On the Equilibrium Points of Doubly Synchronous Binary Asteroid Systems

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Doubly synchronous state is an end state of the binary asteroid systems where the tidal dissipation is balanced by the BYORP effect. A fair amount of doubly synchronous binary asteroids (DSBA) is already discovered among the near-Earth binaries and the main belt binaries. With both the primarys and the secondarys spin periods same as the mutual orbital period, the modelling and dynamics of DSBAS are similar to those of the classical circular restricted three-body problem (CRTBP). Positions of equilibrium points in the DSBAs, along with their stability, have already been studied, either by assuming a sphere plus an ellipsoid or two ellipsoids truncated at 2^{nd} order potentials for the two bodies in the DSBAs.

In this contribution, similar studies have been carried out. We also assume two ellipsoids for the DSBAs. Different from previous works, the mutual potential between the two ellipsoids and the potential between the massless particle and the primary (and the secondary) are truncated at the 4th order [8]. As a further investigation, the closed form for the potential of ellipsoid is also taken into account. First, equations of motion (EOM) for the massless particle in the synodic frame of the DBSA are given. Then, positions of planar equilibrium points are obtained by finding the intersection points among the curves $\partial U/\partial x = 0$, $\partial U/\partial y = 0$, and $\partial U/\partial z = 0$, where U is the potential function in the synodic frame. Last, stability of the equilibrium points is studied by analysing the Hessen matrix of the equilibrium points. The purpose of the current study is to show the role of higher order terms in the potential when the primary and the secondary are highly irregular and close to each other in the DSBAs. In addition we conclude that the spherical harmonic gravitational potential is inaccurate for studying dynamics near the irregular primaries and may thus produce unreal information about the equilibrium points in the vicinity of primaries.

Key Words: Doubly synchronous binary asteroid, Equilibrium point, Stability

1. Introduction

Binary asteroids constitute a large fraction of the asteroid populations. Studies have shown that binaries account for about 15% among the near-Earth asteroids (NEAs) larger than 300 m,[18)] and the same fraction can be expected in the main belt asteroids (MBAs).[15)] Many work has been done in about the dynamics of the binary system that is referred as the Full Two Body Problem (F2BP),[4,9,11,14,17,19,22)] among which a lot focus on the relative equilibria of the binary system, which is the dynamical state for the doubly synchronous binary asteroids. This type of binary asteroids has their respective rotational periods synchronised with the orbital period[15)] and rests in relative equilibrium state.

Dynamics of particles in the F2BP, referred as the Restricted Full Three Body Problem (RFTBP), also receives great interest in the literature. This problem is more complex than the common Restricted Three Body Problem (RTBP) in that at least one of the primaries should be considered as non-spherical and finite in size. Positions of equilibrium points in the DSBAs, along with their stability, have already been studied, either by assuming a sphere plus an ellipsoid[4,5,10,23)] or two ellipsoids truncated at 2^{nd} order potentials[19,24,28,30)] for the two bodies in the DSBAs. For example, Ref. 28) studied the equilibrium points for systems with irregular-shaped yet inertial symmetric primaries and investigate the effect of the asphericity on the locations of the equilibrium points. Ref. 10) studied the equilibrium points and corresponding stability properties of a system comprised of an ellipsoid and sphere in short-axis equilibrium configuration using the closed form for their potentials, then later extended to the long-axis equilibrium configuration by the work of Ref. 3) and Ref. 5). Recently study on the equilibrium points of the double ellipsoid system with similar approach to Ref. 3) has also been done by Ref. 23) but is limited to two examples of binary asteroids.

In this study we first revisit the approach of spherical harmonic gravitational potential for finding equilibrium points in the RFTBP. An innovative method of locating the equilibrium points in the system is devised and applied to both primaries with potentials truncated at 2nd or 4th d/o. Previous results in the literature for the RFTBP are recovered with even more equilibrium points found. Then with the same method the equilibrium points for the double ellipsoid system considering potentials of ellipsoids in closed form are obtained. However, only five equilibrium points, which are similar to those in the RTBP, possibly exist in this scenario while others reported in the literature disappear. Detailed analysis indicates that the mismodelling of the potentials of the primaries by previous work is the main cause of the practical non-existence of the equilibrium points found in the literature.

