PERIODIC ORBITS AND FORMATION FLYING NEAR THE LIBRATION POINTS

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Abstract: *Formation flying along a halo orbit near the L_2 point in the Sun-Earth circular restricted three-body system is considered. The reference orbit is a periodic perturbation of the halo orbit generated by an exosystem, and the output regulation theory is employed for the transition and maintenance. The total velocity change required for the maintenance of the halo orbit is calculated. Then the total velocity change for the maintenance of the perturbed reference orbits with different size is calculated, and the size of the perturbation which does not require an additional velocity change is determined. Reference orbits which are shifted halo orbits in the along-track and cross-track directions are also considered. Varying the size of the shift, the total velocity change is calculated, and the reasonable size of the shift for formaion flying is determined. Finally, formation flying near the L_2 point is considered, where the reference orbit is a periodic orbit whose frequency is equal to that of the linearized equations of motion at the L_2 point. The size of the orbit is determined such that the total velocity change for the maintenance is comparable to that of the maintenance of the halo orbit.*

Keywords: *Sun-Earth Three-Body Problem, Halo Orbit, Formation Flying, Total Velocity Change, Output Regulation.*

1. Introduction

Recently formation flying along a halo orbit near a libration point of the circular-restricted three-body problem (CR3BP) [1] has been studied [2]. As a basic dynamics, the linearized equations of motion along a halo orbit is used, and a control strategy to maintain a satellite near the orbit, which is based on the short-term relaive motion, is proposed. In this case the difficulty lies in the fact that the coefficients of the linearized system depend on the halo orbit and are periodic functions which are only numerically available. The paper [2] is motivated by earlier studies [3, 4] on formation flying in the vicinity of a libration point of the CR3BP with application to spacecraft imaging arrays. Reference orbits in [3, 4] are not natural orbits of the CR3BP but controlled orbits. In this case control implementation is straightforward, but good control strategies need to be designed.

In this paper halo orbits near a collinear libration point are considered, and the maintenance problem by feedback is considered. For this purpose the equations of motion are expressed by the rotating framework whose origin is the libration point. This makes the dynamics semilinear, and the linear control theory can be employed. For the follower's orbit periodic perturbations of the halo orbit are used. To maintain such an orbit, the output regulation thoery for linear systems [Saberi(2000)] is used. Stabilizing feedback controls are designed via the algebraic Riccati equations of the linear quadratic regulator (LQR). As a performance index, ΔV neccessary to maintain a halo orbit or its perturbation for one period is calculated. The frequency of the periodic perturbation is arbitrary, and hence the integer multiple of the frequency of the halo orbit are considered and L_2 in each case is examined. The feedback gain depends on the weight parameters on state and control. Thus varying one of the parameters, ΔV is computed as a function of the parameter.

As for numerical simulations, the Sun-Earth CR3BP is considered, and the Lagrangian point is specified as L_2. As a halo orbit, a periodic orbit with period close to the half of the sun-earth system is considered. The linearized equations of motion at L_2 possess periodic orbits for in-plane and out-of-plane motion. Selecting one of these two frequencies for a periodic perturbation, periodic reference orbit can be generated. The ΔV to maintain such orbit is also calculated.

2. Equations of motion in the CR3BP

Recall that the equations of motion of the circular restricted three-body problem (CR3BP) in the nondimensional form are given by

$$\begin{aligned} X'' - 2Y' - X &= -\frac{1-\rho}{r_1^3}(X+\rho) - \frac{\rho}{r_2^3}(X-1+\rho) + u_x, \\ Y'' + 2X' - Y &= -\frac{1-\rho}{r_1^3}Y - \frac{\rho}{r_2^3}Y + u_y, \\ Z'' &= -\frac{1-\rho}{r_2^3}Z - \frac{\rho}{r_2^3}Z + u_z, \end{aligned} \tag{1}$$

where $\{X, Y, Z\}$ is the rotating frame whose origin is the barycenter of the system, the coordinates of the satellite are normalized by the radius of the circular orbit and time by the rate of the orbit, $\rho = M_2/(M_1+M_2)$, M_1 and M_2 are the masses of the two main bodies with $M_1 > M_2$, (u_x, u_y, u_z) is the control acceleration, and

$$\begin{aligned} r_1 &= [(X+\rho)^2 + Y^2 + Z^2]^{1/2}, \\ r_2 &= [(X-1+\rho)^2 + Y^2 + Z^2]^{1/2}. \end{aligned}$$

