The Effectiveness of Solution Technique on the Autonomous Orbit Determination Accuracy of Lagrangian Navigation Constellation

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The accuracy of autonomous orbit determination of Lagrangian navigation constellation will affect the navigation accuracy for the deep space probes. Because of the special dynamical characteristics of Lagrangian navigation satellite, the error caused by different solution technique will cause totally different orbit prediction accuracy. We apply the RKF78 and RK4 to solve the motion equation of Lagrangian navigation satellites. There is no obvious difference when these two methods are used to calculate the orbits around the Earth-Moon triangular libration points. However the calculation error increases when RKF78 and RK4 are used to calculate the orbits around the Earth-Moon collinear libration points. Although the calculation error will be the order of 1×10^{8} m, it doesn't cause big difference on the AOD with an AOD step of 1 hour. If the AOD step is bigger than 10 hour, the accuracy of autonomous orbit determination using RKF78 is better than the autonomous orbit determination accuracy using RK4.

Key Words: autonomous orbit determination, Lagrangian point, navigation constellation, solution technique

Nomenclature

V	:	velocity
X	:	position
$m_{1,2}$	:	mass of the primaries
$r_{1,2}$	:	distances of spacecraft from primaries
f	:	the state propagation equation
v	:	the observation noise
Subscripts		
1	:	the massive primary
2	:	the secondary primary

1. Introduction

With the deep space exploration becoming a hot spot of aerospace, the demand for a deep space satellite navigation system is becoming increasingly prominent. In 1967, Farquhar first proposed to use an Earth-Moon L_2 libration point satellite to provide navigation information for the far side of the Moon.[1)] Then many researchers discussed the feasibility and performance of a satellite navigation system on periodic orbits around the Earth-Moon libration points. Hill suggested placing navigation constellation on the periodic orbits in the vicinity of libration points of the Earth-Moon system to support deep-space navigation.[2)] Grebow designed a constellation with two satellites located in quasi-periodic orbits around the Earth-Moon L_1 and L_2 to continuously coverage lunar south pole.[3)] Romagnoli and Circi proposed a lunar global positioning system (LGPS) which consists of several satellites orbiting around the Earth-Moon collinear libration points.[4,5)] Zhang and Xu analyzed the architecture and navigation Performance of the Lagrangian point satellite navigation system.[6-8)] Autonomous orbit determination (AOD) is an important performance for the Lagrangian point satellite navigation system. With this ability, the Lagrangian point satellite navigation system can reduce the dependency on ground stations. AOD also can greatly reduce total system cost and assure mission continuity.

The current studies about AOD of Lagrangian navigation satellite are under the circular restricted three-body problem (CR3BP) model.[9-13)] Since the CR3BP is sensitive to the state error and calculation error, different conditions and considerations must be applied to a discussion about AOD accuracy of Lagrangian navigation satellite. The solution technique is an approximation and will always introduce some error in the solution, therefore, the AOD accuracy refers to the accuracy of the solution of the motion equations. To clarify this aspect, the terminology solution technique accuracy refers to the error introduced in the solution of motion equation by the solution technique should be discussed.

Two commonly used methods which are 4 order Runge-Kutta(RK4) method and 7(8) order Runge-Kutta-Fehlberg (RKF78) method are used to solve the motion equation of CR3BP.

There are two types of error involved in a Runge-Kutta method: round-off error and truncation error. Since round-off error depends on the computer on which the algorithm is implemented, round-off error is not considered in the analysis of the algorithm. Truncation error is caused by truncation of infinite Taylor series to form the algorithm which depends on the step size used, the order of the method, and the problem being solved.[14)]

We will analyze the orbit prediction errors caused by the different Truncation errors using RK4 and RKF78 respectively. The prediction error caused by different solution techniques about orbits around the Earth-Moon collinear libration points and triangular libration points are shown.

Then the AOD results with different solution techniques are compared.

2. Dynamical model of Lagrangian navigation satellite

For satellites in Lagrangian point orbits, the equation of the CR3BP should be an appropriate model to describe the satellites' dynamical characteristics. Consider two massive bodies m_1 and m_2 moving under the action of just their mutual gravitation, and let their orbit around each other be a circle of radius r_{12}. As shown in Fig. 1, a non-inertial, co-moving frame of reference o-xyz is defined. The origin of frame *o-xyz* lies at the barycenter of the two-body system. The positive *x* direction goes from m_1 to m_2. The positive *y* axis is parallel to the velocity vector. The *z* axis is perpendicular to the orbital planeis. Now the third body of mass *m* which is vanishingly small compared to the primary masses m_1 and m_2 is introduced. We assume that the mass *m* is so small that it has no effect on the motion of the primary bodies. This is called the restricted three-body problem.