The rest of the paper is organised as follows. Section 2 introduces the dynamical model for the RFTBP, the new method to identify equilibrium points, and the computation of their stability property. Equilibrium points for different primaries considering spherical harmonic potential truncated at 2nd/4th d/o are given in Section 3. Section 4 is devoted to the systematic analysis of the locations and stability properties of equilibrium points obtained by using the closed form of potentials for double ellipsoid system. Section 5 concludes the paper.

2. Dynamical model

The primaries in the RFTBP is modelled as two triaxial ellipsoids, $\mathcal{A}$ and $\mathcal{B}$, with their respective spin axes parallel to the normal direction of their mutual orbit. As a result, the motion of the system is restricted to the orbital plane. The semi-axes and mass of $\mathcal{A}$ and $\mathcal{B}$ are

$$a_A, b_A, c_A, m_A; \quad a_B, b_B, c_B, m_B, \tag{1}$$

and the distance between $\mathcal{A}$ and $\mathcal{B}$ is a. The non-dimensional units of the system is taken as

$$[L] = a, \quad [M] = m_A + m_B, \quad [T] = \left([L]^3/G[M]\right)^{1/2}, \tag{2}$$

Therefore, with $\mu = m_B/(m_A + m_B)$ being the mass fraction of the system. The synodic reference frame $O - xyz$ rotating with the binary system is placed at the barycenter O of the system, as shown in Fig, with x axis connecting from the center of $\mathcal{A}$ O_A to that of $\mathcal{B}$ O_B, z axis aligned with the normal of the orbital plane, and y axis fulfilling the right-handed triads. $\boldsymbol{r}_A = (x_A, y, z)^T$ and $\boldsymbol{r}_B = (x_B, y, z)^T$ are the position vector of the particle at $\boldsymbol{r} = (x, y, z)^T$ relative to the center of $\mathcal{A}$ or $\mathcal{B}$, respectively, and $x_A = x + \mu$, $x_B = x - 1 + \mu$. $r_A = \sqrt{x_A^2 + y^2 + z^2}$ and $r_B = \sqrt{x_B^2 + y^2 + z^2}$. The equations of motion (EOMs) of the particle under the attraction of both primaries in this system is[20)]

$$\begin{cases} \ddot{x} - 2\omega\dot{y} = U_x \\ \ddot{y} + 2\omega\dot{x} = U_y \\ \ddot{z} = U_z \end{cases} \tag{3}$$

in which U_x,U_y and U_z are the partial derivative of the effective potential U with respect to x, y and z, and ω is the rotation rate of the system. The effective potential U is defined as

$$U = \frac{1}{2}\omega^2(x^2 + y^2) + V \tag{4}$$

in which V is the total potential of $\mathcal{A}$ and $\mathcal{B}$ at $\boldsymbol{r} = (x, y, z)$,i.e.,

$$V = (1 - \mu)V_A + \mu V_B, \tag{5}$$

and[13)]

$$\begin{aligned} V_* = &\frac{1}{r_*} - \sum_{l=2(2)}^{\infty} \frac{J_l^* \alpha_*^l}{r_*^{l+1}} P_l(\sin\phi_*) \\ &+ \sum_{l=2(2)}^{\infty} \sum_{m=2(2)}^{l} \frac{J_{lm}^* \alpha_*^l}{r_*^{l+1}} P_{lm}(\sin\phi_*) \cos m\lambda_* \end{aligned} \tag{6}$$

in which the symbol $*$ denotes either A or B, λ_* and ϕ_* are the longitude and latitude of the particle with respect to $\mathcal{A}$ or $\mathcal{B}$. J_l and J_{lm} are the zonal and tesseral harmonic coefficients for the ellipsoids and not equal to zero only when l and m are even integers. P_l and P_{lm} are the Legendre functions and associated Legendre functions. The rotation rate of the system ω is derived analytically in the literature[11,19)] and is related to the different orders of mutual potential considered for the primaries.