Eq. (1) has stationary points known as Lagrangian points L_i satisfying

$$\begin{aligned} X &= \frac{1-\rho}{r_1^3}(X+\rho) + \frac{\rho}{r_2^3}(X-1+\rho), \\ Y &= \frac{1-\rho}{r_1^3}Y + \frac{\rho}{r_2^3}Y, \\ Z &= 0, \end{aligned} \tag{2}$$

and

$$\begin{aligned} &L_1 = (l_1(\rho), 0, 0), \quad L_2 = (l_2(\rho), 0, 0), \quad L_3 = (l_3(\rho), 0, 0), \\ &L_4 = (1/2-\rho, \sqrt{3}/2, 0), \quad L_5 = (1/2-\rho, -\sqrt{3}/2, 0), \end{aligned}$$

where $l_i(\rho)$ are determined by the first equation of Eq. (2). To describe equations of motion near a collinear equilibrium point L_i, it is conveneient to use the coordinate system with center at L_i.

Replacing X, Y, Z by $x + l_i, y, z$, Eq. (1) can be written as

$$
\begin{aligned}
x'' - 2y' - x &= l_i - \frac{1-\rho}{r_1^3}(x + l_i + \rho) - \frac{\rho}{r_2^3}(x + l_i - 1 + \rho) + u_x, \\
y'' + 2x' - y &= -\frac{1-\rho}{r_1^3}y - \frac{\rho}{r_2^3}y + u_y, \\
z'' &= -\frac{1-\rho}{r_1^3}z - \frac{\rho}{r_2^3}z + u_z,
\end{aligned}
\tag{3}
$$

where

$$
\begin{aligned}
r_1 &= [(x + l_i + \rho)^2 + y^2 + z^2]^{1/2}, \\
r_2 &= [(x + l_i - 1 + \rho)^2 + y^2 + z^2]^{1/2}.
\end{aligned}
$$

The linearized equations of (3) at the orign are given as follows

$$
\begin{aligned}
x'' - 2y' - (2\sigma_i + 1)x &= u_x, \\
y'' + 2\bar{x}' + (\sigma_i - 1)y &= u_y, \\
z'' + \sigma_i z &= u_z,
\end{aligned}
\tag{4}
$$

where

$$
\sigma_i = \rho/|l_i(\rho) - 1 + \rho|^3 + (1-\rho)/|l_i(\rho) + \rho|^3 \tag{5}
$$

The state space form of (4) is given by

$$
\boldsymbol{x}' = A\boldsymbol{x} + B\boldsymbol{u}, \tag{6}
$$

where $\boldsymbol{x} = [x \; y \; x' \; y' \; z \; z']^T$, $\boldsymbol{u} = [u_x \; u_y \; u_z]^T$, and

$$
A = \begin{bmatrix}
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
2\sigma_i + 1 & 0 & 0 & 2 & 0 & 0 \\
0 & 1 - \sigma_i & -2 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & -\sigma_i & 0
\end{bmatrix}, \quad
B = \begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 0 \\
0 & 0 & 1
\end{bmatrix}.
$$

The state space form of (3) is then semilinear and is given by

$$
\boldsymbol{x}' = A\boldsymbol{x} + B\boldsymbol{f}(\boldsymbol{x}) + B\boldsymbol{u}, \tag{7}
$$

where

$$
\boldsymbol{f}(\boldsymbol{x}) = \begin{bmatrix}
l_i - 2\sigma_i x - \frac{1-\rho}{r_1^3}(x + l_i + \rho) - \frac{\rho}{r_2^3}(x + l_i - 1 + \rho) \\
\sigma_i y - \frac{1-\rho}{r_1^3}y - \frac{\rho}{r_2^3}y \\
\sigma_i z - \frac{1-\rho}{r_1^3}z - \frac{\rho}{r_2^3}z
\end{bmatrix}.
$$

3. Formation flying near a collinear libration point

The semilinear form (7) is convenient for control purposes. To see this, let $\boldsymbol{x}_f$ be a periodic orbit of the leader near the L_i point given by

$$\boldsymbol{x}_f' = A\boldsymbol{x}_f + B\boldsymbol{f}(\boldsymbol{x}_f), \boldsymbol{x}_f(0) = \boldsymbol{x}_{f0}. \tag{8}$$

