UNCONTROLLED SPACECRAFT FORMATIONS ON TWO-DIMENSIONAL INVARIANT TORI

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ABSTRACT

Within the class of natural motions near libration point regions quasi-periodic trajectories evolving on invariant tori are studied. Those orbits prove beneficial for relative spacecraft configurations with large distances among satellites. In this study properties of invariant tori are outlined, and non-resonant and resonant tori around the Sun/Earth libration point L_1 are computed. A numerical approach to obtain the frequency base and to parametrize a torus in angular phase space is introduced. Initial states for spacecraft formations on the torus' surface are defined. The formation naturally evolve along its surface such that the relative positions within a formation are unaltered and the relative distances and the orientation are closely bounded. An in-plane coordinate frame together with a modified torus motion is introduced and the inner and outer behaviour of the formation's geometry is investigated.

1. Introduction

New mission concepts using large spacecraft formations emphasise the study of multiple spacecraft placed at large relative distances. Projects within the ESA Cosmic Vision are now in the study phase with the aim to trigger new scientific discoveries, such as the detection of asteroids and planets. They are expected to provide high performance for telescope or spectroscopy configurations. The future of fundamental physics done in space lies in the exploitation of the properties of the libration points. The Sun-Earth L_1 libration point region provide a low-acceleration environment that is ideal for spacecraft formations. The centre manifold existing within the vicinity of the libration points provides a variety of natural periodic and quasi-periodic orbits that could prove beneficial for formation flight. Quasi-periodic trajectories, in particular trajectories evolving on an invariant torus, allow for natural formation whose geometry is bounded, and offer more possibilities with respect to periodic orbits.

Much of the available research in formation flying near the libration points focus on the exploitation of periodic orbits for trajectory and formation design. Kolemen and Olikara studied quasi-periodic orbits around the libration points and proposed numerical procedures for their generation and continuation [5, 7]. The algorithm uses a Fourier representation to describe an invariant curve representing the intersection of an invariant torus with a Poincaré section. Schilder investigated invariant objects, in particular two-dimensional tori and their trajectories [8]. They studied invariant tori for a generic dynamical system, but it can be applied with some modifications to the dynamics used in this study. Barden investigate formation flying near libration points in the circular restricted three-body problem (CR3BP) with a focus on the determination of the natural behaviour at the centre manifold [1]. A 'string of pearls' was proposed to demonstrate that quasi-periodic trajectories evolving on an invariant torus are useful for formation flying. Hértier explores quasi-periodic Lissajous trajectories near a given reference orbit in the vicinity the Sun-Earth libration point L_2 for the placement of large formations of spacecraft [2, 3]. They derive natural regions where the geometry of the formation is maintained. Further analyses considered natural and non-natural arcs for formation applications [4]. Most of the formation flying missions have been considering spacecraft at a relatively small distance from the reference orbit. However, observatory and interferometry missions in space have been the motivation for the analysis of large formations, in particular on quasi-periodic trajectories on invariant tori.

In this study, the primary goal is to characterize the natural motion of spacecraft on a two-dimensional invariant torus and indicate properties of the motion that are beneficial for formation flying missions. A two-dimensional invariant torus can be described as a set of orbits that start on a surface and stay on that surface during the dynamical evolution. Within the class of natural motions on two-dimensional tori the focus is set on quasi-halo orbits enabling relative spacecraft configurations with large distances among satellites. With the aim of identifying these orbits, a numerical approach was developed to parametrize invariant tori and determine their frequency base. Formations on the surface of a torus are examined, and formation snapshots are introduced to explain their shape and orientation in space. The variation of the formation's geometry depends on the selection of the initial states on the torus, and therefore on the distribution of the spacecraft on the torus' surface. The appropriate orientation of the cutting plane comes from the linear subspaces of the Monodromy matrix. A sophisticated solution is derived from the parametrization of the torus. Quasi-periodic orbits in the vicinity of Lagrange point L_1 in the Sun-Earth system were studied in detail. This analysis details and expands the understanding of the natural dynamics and points out the advantages of the surface of a torus suitable for formation flight.

2. Dynamical Representation

The dynamical reference model used is the circular restricted three body problem (CR3BP). It assumes that the Earth travels around the Sun in a circular orbit, whereas the spacecraft is modelled as a massless particle moving under the gravitational forces of the Sun (primary body) and the Earth (secondary body). In the CR3BP, the motion of the spacecraft is described in terms of rotating coordinates relative to the barycentre of the system primaries. In this frame, the rotating x-axis is directed from the primary to the secondary body. The non-linear equations of motion are written as

$$\begin{aligned} \ddot{x} &= 2\dot{y} + \Omega_x + u_x \\ \ddot{y} &= -2\dot{x} + \Omega_y + u_y \\ \ddot{z} &= \Omega_z + u_z \end{aligned} \tag{1}$$

$$\Omega = \frac{1}{2}(x^2+y^2) + \frac{1-\mu}{r_1} + \frac{\mu}{r_2} + \frac{1}{2}\mu(1-\mu) \tag{2}$$

where r_1 represents the distance from the spacecraft to the larger primary, and r_2 to the larger primary. The non-dimensional time is the inverse of the mean motion of the primaries, t^*. The characteristic length l^* is the distance between the two primaries and the characteristic mass m^* is the total mass of the system, see Eq. 3.

$$
\begin{aligned}
l^* &= 1AU \\
t^* &= \left(\frac{l^{*3}}{Gm^*}\right)^{-1/2} \\
m^* &= m_{Sun} + m_{Earth/Moon}
\end{aligned}
\tag{3}
$$

This study focus on the motion around the first Sun/Earth libration point assuming that the secondary body is a single point mass combining the Earth/Moon two-body system. Five equilibrium points named $(L_1 - L_5)$ exist in the CR3BP. The value for the mass ratio is $\mu = 3.0406e-06$. In the vicinity of each equilibrium point subspaces can be identified via linearisation of the equations of motion about the libration points. The subspaces are spanned by eigenvectors of the Monodromy matrix, which is a state transition matrix (STM) for a periodic orbit, mapping the initial state vector to the final state vector after one period. An independent basis B is established with

$$
B = (e_s, e_u, e_{r1}, e_{r2}, e_5, e_6)^T \tag{4}
$$

where $span(e_{r1}, e_{r2})$ is a periodic subspace, spanning the plane of the imaginary eigenvectors, e_s correspond to the stable direction and e_u to the unstable direction. The periodic subspaces build the centre manifold, which is associated with families of periodic orbits, whereas many of these periodic orbits also possess centre components that correspond to quasi-periodic motions.