The system admits equilibrium points, which can be obtained by setting $\ddot{x} = \ddot{y} = \ddot{z} = \dot{x} = \dot{y} = \dot{z} = 0$ in Eq. (3), hence

$$U_x = U_y = U_z = 0. \tag{7}$$

Before solving directly from $U_x = U_y = U_z = 0$, the curves representing $U_x = 0$, $U_y = 0$ and $U_z = 0$ can be easily plotted and the equilibrium points should be intuitively detected by the intersections of the curves. This method is later adopted in the following sections to identify the number of equilibrium points and their approximate locations. Because of the symmetry of the system, the equilibrium points can only be located in the $x - y$ plane or $x - z$ plane. The variational equation with respect to the equilibrium points is

$$\begin{cases} \ddot{\xi} - 2\omega\dot{\eta} = U_{xx}^0\xi + U_{xy}^0\eta + U_{xz}^0\zeta \\ \ddot{\eta} + 2\omega\dot{\xi} = U_{xy}^0\xi + U_{yy}^0\eta + U_{yz}^0\zeta \\ \ddot{\zeta} = U_{xz}^0\xi + U_{yz}^0\eta + U_{zz}^0\zeta \end{cases} \tag{8}$$

Here, the superscript 0 indicates that the second partial derivative is evaluated at the equilibrium points. The stability of the equilibrium points can be analysed from the eigenvalues of the matrix $\boldsymbol{\Phi}$

$$\boldsymbol{\Phi} = \begin{bmatrix} \boldsymbol{0}_{3\times3} & \boldsymbol{I}_{3\times3} \\ \boldsymbol{G} & \boldsymbol{K} \end{bmatrix}_{6\times6} \tag{9}$$

in which $\boldsymbol{0}_{3\times3}$ is the zero matrix, $\boldsymbol{I}_{3\times3}$ is the identity matrix,

$$\boldsymbol{G} = \begin{bmatrix} U_{xx}^0 & U_{xy}^0 & U_{xz}^0 \\ U_{xy}^0 & U_{yy}^0 & U_{yz}^0 \\ U_{xz}^0 & U_{yz}^0 & U_{zz}^0 \end{bmatrix}, \quad \boldsymbol{K} = \begin{bmatrix} 0 & 2\omega & 0 \\ -2\omega & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{10}$$

3. Spherical Harmonic Model Approximation

3.1. Second Order Case

We start by considering the case when the total potential is truncated at 2nd order and degree, i.e.,

$$V_*^{2\text{nd}} = \frac{1}{r_*} - \frac{J_2^*\alpha_*^2}{r_*^3}\left(\frac{3}{2}\frac{z^2}{r_*^2} - \frac{1}{2}\right) + \frac{J_{22}^*\alpha_*^2}{r_*^3}\frac{3(x_*^2 - y^2)}{r_*^2}. \tag{11}$$

Four doubly synchronous binary asteroid systems are selected as examples. The system parameters for Fig. 1~4 are listed in Tab. 1. It is clearly shown that there exist at least five equilibrium points in the system. Similar to the traditional CRTBP, they are all in the $x - y$ plane, with three of them lying on the line connecting the binaries called collinear equilibrium points, and the other two located above and below the line called triangular equilibrium points. When the asteroids have small asphericity (Fig. 1), other intersections of $U_x = 0$ and $U_y = 0$ (or $U_z = 0$) can also be found, but are inside the body of the asteroids. As the asteroids become more elongated (Fig. 2 and Fig. 3), these intersections emerge above the surfaces of the asteroids, rending them equilibrium points in reality. Hence, the number of equilibrium points turn to be 9 or even 13 (the number could also be 7 or 11 with the right combinations of asteroid parameters, which are not discussed in this study). The new equilibrium points E_y^i located in the $x - y$ plane are similar to the center equilibrium points obtained in the single triaxial ellipsoidal asteroid system. However, the new equilibrium points E_z^i located in the $x - z$ plane right above the spin axes of the binaries are unique for the binary system with asphericity considered.

Mass fraction of the binary system is also varied to illustrate its effect on the evolution of the equilibrium points shown in

Table 1. System parameters for the doubly synchronous asteroid systems shown in figures.