To maintain this orbit, a atandard way is to linearize (7) along $\boldsymbol{x}_f$ and apply the LQR theory for periodic systems. The resulting equation has periodic coefficients depending on $\boldsymbol{x}_f$ and a feedback control is designed by the differential Riccati equation . However, the leader can maintain this orbit by the feedback

$$\boldsymbol{u} = -F\boldsymbol{e} + \boldsymbol{f}(\boldsymbol{x}_f) - \boldsymbol{f}(\boldsymbol{x}), \tag{9}$$

where $\boldsymbol{e} = \boldsymbol{x} - \boldsymbol{x}_f$ and F is an arbitrary feedback gain such that $A - BF$ is stable. In fact

$$\boldsymbol{e}' = (A - BF)\boldsymbol{e}. \tag{10}$$

Now consider formation flying where the follower is required to track $(x_f + w_1, y_f + w_2, z_f + w_3)$. Here (x_f, y_f, z_f) is the periodic trajectory of the leader given by (8) and (w_1, w_2, w_3) is a periodic (or quasiperiodic) trajectory generated by an exosystem

$$\boldsymbol{w}' = S\boldsymbol{w},\ \boldsymbol{w}(0) = \boldsymbol{w}_0 \tag{11}$$

where

$$S = \begin{bmatrix} 0 & s_1 & 0 & 0 \\ s_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -\omega_2^2 & 0 \end{bmatrix},\ s_1 s_2 = -\omega_1^2, \omega_2 > 0. \tag{12}$$

This problem can be solved if and only if the regulator equation associated with (6)

$$\begin{aligned} &A\Pi - \Pi S + B\Gamma = 0, \\ &C\Pi + D = 0, \end{aligned} \tag{13}$$

has a solution [Saberi(2000)], where

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix},\quad D = -\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}. \tag{14}$$

In this case it is known [7] that the regulator equation (12) has a solution. In fact the solution is given by

$$\Pi = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & s_1 & 0 & 0 \\ s_2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},\quad \Gamma = \begin{bmatrix} s_1 s_2 - 2s_2 - 2\sigma - 1 & 0 & 0 \\ 0 & s_1 s_2 + 2s_1 - 1 + \sigma & 0 \\ 0 & 0 & -\omega_2^2 + \sigma \end{bmatrix}. \tag{15}$$

The feedback control

$$\boldsymbol{u} = -F\boldsymbol{x} + (\Gamma + F\Pi)\boldsymbol{w} \tag{16}$$

fulfills the output regulation for the linear system (6), i.e.,

$$\boldsymbol{z} = C\boldsymbol{x} + D\boldsymbol{w} \to 0 \text{ as } \tau \to \infty. \tag{17}$$

Now consider the same tracking problem for the semilinear system (7). Combining (7) and (8), the dynamics for the error $\boldsymbol{e}$ is given by

$$\boldsymbol{e}' = A\boldsymbol{e} + B(\boldsymbol{f}(\boldsymbol{x}) - \boldsymbol{f}(\boldsymbol{x}_f)) + B\boldsymbol{u} \tag{18}$$

Define the output

$$\boldsymbol{z} = C\boldsymbol{e} + D\boldsymbol{w}. \tag{19}$$

The tracking problem above is equivalent to the output regulation problem (18) and (19).

Theorem 3..1. The output regulation problem (18) and (19) can be solved by the modified control

$$\boldsymbol{u} = -F\boldsymbol{e} + \boldsymbol{f}(\boldsymbol{x}_f) - \boldsymbol{f}(\boldsymbol{x}) + (\Gamma + F\Pi)\boldsymbol{w}. \tag{20}$$

Proof. In fact

$$\begin{aligned}
(e - \Pi\boldsymbol{w})' &= A\boldsymbol{e} + B\boldsymbol{f}(\boldsymbol{x}) - B\boldsymbol{f}(\boldsymbol{x}_f) + B\boldsymbol{u} - \Pi S\boldsymbol{w} \\
&= (A - BF)\boldsymbol{e} + B(\Gamma + F\Pi)\boldsymbol{w} - \Pi S\boldsymbol{w} \\
&= (A - BF)(\boldsymbol{e} - \Pi\boldsymbol{w}) + (A\Pi - \Pi S + B\Gamma)\boldsymbol{w} \\
&= (A - BF)(\boldsymbol{e} - \Pi\boldsymbol{w}).
\end{aligned}$$