3. Quasi-Periodic Objects

The equations of motion Eq. 1 in the CR3BP can be considered generically as an autonomous ordinary differential equation, assuming that those equations possess a quasi-periodic orbit as a solution that reside on an invariant tori about the corresponding periodic orbit. In other words the closure of the quasi-periodic orbit is an invariant torus. It is often preferable to regard the torus directly as an invariant object, independent of a particular trajectory on its surface, see Fig. 1 (right) [6]. The propagation of multiple trajectories densely fill and describe the surface of the invariant torus. Fig. 1 (right) represents the surface to illustrate the concept.

3.1. Quasi-Periodic Motion on Invariant Tori

The objective is to gain a better understanding of the motion of a spacecraft on a torus. This knowledge of the natural flow is very useful for trajectory and formation design. Quasi-periodic solutions of a non-linear system are described as the motion on a p-dimensional torus that is associated with p different internal frequencies. All trajectories of this flow are quasi-periodic

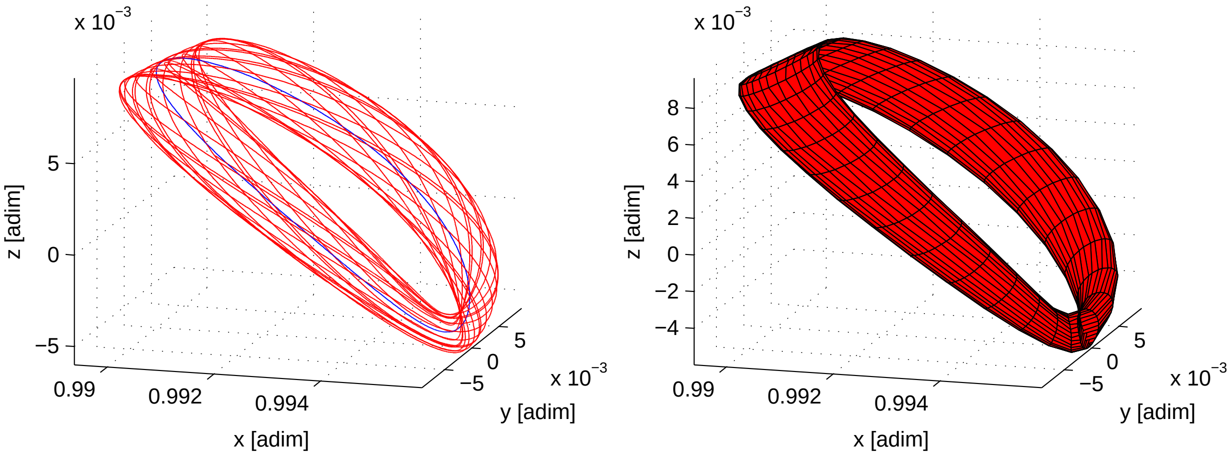


Figure 1. Periodic (left, blue) versus quasi-periodic (left, red) orbit. Invariant object (right), surface densely filled by quasi-periodic trajectory.

functions of time and their properties strongly depend on arithmetical properties of the frequency base. In case of a 2-dimensional torus (2-torus) the frequency base has two entries.

$$\omega = \{\omega_1, \omega_2\} \tag{5}$$

The parallel flow on an invariant torus with the frequency base ω is non-resonant, if the basic frequencies are rationally independent (no non-trivial linear combination with integers is equal to zero). In this case each trajectory densely covers the torus' surface. In the other case, when the frequencies are rationally dependent (integer k exist to solve the Eq. 6) the torus is called resonant.

$$k_1\omega_1 + k_2\omega_2 = 0 \tag{6}$$

The motion is directly linked to the frequency base of the torus, and can be described by a particle that is longitudinally moving about the torus structure with the frequency ω_1, while rotating with the frequency ω_2. The motion is visually described in Fig. 2. The time needed for one rotation is

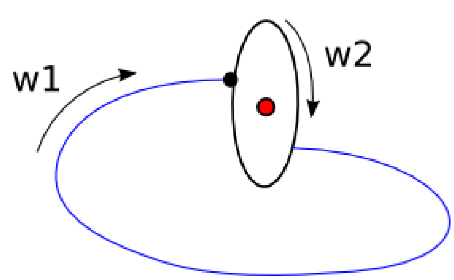


Figure 2. Visualization of the motion on a torus with a two-dimensional frequency base. The black circle represents a cross section of the torus.

$$T_i = \frac{2\pi}{\omega_i} \tag{7}$$

The rotation number of a torus is defined as

$$\rho = 2\pi \frac{\omega_1}{\omega_2} \tag{8}$$

which uniquely defines a torus with a given energy. The rotation number represents the average movement in the ω_2 direction when one revolution is done in ω_1 direction.

3.2. Obtaining Quasi-Periodic Solutions

A convenient way to obtain quasi-periodic solutions is to use a numerical approach which is based on a Poincaré section. For a description of this method, see [5]. The method uses a reduction of the original system, where periodic orbits are represented by equilibrium points, while quasi-periodic orbits are represented by closed curves. In other words this method reduces the calculation of quasi-periodic orbits to a search for periodic orbits that return to a closed curve on a section plane. Those curves are modelled by truncated Fourier coefficients in position and velocity space.