Parameters	a_A (m)	$a_A : b_A : c_A$	a_B (m)	$a_B : b_B : c_B$
Fig. 1,7	1000	1 : 0.9 : 0.8	1000	1 : 0.9 : 0.8
Fig. 2,5,8	1000	1 : 1 : 0.3	1000	1 : 1 : 0.3
Fig. 3,6,9	1000	1 : 0.4 : 0.3	1000	1 : 0.4 : 0.3
Fig. 4,10	1000	1 : 0.4 : 0.3	250	1 : 0.4 : 0.3

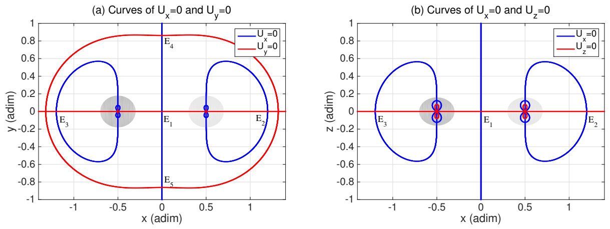

Fig. 1. Equilibrium points $E_1 \sim E_5$ in the $x-y$ plane for the doubly synchronous asteroid with mass fraction $\mu = 0.5$. The dark shaded and light shaded regions represent the projections in the $x-y$ $(x-z)$ plane of $\mathcal{A}$ and $\mathcal{B}$, respectively. The same representations apply in the following figures.

Fig. 3~Fig. 4. An obvious fact, which is similar to CRTBP, is that the collinear equilibrium points E_1 and E_2 along with the triangular equilibrium E_4 and E_5 move towards the secondary as the mass fraction becomes small. The positions and stability properties of the equilibrium points shown in Fig. 1~Fig. 4 are listed in Appendix B. The superscripts "S" and "U" denote that the equilibrium points are stable and unstable, respectively. All the equilibrium points of the binary systems are unstable except for the triangular equilibrium points of the systems corresponding to mass fraction $\mu < \mu_0$ (Fig. 4). For the CRTBP, $\mu_0 \approx 0.03852$ is the well-known Routh limit.[26] For the synchronous binary systems, μ_0 is dependent on the system parameters, which should deviate from the Routh limit.

3.2. Fourth Order Case

Potential terms up to the fourth order are considered for both the ellipsoids for plotting the U_x, U_y and U_z curves as shown in

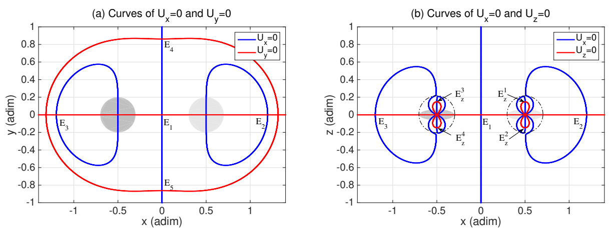

Fig. 2. Equilibrium points $E_1 \sim E_5$ in the $x-y$ plane and $E_z^1 \sim E_z^4$ in the $x-z$ plane for the doubly synchronous asteroid with mass fraction $\mu = 0.5$.

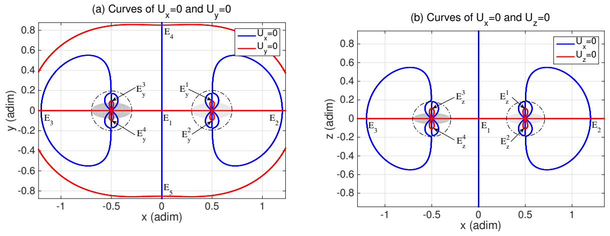

Fig. 3. Equilibrium points $E_1 \sim E_5$, $E_y^1 \sim E_y^4$ in the $x-y$ plane and $E_z^1 \sim E_z^4$ in the $x-z$ plane for the doubly synchronous asteroid with mass fraction $\mu = 0.5$.

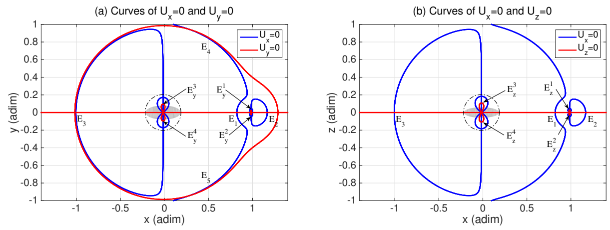

Fig. 4. Equilibrium points $E_1 \sim E_5$, $E_y^1 \sim E_y^4$ in the $x-y$ plane and $E_z^1 \sim E_z^4$ in the $x-z$ plane for the doubly synchronous asteroid with mass fraction $\mu = 0.016$.