Hence $(e - \Pi\boldsymbol{w}) \to 0$. Now

$$\begin{aligned}
\boldsymbol{z} &= C\boldsymbol{e} + D\boldsymbol{w} \\
&= C(\boldsymbol{e} - \Pi\boldsymbol{w}) + (C\Pi + D)\boldsymbol{w} \\
&= C(\boldsymbol{e} - \Pi\boldsymbol{w}) \to 0,
\end{aligned}$$

which implies that the follower tracks $(x_f + w_1, y_f + w_2, z_f + w_3)$ and stays in the vicinity of (x_f, y_f, z_f).

Remark 3..1. The frequencies of S can be chosen as those of the linearized equations (4).

It is natural to use the periodic orbit $\boldsymbol{x}_f$ as the reference orbit of the leader. However, reference orbits can be generated by the linear system (6). Recall that the out-of-plane motion of (6) is periodic, and the in-plane motion also has periodic solutions. Hence quasi-periodic orbits can be generated. By adjusting one of the frequencies, periodic orbits can be generated. To keep the leader (and/or follower) in such an orbit, one can choose the trivial solution $\boldsymbol{x}_f = 0$ and apply to (7) the control law

$$\boldsymbol{u} = -F\boldsymbol{x} - \boldsymbol{f}(\boldsymbol{x}) + (\Gamma + F\Pi)\boldsymbol{w}. \tag{21}$$

In this case the output regulation problem is defined by

$$\begin{aligned} \boldsymbol{x}' &= A\boldsymbol{x} + B\boldsymbol{f}(\boldsymbol{x}) + B\boldsymbol{u}, \\ z &= C\boldsymbol{x} + D\boldsymbol{w}. \end{aligned} \tag{22}$$

Corollary 3..1. The output regulation problem (22) can be solved by the control (21).

4. Simulation results

In this section the control laws (9), (20) and (21) are applied to the Sun-Earth CR3BP, and ΔV necessary for the maintenance of a periodic orbit is calculated. The Lagrangian point is assumed to be L_2. In this case $\rho = 3.0542 \times 10^{-6}$ and $\sigma = 1.0104$. The period and the radius of the Sun-Earth system are $T_0 = 365.26$ [d] and $R_0 = 1.4960 \times 10^8$ [km]. The initial condition of the reference orbit $\boldsymbol{x}_f$ is $\boldsymbol{x}_{f0} = [-1.6623 \times 10^{-3} \;\; 0.0000 \;\; 0.0000 \;\; 9.8104 \times 10^{-3} \;\; 1.0000 \times 10^{-4} \;\; 0.0000]$ with period $T = 3.1026$. The trajectory is given by Fig. 1, which is unstable and within two periods it diverges away. Thus the reference orbit is the periodic extension of the trajectory of the first period, and to maintain this orbit feedback control (9) is necessary. The feedback control F is designed by the Riccati equation

$$A'X + XA + Q - XBR^{-1}B'X = 0 \tag{23}$$

of the linear quadratic regulator and is given by $F = R^{-1}B'X$, where $Q = I$ and $R = I$. The ΔV required to maintain this orbit for one period is $\Delta V_1 (= 2.6249 \times 10^{-4}) = 7.8232$ [m/s]. As the follower's orbit, it is assumed that $w_1 = a\cos\omega t, w_2 = a\sin\omega t, w_3 = b\cos\omega t$, where $a = b = 1.0000 \times 10^{-6}, \omega = k\omega_f, k = 0.5, 1, 2, \cdots, 20$ and $\omega_f = 2\pi/T$. The ΔV required for the maintenance of the reference orbit for one period T are shown in Fig. 2. The ΔV remains constant for $k = 0.5 - 4$, and it is comparable to that of the halo orbit. Thus the fuel consumption for the leader and the follower is approximately equal. Fig. 3 shows the controlled trajectory in the $X - Y$-plane for $k = 2, 4, 10, 20$ and $a = 0.0002$. The case of different reference orbits are shown in Fig. 4, where shifted halo orbits in Y- and Z-directions are considered. The ΔV required to maintain these orbits for one period are shown in Table. 1 and 2. The case $d = 10^{-6}$ corresponds to the shift of ≈ 150 [km] in Y-direction, and the ΔV required for the maintenance for one period is 9.0097 [m/s] which is comparable to that of the maintenance of the halo orbit. In the case of the shift in the Z-direction, the ΔV required for the maintenance for one period is 7.8522 [m/s], which is smaller. Furtheremore, the increase of the shift d does not affect the ΔV very much as seen from Table2.