3.3. Families of Quasi-Periodic Invariant Tori

The four-dimensional centre manifold around L_1 is occupied by quasi-periodic orbits of two different families, the Lissajous family around vertical Lyapunov orbits, and the two-dimensional invariant tori around halo orbits. In other words, periodic orbits in the libration point regions are surrounded by a variety of quasi-periodic orbits. The focus in this paper is set on quasi-periodic orbits that have an underlying halo orbit. The numerical methods outlined in the previous section allow to compute individual trajectories and the corresponding invariant tori. One well-known type of periodic

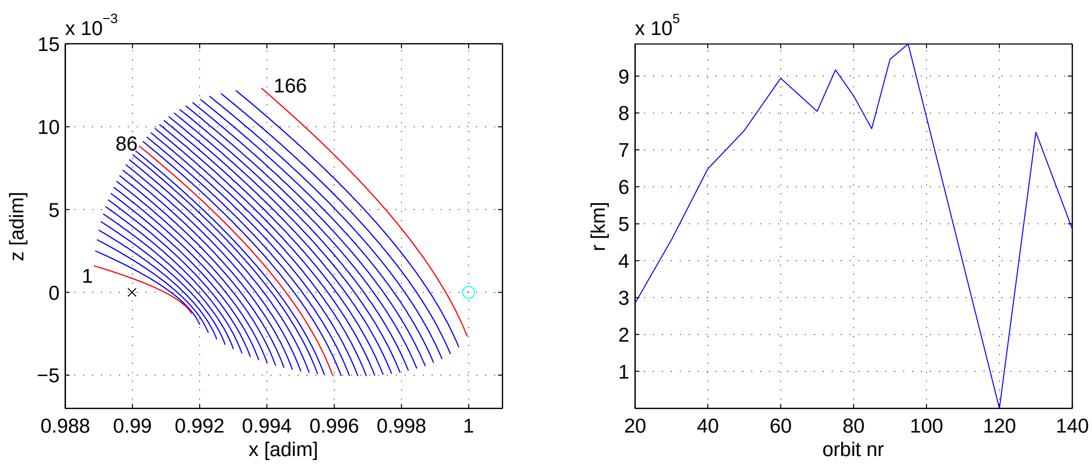


Figure 3. Northern halo family around L_1 in synodic reference frame (left). Maximal size of tori around halo orbits (right).

trajectory is the set of halo orbit families that are symmetric across the xz rotating plane. These orbits serve as periodic trajectories for the calculation of families of invariant tori. The northern halo orbit family near the L_1 Sun-Earth libration point appears in Fig. 3. The circle indicates the position of the Earth, and the cross the libration point L_1. By applying this method to a wide range of periodic halo orbits of the Northern L_1 family, the family of invariant tori are computed. Tab. 1 shows the orbital period, and size parameters for a small, medium and large halo orbit within the family, those orbits are highlighted in red in Fig. 3.

Halo orbit	orb #	T_{period} [adim]	A_x [adim]	A_z [adim]
small	1	3.058	0.00145	0.00144
medium	86	2.945	0.00288	0.00692
large	166	2.193	0.00306	0.00749
Torus	A	ω_1	ω_2	ρ
small	5.858e-07	2.120	1.554	8.574
medium	2.535e-06	2.126	1.559	8.568
large	9.243e-06	2.087	1.538	8.529

Table 1. Properties of three halo orbits within the L_1 family (top). Frequencies and size of tori shown in Fig. 4 (bottom).

The continuation parameter for the family is the area that is confined by the invariant curve. This parameter indicates the maximal size of existing tori. Later, the maximal mean distance for spacecraft formations placed on the torus can be derived. The size is plotted as a function of the orbit number within the halo family in Fig. 3 (right). The area under the curve in Fig. 3 (right) show possible geometries. The existence of tori strongly depends on the periodic orbit. For orbit number 120, the continuation procedure fails, due to a resonance.

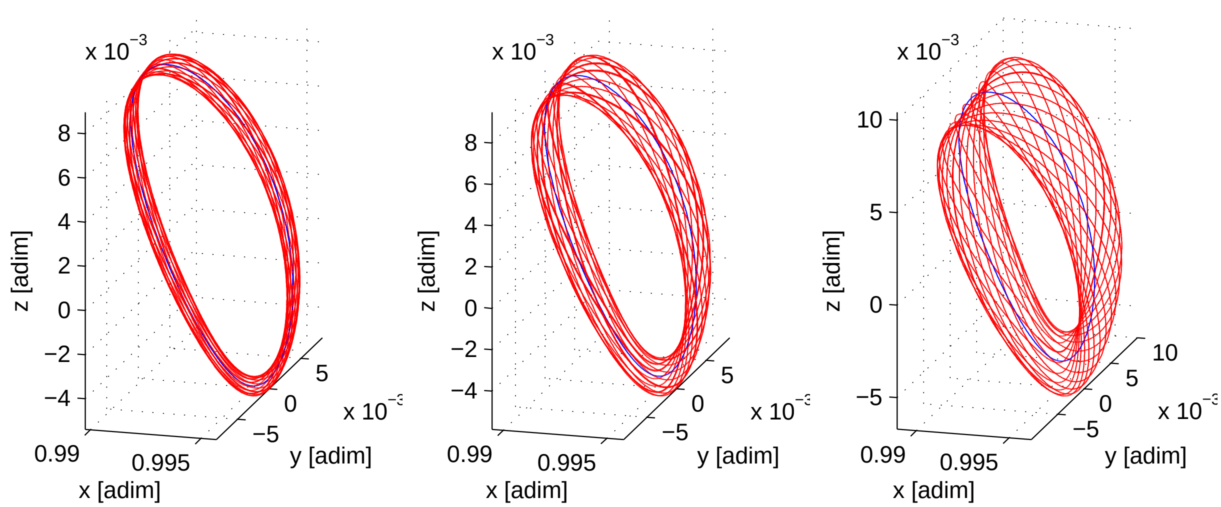


Figure 4. Trajectories in the synodic reference frame emanating around three different sized tori.

The arithmetical properties of the frequency base defines if the torus is densely fill by the quasi-periodic trajectory or if a resonance within the motion exists. The torus is not filled by the trajectory and the Poincaré section plane is not described by a curve, but a few discrete points. Three invariant tori from the family around the periodic orbit number 80 out of the halo orbits are shown in Fig. 4. The size, frequency base and cross section for all three tori are put together in the bottom part of Tab. 1. The frequency ω_1 and the fraction $\frac{\omega_1}{\omega_2}$ are plotted for the family of tori in Fig. 5. For $\frac{\omega_1}{\omega_2} = 1.5$ a resonant and near resonant are highlighted and the corresponding trajectories on the tori are plotted in Fig. 6.