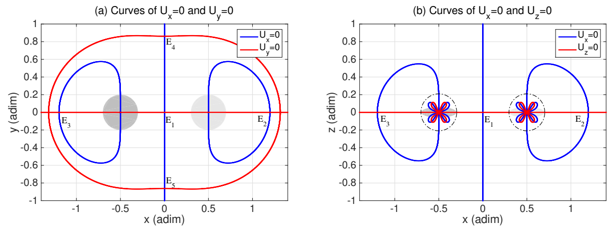

Fig. 5. Equilibrium points $E_1 \sim E_5$ in the $x-y$ plane and new equilibrium points in the $x-z$ plane for the doubly synchronous asteroid with mass fraction $\mu = 0.5$.

Fig. 5 and Fig. 6. Comparing Figure 5 and Figure 2 (or Figure 6 and Figure 3), which corresponds to the same system, except for the five equilibrium point E_i $(i = 1, \ldots, 5)$, the number and positions of the other equilibrium points have totally changed. This is obviously irrational as it might lead to the conclusion that the equilibrium points for the RFTBP depend on the potentials taken to model the primaries.

Two issues have been identified in the traditional work that models the primaries in the RFTBP by means of spherical harmonic gravitational potentials based on the analysis in Section 3.1 and 3.2.

The first issue is that most of the additional new equilibrium points found in Section 3.1 and 3.2 and in the literature are quite close to the centers of the primaries (e.g. $z \sim O\left(\sqrt{3J_2}\right)$ according to the analytical approximation for the out-of-plane equilibrium points derived in Ref. 8)). When the real shape of the primaries are taken into account, these equilibrium points actually do not exist (see Figure 1).

The second issue is that the spherical harmonic gravitational potential may be inappropriate to model the irregular primaries in the RFTBP, especially in determining possible equilibrium points near the primaries such as those work by Ref. 28). We will show in the next section that the spherical harmonic gravitational potential actually provide misleading information on the existence of the additional equilibrium points.

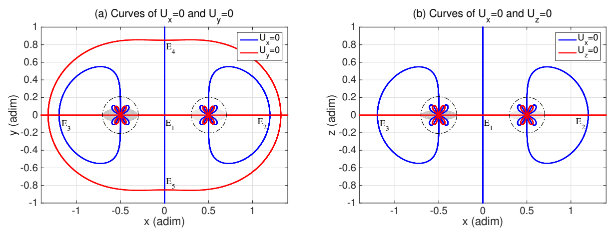

Fig. 6. Equilibrium points $E_1 \sim E_5$ in the $x-y$ plane and new equilibrium points in the $x-z$ plane for the doubly synchronous asteroid with mass fraction $\mu = 0.5$.

Table 2. Positions and stability properties of the equilibrium points shown in the figures.

