time

close to each other, we may use some local control laws to perform small maneuvers to ensure appropriate separation.

Another option consists of changing the configurations at t_i so that there exists a sequence of reconfigurations such that the relative distance between the spacecraft stay larger than a given safe distance. This can easily be done using our approach since F_1 is already known. Solving a new design would only require 120 function evaluations as the same generating function can be used.

5. THE DEPLOYMENT PROBLEM IN THE HILL THREE-BODY PROBLEM

5.1 Problem settings

The Hill three-body problem is a three-body problem in which three main assumptions are made: 1) One of the three bodies has negligible mass compared to the other two-bodies. 2) One of the two massive bodies is in circular orbit about the other one. 3) One of the two massive bodies has larger mass than the other one. These hypothesis hold to study the motion of a spacecraft under the influence of the Sun and the Earth for example. Under these assumptions, the normalized Lagrangian for this system is

$$L(q, \dot{q}) = \frac{1}{2}(\dot{q}_x^2 + \dot{q}_y^2) + \frac{1}{\sqrt{q_x^2 + q_y^2}} + \frac{3}{2}q_x^2 - (\dot{q}_x q_y - \dot{q}_y q_x)$$

where $(q_x, q_y) = (x, y)$. This problem has 2 equilibrium points, L_1 and L_2 whose coordinates are $L_1(-\left(\frac{1}{3}\right)^{1/3}, 0)$ and $L_2(\left(\frac{1}{3}\right)^{1/3}, 0)$.

We consider several spacecraft at L_2 at the initial time and solve the deployment problem. In other words, we want to find the optimal control laws that drive the formation from L_2 to a given configuration at $t = T$. We assume continuous thrusts and no thrust constraints. Thus, for each of the spacecraft, we need to solve an optimal control problem formulated as:

$$\min_{U=(u_x,u_y)} J = \min_{U=(u_x,u_y)} \frac{1}{2}\int_{t=0}^{t=T} (u_x^2 + u_y^2)dt\,, \quad (15)$$

subject to the dynamics:

$$\frac{\partial L}{\partial q}(q,\dot{q}) - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}}(q,\dot{q}) = U\,, \quad (16)$$

and the boundary conditions:

$$X(t=0) = X_{L_2} = (3^{-1/3}, 0, 0, 0)\,,\ X(t=T) = X_T\,,$$

where $X = (q_x, q_y, \dot{q}_x, \dot{q}_y)$ and $U = (u_x, u_y)$. Necessary conditions for optimality can be found from Pontryagin's maximum principle:

$$\dot{X} = \frac{\partial H}{\partial P}\,,\ \dot{P} = -\frac{\partial H}{\partial X}\,,\ \frac{\partial H}{\partial U} = 0\,, \quad (17)$$

where $P = (p_1, p_2, p_3, p_4)$ and

$$H(X,P,U) = P^T\dot{X} + \frac{1}{2}u_x^2 + \frac{1}{2}u_y^2\,.$$

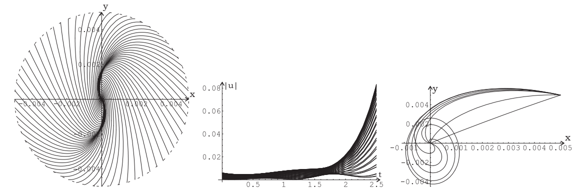
Then, from $\frac{\partial H}{\partial U} = 0$, we find the optimal control feedback law: $u_x = -p_3\,,\ u_y = -p_4$. We substitute $U = (u_x, u_y)$ into H to obtain $\bar{H}(X,P) = H(X,P,U(X,P))$. Thus, the necessary conditions for optimality now define a *Hamiltonian* position to position boundary value problem that can be solved using F_1:

$$\dot{X} = \frac{\partial \bar{H}}{\partial P}, \dot{P} = -\frac{\partial \bar{H}}{\partial X}, X(0) = X_{L_2}, X(T) = X_T \quad (18)$$

Eqns. (6) provide the value of the co-state P at the initial and final times. Then, the optimal trajectory is found by integrating Hamilton's equations (Eqns. (18)).

5.2 The deployment problem

In Fig. 5(a) and 5(b), we plot the optimal control trajectories and the norm of the optimal control laws for different final positions X_T on a circle of radius $r = 0.05$ ($10,700$ km in the Earth-Sun system) and a transfer time of $t = 2.5$ (i.e., about 145 days in the Earth-Sun system). We observe that some values of the final position requires less fuel, they correspond to $X_T = r\cos(\theta) + r\sin(\theta)$ where $\theta = \{19\pi/32, 51\pi/32\}$. Similarly, we may vary the transfer time. In Fig. 5(c), we plot the optimal trajectories for $T \in \{0.1, 1.1, 2.1, 3.1, 4.1, 5.1, 6.1\}$ (i.e., from 6 to 290 days). As T increases, the trajectory wraps around L_2 so that the spacecraft takes advantage of the geometry of the Libration point.



(a) Optimal trajectories (b) Norm of the optimal control laws (c) Optimal trajectories

Figure 5: The deployment problem

In this manner, we can explore the best deployment sequence. Depending on the final configuration geometry (e.g., the spacecraft must be equally spaced on a circle of radius r) and mission specifications, we are able to choose the optimal transfer time and final configuration to minimize the fuel expenditure by evaluating a set of functions.

6. CONCLUSIONS

To conclude, we have been able to obtain semi-analytic description of the phase flow of complex dynamical models that solves two-point boundary value problems. Such a description of the phase space is superior in many ways to the traditional approach based on the initial value problem. In previous work [4], we successfully applied this method to compute periodic and stable configurations.

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