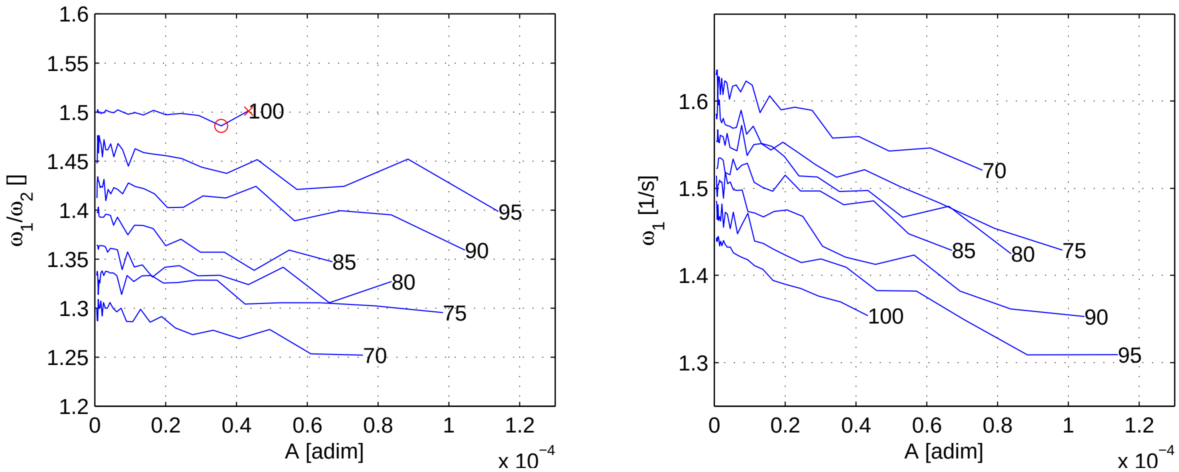


Figure 5. Relation between ω_1 and ω_2 (left) and ω_1 (right) plotted as a function of the continuation parameter of the torus family A for several periodic orbits (numbered).

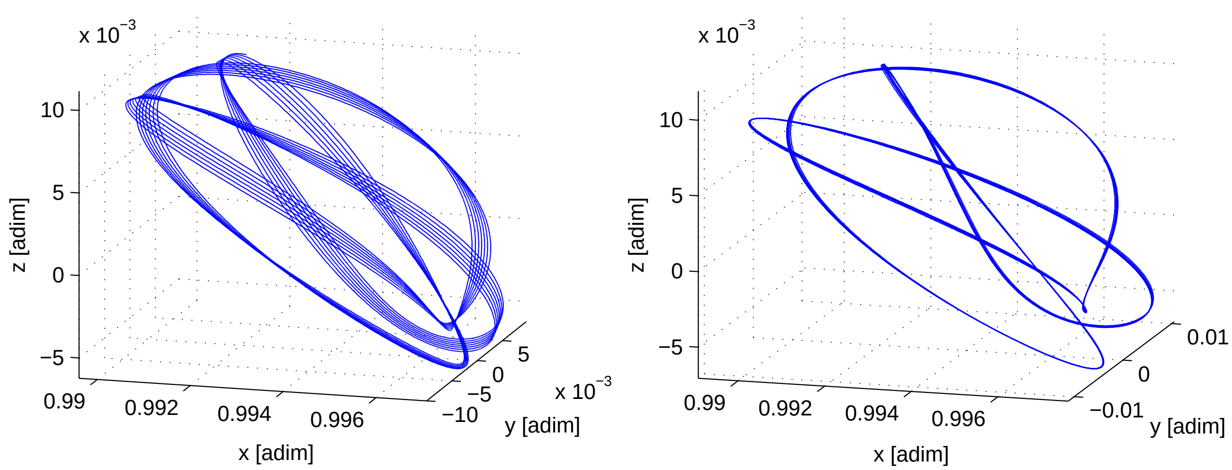


Figure 6. Resonance trajectories on the surface of a torus. Near resonance (left), resonant (right).

4. Parametrization of an Invariant Torus

Several tools are available to study the properties of fixed points or periodic orbits, such as Poincaré maps. The study of quasi-periodic motions in particular on an invariant torus is difficult and no numerical tools are available. One issue is to get a parametrization of the torus for further studies. In order to find a parametrization for a torus, a point on the surface of the torus is described by two angular coordinates. A transformation from Cartesian coordinates to torus coordinates (θ_1, θ_2, u) is introduced. The aim of our approach is a numerical approximation of the parametrization of the original system starting from an already calculated torus. Any quasi-periodic solution $x(t)$ can be written as

$$x(t) = u(\omega t) \tag{9}$$

where u is a torus function, whereas ω are the basic frequencies of a solution $x(t)$ as introduced in Eq. 5. This transformation maps the torus in an area $[0, 2\pi)^2$ in the parameter space. After the transformation in torus coordinates an equation to determine u is required, which is a solution of the invariance condition.

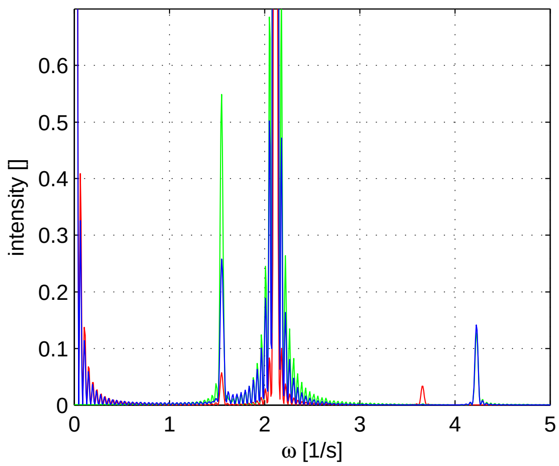
$$f(u, \theta) = \sum_{i=1}^{m} \Omega_i \frac{\partial u}{\partial \theta_i} + \sum_{i=m+1}^{p} \omega_i \frac{\partial u}{\partial \theta_i} \tag{10}$$

An approach to obtain the parametrization in torus coordinates is presented in the following, together with tools derived from the parametrization.

4.1. Numerical Approach from a Trajectory to the Torus Function

A robust method to compute trajectories that ly on an invariant torus is already available, the presented method investigates the trajectory and derives the torus parametrization. The system frequencies are determined from trajectories by the means of Fourier spectral methods. Fig. 7 (left) shows the results of the Fourier analysis, using a quasi-periodic trajectory evolving on a torus for $t = 30$ (dimensionless time units). The two distinctive peaks in the spectrum correspond to the system frequencies introduced in Eq. 5. The highest peak is longitudinal frequency ω_1, whereas the smaller peak at $1.5\frac{1}{s}$ correspond to the rotational frequency ω_2. The peaks above $3\frac{1}{s}$ are the double frequencies.