Figures	Fig. 1	Fig. 2	Fig. 3	Fig. 4
$\mathcal{A}$	(-0.5, 0.0, 0.0)	(-0.5,0.0,0.0)	(-0.5,0.0,0.0)	(-0.01538,0.0,0.0)
$\mathcal{B}$	(0.5, 0.0, 0.0)	(0.5,0.0,0.0)	(0.5,0.0,0.0)	(0.98462,0.0,0.0)
E_1	$(0.0, 0.0, 0.0)^U$	$(0.0,0.0,0.0)^U$	$(0.0,0.0,0.0)^U$	$(0.82161,0.0,0.0)^U$
E_2	$(1.19804, 0.0, 0.0)^U$	$(1.19781, 0.0, 0.0)^U$	$(1.19730, 0.0, 0.0)^U$	$(1.16733, 0.0, 0.0)^U$
E_3	$(-1.19804, 0.0, 0.0)^U$	$(-1.19781, 0.0, 0.0)^U$	$(-1.19730, 0.0, 0.0)^U$	$(-1.00608, 0.0, 0.0)^U$
E_4	$(0.0, 0.86200, 0.0)^U$	$(0.0, 0.86192, 0.0)^U$	$(0.0, 0.85170, 0.0)^U$	$(0.28293, 0.94299, 0.0)^S$
E_5	$(0.0, -0.86200, 0.0)^U$	$(0.0, -0.86192, 0.0)^U$	$(0.0, -0.85170, 0.0)^U$	$(0.28293, -0.94299, 0.0)^S$
E_z^1	—	$(0.49986, 0.0, 0.14755)^U$	$(0.49997, 0.0, 0.10837)^U$	$(0.98461, 0.0, 0.02709)^U$
E_z^2	—	$(0.49986, 0.0, -0.14755)^U$	$(0.49997, 0.0, -0.10837)^U$	$(0.98461, 0.0, -0.02709)^U$
E_z^3	—	$(-0.49986, 0.0, 0.14755)^U$	$(-0.49997, 0.0, 0.10837)^U$	$(-0.01538, 0.0, 0.10844)^U$
E_z^4	—	$(-0.49986, 0.0, -0.14755)^U$	$(-0.49997, 0.0, -0.10837)^U$	$(-0.01538, 0.0, -0.10844)^U$
E_y^1	—	—	$(0.49998, 0.09617, 0.0)^U$	$(0.98461, 0.02403, 0.0)^U$
E_y^2	—	—	$(0.49998, -0.09617, 0.0)^U$	$(0.98461, -0.02403, 0.0)^U$
E_y^3	—	—	$(-0.49998, 0.09617, 0.0)^U$	$(-0.01538, 0.09617, 0.0)^U$
E_y^4	—	—	$(-0.49998, -0.09617, 0.0)^U$	$(-0.01538, -0.09617, 0.0)^U$

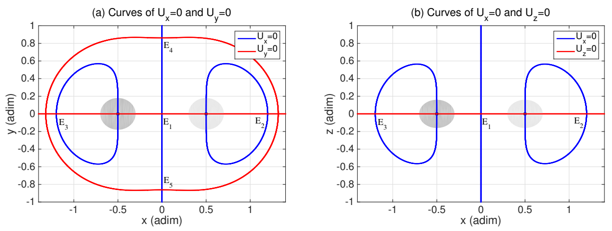

Fig. 7. Equilibrium points E_1 ~ E_5 in the $x-y$ plane for the doubly synchronous asteroid with mass fraction μ = 0.5.

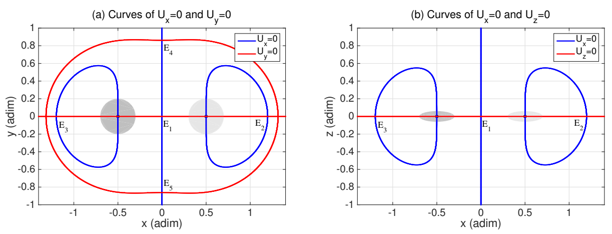

Fig. 8. Equilibrium points E_1 ~ E_5 in the $x-y$ plane for the doubly synchronous asteroid with mass fraction μ = 0.5.

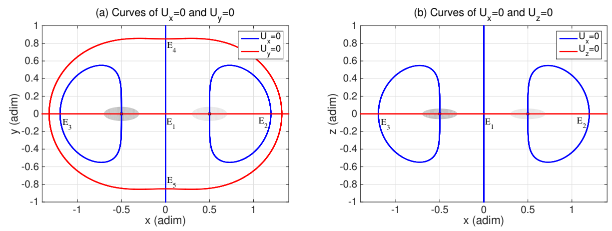

Fig. 9. Equilibrium points E_1 ~ E_5 in the $x-y$ plane for the doubly synchronous asteroid with mass fraction μ = 0.5.

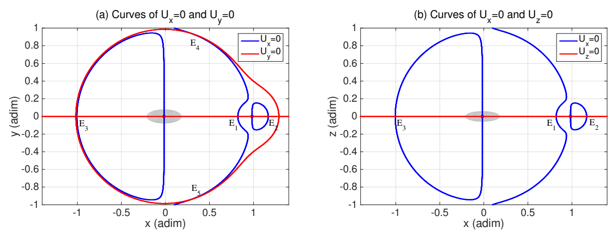

Fig. 10. Equilibrium points E_1 ~ E_5 in the $x-y$ plane for the doubly synchronous asteroid with mass fraction μ = 0.016.