Finally assume that $\boldsymbol{x}_f = 0$, and consider the control law (21). The frequency of the in-plane motion is $\omega_{in} = 0.9901$ and that of the out-of-plane is $\omega_{out} = 0.9951$. In this case it is assumed that $\omega_1 = \omega_2 = \omega_{in}$ and $a = b = 1.0000 \times 10^{-6}$. The ΔV required to maintain this small orbit for one period is 1.1916 [m/s] and is 15% of the halo orbit maintenance. Hence such orbits are useful for missions close to the Lagrangian point.

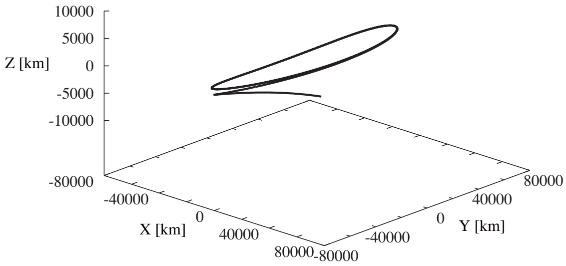

Fig. 1. Uncontrolled trajectory for 2 periods.

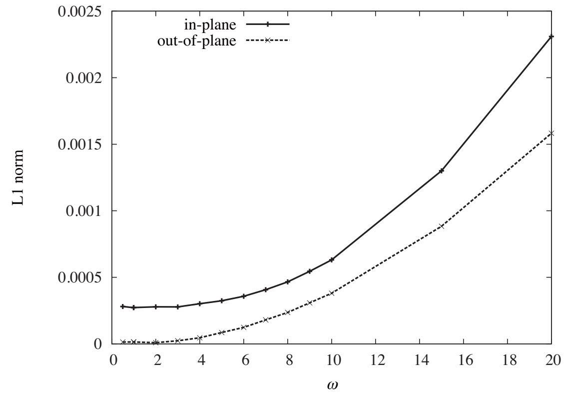

Fig. 2. ΔV **vs.** ω **(**$a = b = 1.0 \times 10^{-6}$**).**

Table 1. L_1**-norm for maintenance of shifted halo orbits in** Y**-direction.**

d	L_1	L_1(in)	L_1(out)	L_1[m/s]	L_1(in)[m/s]	L_1(out)[m/s]
0.000001	0.0003023	0.0003021	0.0000054	9.009665	9.005839	0.159879
0.000010	0.0008693	0.0008692	0.0000105	25.911608	25.907552	0.313093
0.000100	0.0081282	0.0081279	0.0000649	242.270162	242.259841	1.934422

5. Conclusions

The formation flying near a halo orbit in the Sun-Earth system is considered. The reference orbit is a periodic perturbation of the halo orbit generated by an exosystem and the output regulation theory is employed. The halo orbit is unstable and a small error in the initial condition results in a large deviation from the orbit within a few periods. The ΔV required for the halo orbit maintenance is calculated. The maintenance of the reference orbit with perturbation of order up to 10^{-3} requires ΔV is comparable to that of the halo orbit maintenance. The case of reference orbits which are shifted halo orbits in the along-track and cross-track directions is also considered, and the ΔV is comparable to that of the halo orbit maintenance when the shift distance is of order 10^{-3} or less.