With the knowledge of the two torus frequencies the parametrization can be obtained. Assuming a discretization of the two-dimensional domain of 30 elements in the direction of θ_1 and θ_2, this leads to 30^2 discretization points where the function u is evaluated. The trajectory is scanned and u is obtained at appropriate time steps, where t is



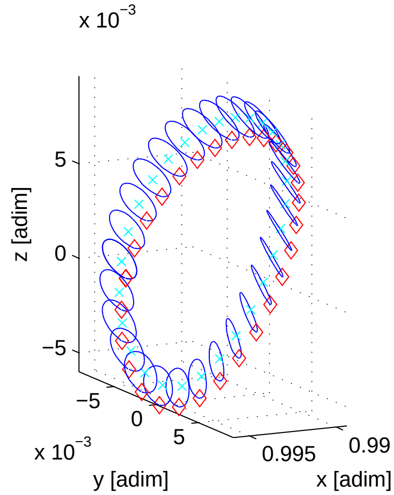


Figure 7. Spectrum from the Fourier analysis (left). Invariant circles along torus, obtained within the the parametrization process (right).

$$t = i\frac{2\pi}{\omega_1} + j * t_{step} \tag{11}$$

with

$$t_{step} = \frac{2\pi}{\omega_1}/30. \tag{12}$$

The results are invariant curves representing cross sections of the torus, they are visualized in Fig. 7 (right). The cross defines the centre of the curves, whereas the the red markers highlight the zero direction from where the angle θ_1 is measured. An indication for wrong system frequencies is if the points on the invariant curves are wide spread and no structure is visible. The value u for the corresponding values of θ_1 and θ_2 are expressed by trigonometric polynomials

$$u(\theta) = \frac{a_0}{\sqrt{2}} + \sum_{n=1}^{n_{max}} (a_n cos(n\theta) + b_n sin(n\theta)) \tag{13}$$

4.2. Properties of the Parametrization

The parametrization of a torus is a tool to study the quasi-periodic motion from a dynamical system perspective. The parametrization provides the entire set of trajectories on the torus. Orbits on the neighbourhood of others can be easily computed. The parametrization reveals invariant curves at equidistant time steps on the torus suitable for formations, see Fig. 7. The blue circles are invariant curves, the zero directions for the angle θ_2 are indicated by red markers. A torus mesh is easily created from the parametrization enabling distance evaluations between points on the torus and the

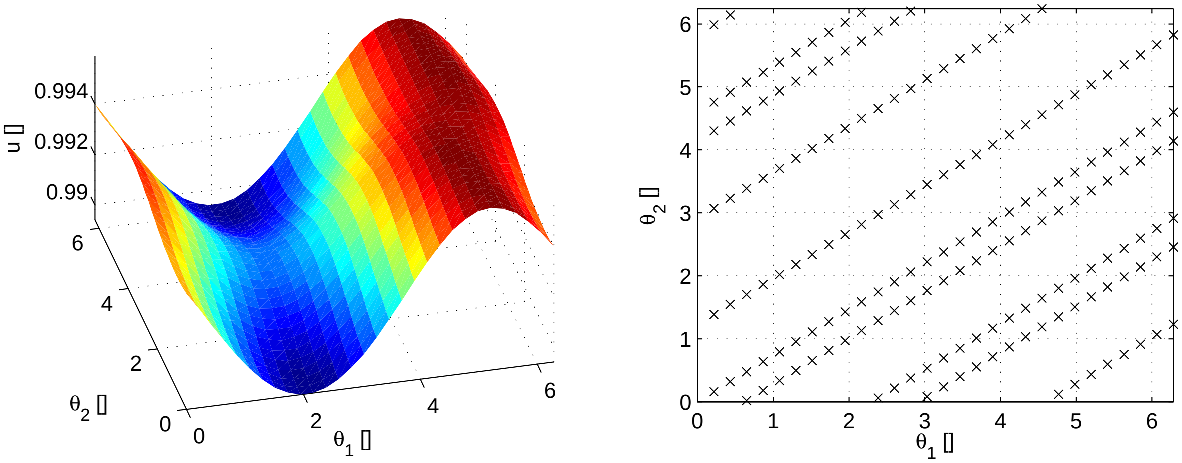


Figure 8. Torus function u plotted in two-dimensional angular phase space (left). Characteristic represent an orbit on the torus surface (right).

nearby space. The x-component of the torus function is plotted in angular phase space in Fig. 8 (left). A linear characteristic in angular phase space is the map of a quasi-periodic trajectories on the torus. Formation distance and phasing consideration can be extracted from the pattern of the characteristics in Fig. 8 (right).

5. Relative Spacecraft Configurations on Invariant Tori

After gaining a fundamental understanding of the motion associated with invariant tori, aspects of formation flying can be studied. The surface of a torus defines a natural region suitable for spacecraft formations. In this analysis the behaviour of spacecraft placed on the surface is explored. A quasi-periodic trajectory is shown in Fig. 9 (grey). The initial states for a spacecraft formation are defined at the intersection of a cutting plane with the torus. The evolution of the formation is indicated by red points at four moments in time.

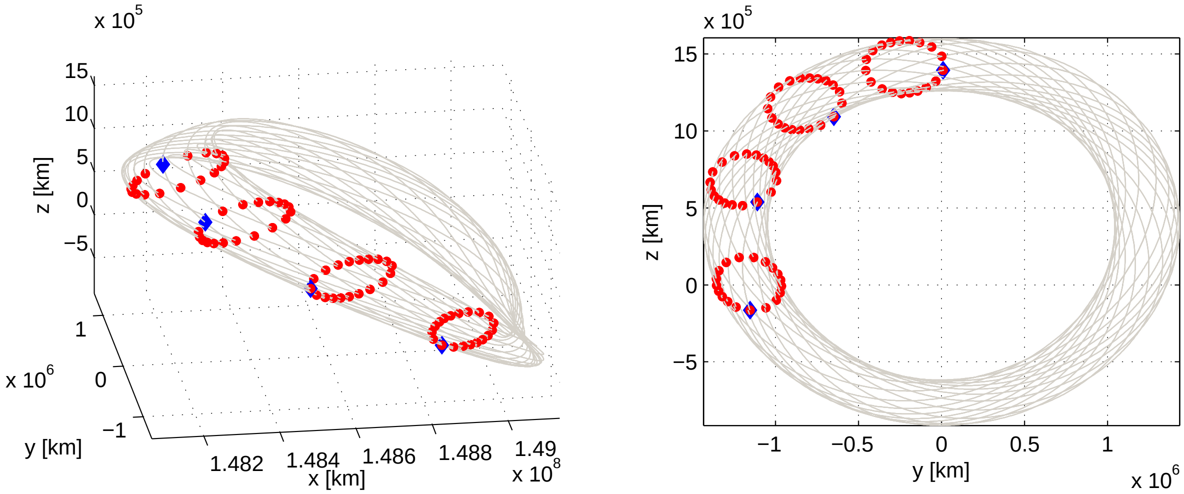


Figure 9. Quasi-periodic trajectory shown in the synodic reference frame (grey lines). Four snapshots of the formation in time (red dots).

