Unified Formulation for Element-Based Indirect Trajectory Optimization

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One of the greatest challenges when designing optimal trajectories using the indirect method is the treatment of the costate. These variables, introduced by the two point boundary value problem (TPBVP) derived from the first-order necessary optimality conditions and linked to their corresponding state variables, lack a physical interpretation in general. This complicates the task of finding an initial guess for the solution of the TPBVP, required if it is to be solved using iterative numerical algorithms. Furthermore, the introduction of the costate into the state equations through the control term increases their complexity and makes it more difficult to derive variational equations for the accurate integration of the state transition matrix, essential to increase the performance of numerical shooting methods.
When dynamics are formulated using Cartesian coordinates, these issues are alleviated by the well-known result of the primer vector [1]. By applying the Pontryagin Maximum Principle (PMP), it is possible to verify that the direction of thrust is given by a vector whose components are the costate variables associated to velocity. This leads to a fairly simple expression of the control involving only the costate, while also providing some physical sense for at least part of the costate. However, there are practical cases where formulations for dynamics other than Cartesian may be more convenient. For instance, using orbital elements allows to express constraints in a natural way as fixed initial and final values of the state when moving from one orbit to another without specifying the specific departure and arrival points. Unfortunately, the primer vector result is not preserved for a general formulation of dynamics, and thrust orientation becomes a function of both state and costate. Nevertheless, by carefully manipulating the equations it is possible to write them in a more convenient way, partially isolating the different contributions of state and costate.
This paper proposes a unified, efficient formulation of the indirect equations for trajectory optimization with general dynamical models in the form of Equation (1), where $\mathbf{x}$ is the state (excluding mass), t is time (in general, the independent variable), m is mass, $\mathbf{f}$ is thrust orientation, and T is thrust magnitude. Applying the PMP to get the optimal thrust orientation and substituting it back into the equations, it is possible to identify a quadratic form of the costate (excluding the term related to mass). The matrix for this quadratic form only depends on the state, and is obtained by multiplying matrix $\mathbf{B}$ by its transpose. Note that this new matrix is, by construction, square (with the same size as $\mathbf{x}$), symmetric and (semi)definite positive. Moreover, it is possible to verify that the contribution of the costate in all equations (including the variational ones) is reduced to this quadratic form and its derivatives, whose calculation is simplified by the separation of the state and costate contributions in the matrix and vector of the quadratic form respectively. This allows to develop an efficient generic framework for any formulation in the form of Equation (1), where only matrix $\mathbf{BB}^{\mathrm{T}}$ and its derivatives will depend on the chosen formulation. On the other hand, this expressions also invite to consider possible geometrical interpretations. The effect of thrust on each component of the state equations is given as the projection of the costate over a space defined by the state through metric tensor $\mathbf{BB}^{\mathrm{T}}$, the primer vector being now included as particular case with constant metric tensor. This interesting mathematical structure is explored for possible practical applications.

$$\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t}=\boldsymbol{A}(\boldsymbol{x},t)+\frac{T}{m}\boldsymbol{B}(\boldsymbol{x},t)\boldsymbol{f} \qquad (1)$$

References

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