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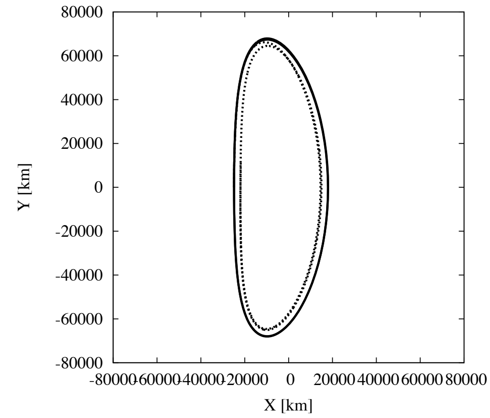
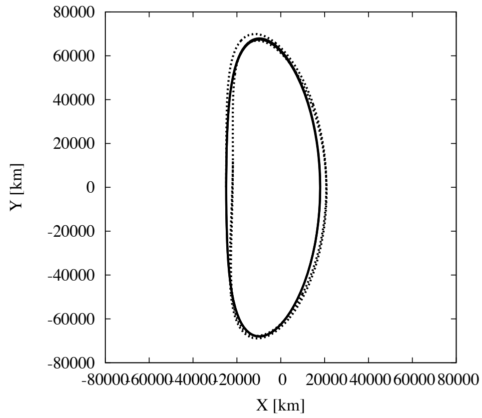
Table 2. L_1**-norm for maintenance shifted halo orbits in** Z**-direction.**

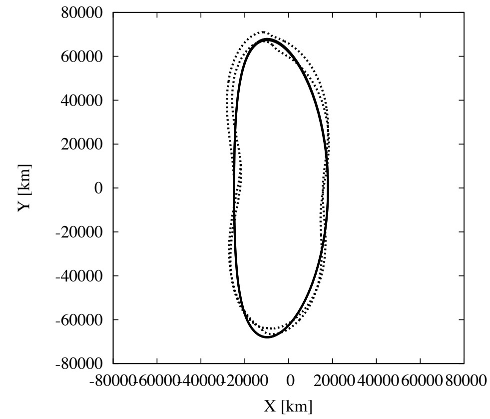
d	L_1	L_1(in)	L_1(out)	L_1[m/s]	L_1(in)[m/s]	L_1(out)[m/s]
0.000001	0.0002634	0.0002624	0.0000053	7.852224	7.820001	0.158423
0.000010	0.0002730	0.0002625	0.0000271	8.135598	7.825000	0.806314
0.000100	0.0004179	0.0002642	0.0002569	12.454742	7.874737	7.657699



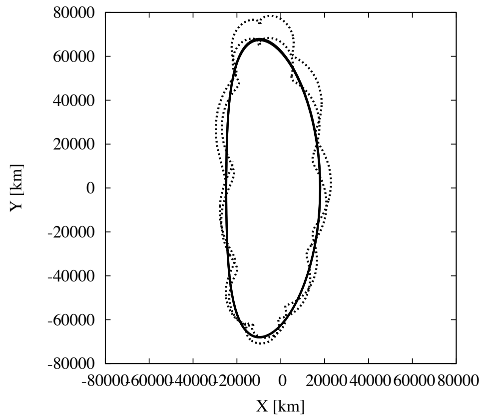
(a) $\omega = 2$.



(b) $\omega = 4$.

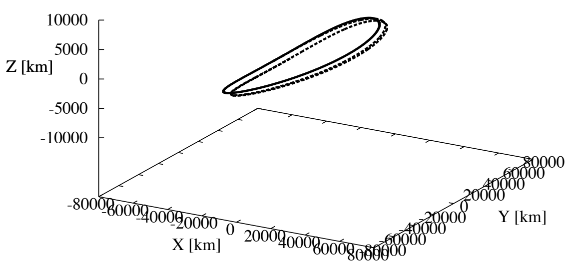


(c) $\omega = 10$.

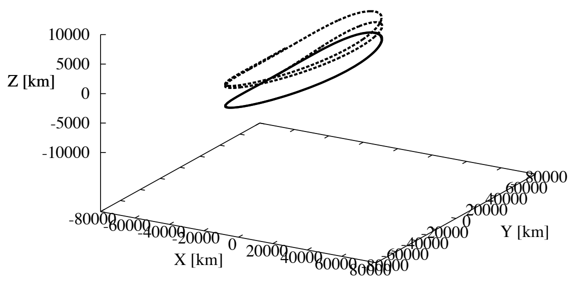


(d) $\omega = 20$.

Fig. 3. Controlled trajectory ($a = 2.0 \times 10^{-4}$).



(a) Shifted trajectory in Y-direction.(d=0.0002)



(b) Shifted trajectory in Z-direction.(d=0.002)

Fig. 4. Shifted halo orbits.

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