Initially, the set of position vectors describe a planar curve, The motion in the yz-plane as seen from Earth is almost circular and bounded as time proceeds, see Fig. 9 (right). As the spacecraft proceed in time, they move longitudinally along the underlying halo orbit, and describe latitudinally a winding motion. These components are significant aspects of the natural motion and effect the evolution of the curve in time. The curve's shape contracts and expands, and the orientation of the plane in space changes as the trajectories are propagated forward in time. Phasing conditions are introduced and the evolution of the formation is studied in the following.

5.1. Introduction of an In-plane Coordinate Reference Frame

Connecting the spacecraft within the formation at arbitrary time will create a curve that represents a snapshot of the formation. Those curves represent the orientation and shape of the formation in time. In the left plots of Fig. 10 the shape and orientation of the formation is shown in the rotating reference frame with an moving origin at the periodic centre orbit. The colour in the plots correspond to the time, the time step between the formation snapshots is $t = 0.05$, dark blue equals $t = 0$, and orange to $t = 3.014$. The difference between the upper and lower plots are the orientation of the cutting plane, where the formations is initially placed. In Fig. 9 the blue points indicate this defined zero direction for each snapshot. The evolution of the curve, and therefore of the formation, in this reference frame over one orbital revolution can be characterized by in-plane and out-of-plane behaviour. The out-of-plane parameters describe the orientation of the formation, whereas the shape

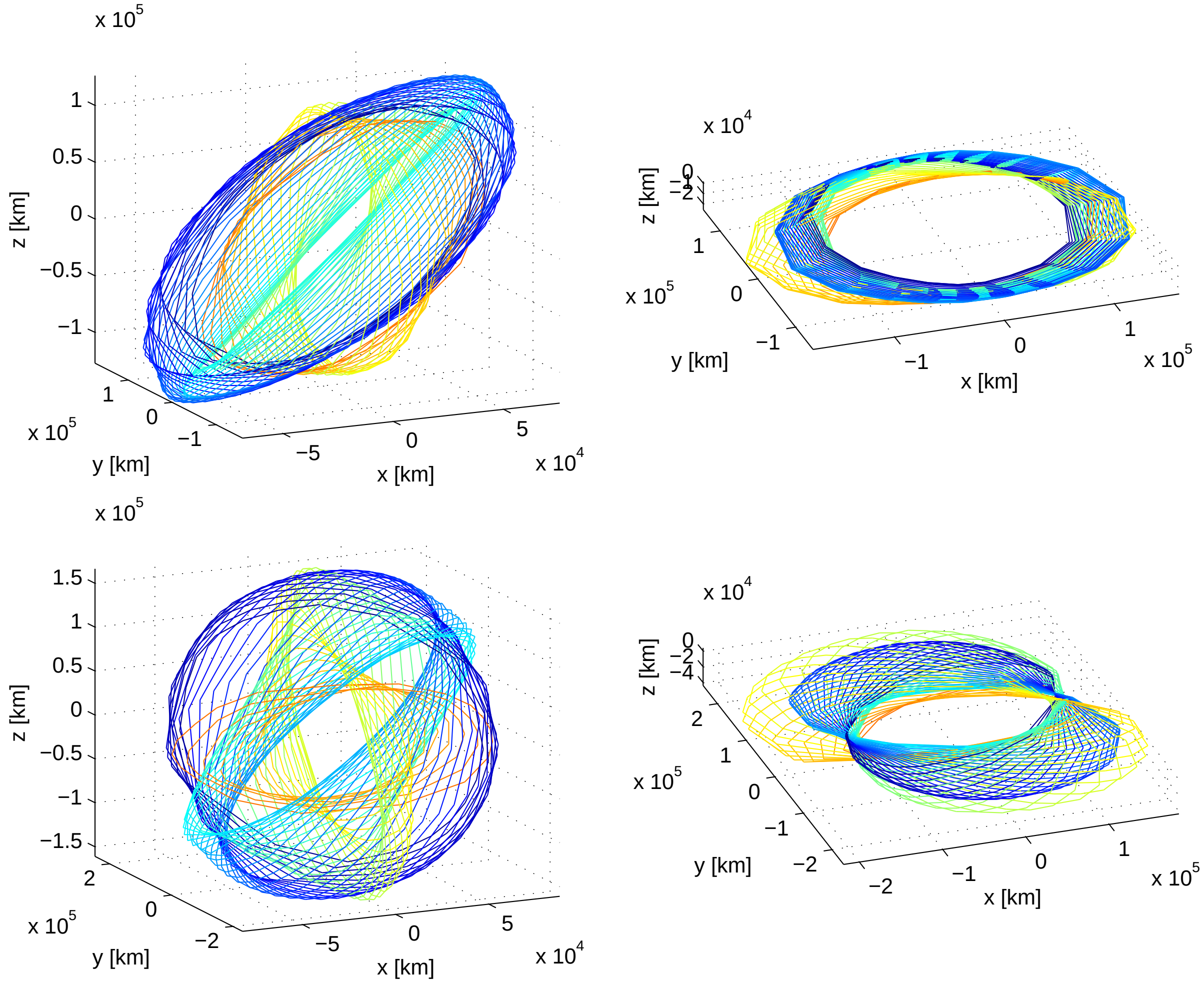


Figure 10. Formation snapshots in the rotating reference frame with a local origin (left), and in the in-plane reference frame (right).

is described by the in-plane parameters. A co-planar reference frame is introduced that lies within the formation plane with its origin at the underlying periodic orbit. The x-direction is arbitrary chosen and defines a zero angle direction. Fig. 10 (right) show the transformation of the formations snapshots into the newly defined in-plane coordinate frame.