4. Double Ellipsoid Model

Closed form for the ellipsoid potential is taken into account to model the binary asteroid system. Details for the computation of the ellipsoid potential are given in Ref. 21) and Ref. 3), and are omitted here. The same sets of system parameters as those adopted in Section 3.1 are chosen for the RFTBP (Table 1), and the plots of the corresponding $U_x = 0$, $U_y = 0$ and $U_z = 0$ curves are shown in Fig. 7~Fig. 10 for comparison with Fig. 2~6. The distance between $\mathcal{A}$ and $\mathcal{B}$ a is fixed to be 5000 m in these figures. The rotation rate of the system ω is taken to be consistent with the result assuming mutual potential of the ellipsoid truncated at 4th d/o.

It is evident from Figure 7~Figure 10 that except for the five equilibrium points E_i ($i = 1, \ldots, 5$) all the additional equilibrium points found previously no longer exist. This proves that the spherical harmonic gravitational potential is not appropriate in modelling the irregular primaries near their surfaces, otherwise, misleading results can be obtained. However, the spherical harmonic gravitational potential still captures the dynamics of the system far from the primaries.

Comparing the locations of the five equilibrium points in Figure 7~Figure 10 which are listed in Table 3 with those listed in Table 2, the differences are found to be less than ~ 10^{-2} in nondimensional unit and decrease to as small as ~ 10^{-4} as the asphericity of the primaries becomes smaller or the mass fractions get larger. In all of the scenarios, the stability properties of the corresponding equilibrium points in the two models remain the same.

5. Conclusion

We have studied the equilibrium points in the RFTBP where both the primaries are ellipsoids. We revisit the approach of spherical harmonic gravitational potential for modelling the primaries when determining the equilibrium points in the RFTBP, which is adopted by many existing work in the literature. By comparison with the results obtained by considering the ellipsoid potentials of the primaries in closed form, it is shown that the previous approach is problematic and may produce false

Table 3. Positions and stability properties of the equilibrium points shown in the figures.

Figures	Fig. 7	Fig. 8	Fig. 9	Fig. 10
$\mathscr{A}$	(-0.5, 0.0, 0.0)	(-0.5,0.0,0.0)	(-0.5,0.0,0.0)	(-0.01538,0.0,0.0)
$\mathscr{B}$	(0.5, 0.0, 0.0)	(0.5,0.0,0.0)	(0.5,0.0,0.0)	(0.98462,0.0,0.0)
E_1	$(0.0, 0.0, 0.0)^{U}$	$(0.0,0.0,0.0)^{U}$	$(0.0,0.0,0.0)^{U}$	$(0.82155,0.0,0.0)^{U}$
E_2	$(1.19802, 0.0, 0.0)^{U}$	$(1.19775, 0.0, 0.0)^{U}$	$(1.19716, 0.0, 0.0)^{U}$	$(1.16738, 0.0, 0.0)^{U}$
E_3	$(-1.19802, 0.0, 0.0)^{U}$	$(-1.19775, 0.0, 0.0)^{U}$	$(-1.19716, 0.0, 0.0)^{U}$	$(-1.00607, 0.0, 0.0)^{U}$
E_4	$(0.0, 0.86190, 0.0)^{U}$	$(0.0, 0.86163, 0.0)^{U}$	$(0.0, 0.85070, 0.0)^{U}$	$(0.28712, 0.94151, 0.0)^{S}$
E_5	$(0.0, -0.86190, 0.0)^{U}$	$(0.0, -0.86163, 0.0)^{U}$	$(0.0, -0.85070, 0.0)^{U}$	$(0.28712, -0.94151, 0.0)^{S}$

conclusions on the existence of the equilibrium points. This is due to the finite mass distribution that should be taken into account for the RFTBP and the mismodelling of the spherical harmonic gravitational potential for irregular bodies in the vicinity of their surfaces.

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