In this analysis the assumption holds that the formation snapshots in the in-plane coordinate frame are planar. The orientation of the normal vector with respect to the rotating reference frame is expressed in two angles, the declination and right ascension, see Fig. 11 (right). With these angles the plane stability can be defined and the orientation and shifting rate of the formation plane is given. A transformation matrix between the synodic and the coplanar reference frame is derived from those angles. The shape of the formation snapshots, and therefore of the formation is evaluated by the distance between the spacecraft.

The distance between three spacecraft within the formation are shown in Fig. 12 (right). The paths are plotted for approximately $t = 9$ dimensionless time units, which is equivalent to three revolutions

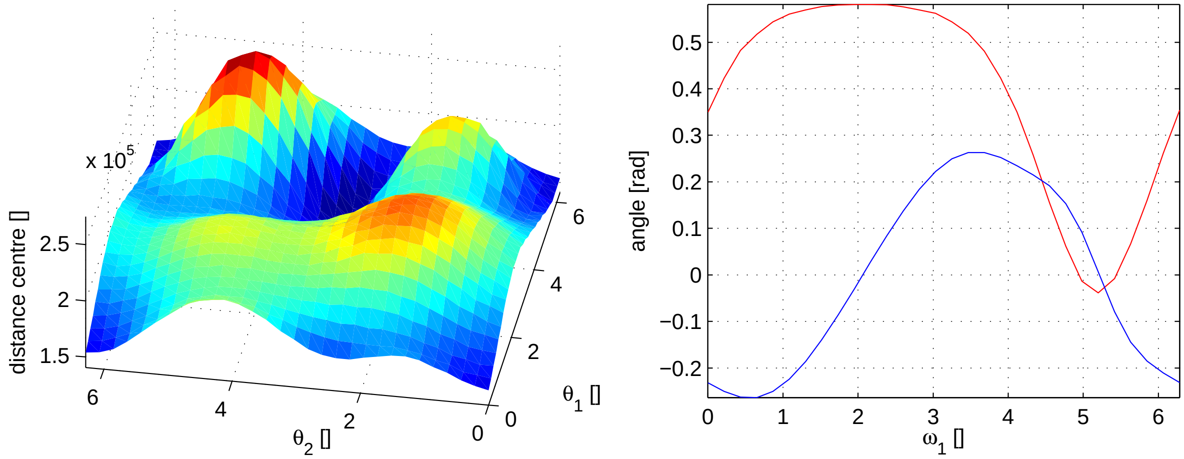


Figure 11. In- and out-of-plane parameter describing the formation snapshots. Distance to the centre orbit (left), orientation of the formation plane defined by declination (blue, right) and right ascension (red, right).

longitudinally along the torus.

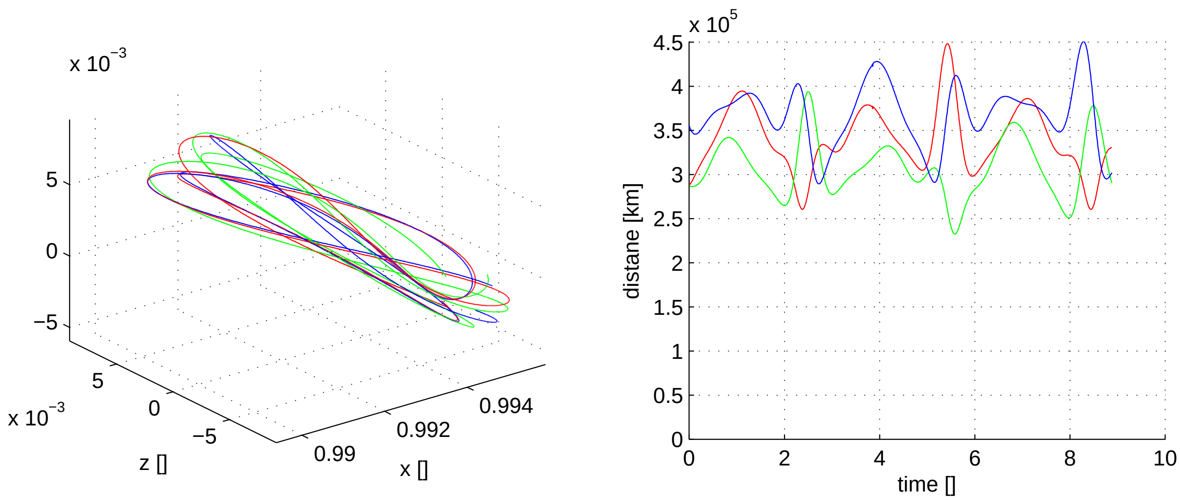


Figure 12. Trajectories of three spacecraft within the formation (left). Distance between the spacecraft (right).

5.2. Phasing Multiple Spacecraft

The variation in the distance between the spacecraft and the orientation in space depends on the phasing/initial placement of the spacecraft on the torus' surface, as seen in Fig. 10. It is important that the spacecraft keep their relative distance within the formation and that the orientation is bounded. Depending on the initial set of state vectors, it is possible for the formation to evolve along the surface of the torus such that the relative positions of each spacecraft in the formation are unaltered and the relative distances are closely bounded.

A linear approximation for this phasing problem is derived from the the Monodromy matrix and their periodic subspaces. Their eigenvalues and the associated eigenvectors indicate the linear stability of the halo orbit and characterize the nearby motion. Specially, the two-dimensional subspace spanned by the two complex eigenvectors define the plane for the initial states of the formation that stay bounded in the linear system. An exact solution is based on the torus parametrization and their invariant curves. The planar approximation of the invariant curves in Fig. 7 (right) coincide with the subspaces described above. Phasing conditions are pointed out, and natural regions are identified for formations on the torus where the variations of the distance and orientation between spacecraft are minimal.

5.3. Modified Torus Motion

In the previous, a natural solution for multiple spacecraft moving in a relative configuration within this dynamical system on a torus structure is presented. No manoeuvres are necessary and all spacecraft proceed naturally and along their paths and their motion is constrained within limits. The type of natural motion is an option for formation flying. However, some missions may requiring a tight pre-specified formation forcing a specified configurations.

The knowledge of the natural motion on the torus, in particular the frequencies and the plane orientation from the parametrization, is used to propose a modified torus motion in the in-plane coordinate frame as following

$$\begin{aligned} x &= r \cdot sin(\omega_2 t + \phi_0) \\ y &= r \cdot cos(\omega_2 t + \phi_0) \\ z &= 0 \end{aligned} \tag{14}$$

where ω_2 is the rotating frequency introduced for the torus motion, describing the winding motion component. The radius of the desired formation is defined by r, whereas the phasing ϕ varies between the spacecraft in the formation. Formulation the equations of motion in the inertial reference the following transformation is required

$$\begin{aligned} r_i &= R(t) \cdot r_{ip} \\ v_i &= R(t) \cdot v_{ip} + \omega \times r_i \\ a_i &= R(t) \cdot a_{ip} + 2\omega \times (R(t) \cdot v_{ip}) + \omega \times (\omega \times r_i) + \frac{d\omega}{dt} \times r_i \end{aligned} \tag{15}$$

$$[\omega]_\times = \frac{dR}{dt} R^T = \begin{bmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{bmatrix} \tag{16}$$

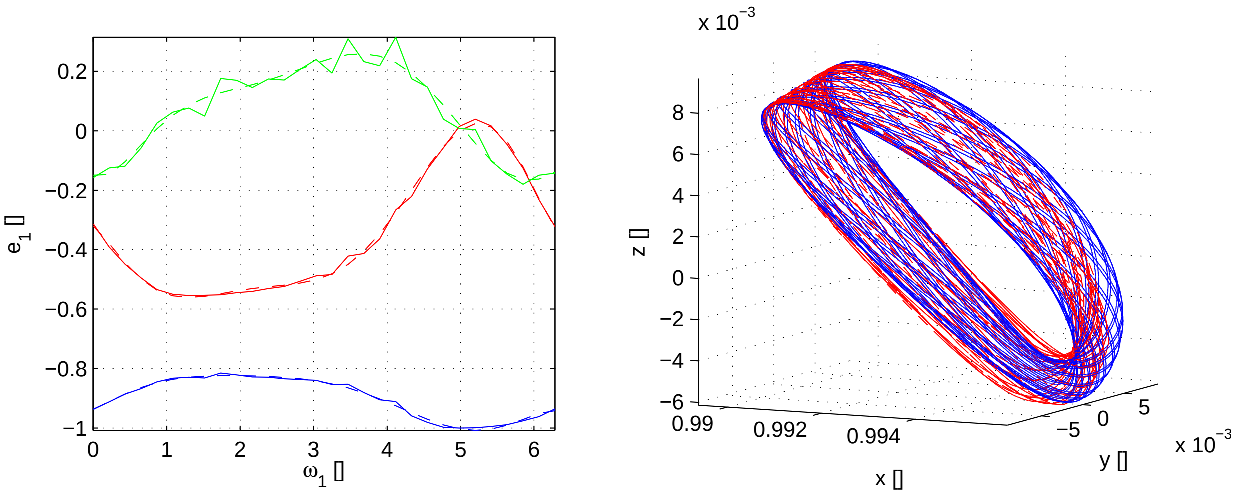


Figure 13. Vector components of e_1 of the rotation matrix (left). Natural trajectory on torus versus modified motion described by Eq. 15 (right).

where ω is the angular velocity vector, and $R(t)$ the time-invariant rotation matrix. The derivatives of the angular velocity vector, the rotational matrix are required to solve the equation. They are obtained numerically in this study. Furthermore, an interpolation of the rotation matrix at t is mandatory, and the Gram Schmidt procedure applied to assure orthogonality.

$$R(t) = [e_1(t), e_2(t), e_3(t)] \tag{17}$$

where e_1 is the unit vector pointing in the zero direction from the torus parametrization, e_2 the normal vector to the formation plane, and e_3 complements the system. The components e_1 is shown in Fig. 13 (left). The variations between the torus motion and the motion introduced in this section depend on the set of parameters in Eq. 14 and Eq. 15, as seen in Fig. 13 (right). Forcing a specified configuration it becomes necessary to insert manoeuvres to maintain the formation.

6. Summary and Concluding Remarks

In the present study, natural quasi-periodic trajectories near libration point regions evolving on quasi-periodic invariant tori were studied. Quasi-periodic trajectories were computed using a numerical method, starting around periodic halo orbits in the CR3BP. A numerical method to obtain the torus frequency base is proposed. The parametrization in the two-dimensional angular phase space of the torus' surface is introduced which enables to further study the motion with tools known from the dynamical system theory. By introducing a certain phasing of spacecraft on the torus, trajectories that are particularly suitable for satellite formations emerge. The relative positions of each spacecraft in the formation are unaltered and the relative distances and orientation are closely bounded. An in-plane coordinate frame is introduced and the inner and outer behaviour is investigated, with the aim to introduce an analytical model for the torus motion. Another field for future work are the implementation of manoeuvres to find connecting orbits between different sized tori.

Acknowledgements

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7. References

[1] BT Barden and KC Howell. Fundamental motions near collinear libration points and their transitions. *Journal of the Astronautical Sciences*, 46(4):361–378, 1998.

[2] A Héritier and KC Howell. Natural regions near the Sun-Earth libration points suitable for space observations with large formations. *Knowledge Creation Diffusion Utilization*, pages 1–18, 2011.

[3] A Héritier and KC Howell. Regions Near the Libration Points Suitable to Maintain Multiple Spacecraft. pages 1–20, 2012.

[4] KC Howell. Formations near the libration points: Design strategies using natural and non-natural arcs. *of GSFC 2nd International Symposium on*, pages 1–25, 2004.

[5] Egemen Kolemen, N. Jeremy Kasdin, and Pini Gurfil. Multiple Poincaré sections method for finding the quasiperiodic orbits of the restricted three body problem. *Celestial Mechanics and Dynamical Astronomy*, October 2011.

[6] ZP Olikara and KC Howell. Computation of Quasi-Periodic Invariant Tori in the Restricted Three-Body Problem. *Paper No. AAS*, pages 1–15.

[7] ZP Olikara and DJ Scheeres. Computing families of quasi-periodic tori in autonomous hamiltonian systems. *Proc. of the seventh EUROMECH*, (July):24–29, 2011.

[8] Frank Schilder, Hinke M Osinga, and Werner Vogt. Continuation of Quasiperiodic Invariant Tori. (October), 2004.