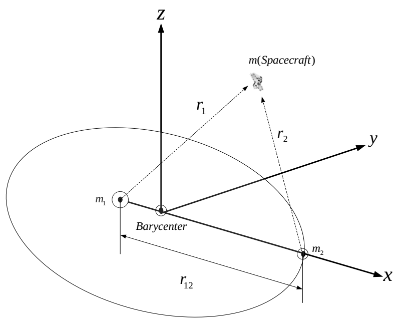

Fig. 1. Diagram of the circular restricted three-body problem in *o-xyz* frame.

The non-demensional equations of motion for the CR3BP are shown as following.[15)]

$$\begin{cases} \ddot{x} - 2\dot{y} = x - (1-\mu)\dfrac{x+\mu}{r_1^3} - \mu\dfrac{x+\mu-1}{r_2^3} \\ \ddot{y} + 2\dot{x} = \left(1 - \dfrac{1-\mu}{r_1^3} - \dfrac{\mu}{r_2^3}\right)y \\ \ddot{z} = \left(\dfrac{\mu-1}{r_1^3} - \dfrac{\mu}{r_2^3}\right)z \end{cases} \qquad (1)$$

Where

$$\begin{aligned} r_1 &= \sqrt{(x+\mu)^2 + y^2 + z^2}, \\ r_2 &= \sqrt{(x+\mu-1)^2 + y^2 + z^2}, \\ \mu &= \frac{m_2}{m_1 + m_2}。 \end{aligned} \qquad (2)$$

Eq. (1) has five equilibrium points which called Lagrange points (or libration point). Three Lagrange points on the *x* axis are unstable, and two Lagrange points which form an equilateral triangle with the two primary bodies in the *x-y* plane are stable. Many interesting periodic orbits exist in the vicinity of the Lagrange points no matter whether it is stable. The Lagrange navigation satellite constellation discussed in this paper is distributed in these periodic orbits.

3. Analysis of the error caused by different solution techniques

We can use Eq. (1) to predict the state of Lagrangian satellite. Eq. (1) can be written as

$$\dot{\boldsymbol{X}} = f(\boldsymbol{X}, t), \qquad (3)$$

Where

$$\boldsymbol{X} = [x \quad y \quad z \quad \dot{x} \quad \dot{y} \quad \dot{z}]^T. \qquad (4)$$

Usually the prediction is part of the process of AOD of Lagrangian satellite. For example, when the Extended Kalman Filter (EKF) is applied in the autonomous orbit determination, an important basic equation is the motion equation.[16)]

$$\boldsymbol{X}_{k,k-1} = f\left(\hat{\boldsymbol{X}}_{k-1}, t_{k-1}\right) \qquad (5)$$

The first step of the AOD is to solve the equation of motion. Since the errors introduced by the solution technique can be controlled at some level, they are seldom considered as the dominant error source for AOD of satellites on near Earth orbits. However the errors introduced by different solution techniques for Lagrangian satellite motion equation show big differences under the same initial conditions. Therefore we should discuss the effectiveness of solution error on the AOD.

In this section RK4 and RKF78 are used to solve the Eq. (3) respectively. We will compare the results obtained from the two methods.

The 4 order Runge-Kutta formaula takes the form.[17)]

$$\boldsymbol{X}_{n+1} = \boldsymbol{X}_n + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) \qquad (6)$$

$$\begin{cases} k_1 = h \cdot f(t_n, \boldsymbol{X}_n) \\ k_2 = h \cdot f\left(t_n + \dfrac{1}{2}h, \boldsymbol{X}_n + \dfrac{1}{2}k_1\right) \\ k_3 = h \cdot f\left(t_n + \dfrac{1}{2}h, \boldsymbol{X}_n + \dfrac{1}{2}k_2\right) \\ k_4 = h \cdot f(t_n + h, \boldsymbol{X}_n + k_3) \end{cases} \qquad (7)$$

It is difficult to estimate the truncation error of RK4. Fehlberg proposed an improved Runge-Kutta method which is called Runge-Kutta-Fehlbert(RKF). RKF can control the step size according to the truncation error. The seventh-order Runge-Kutta (RKF7(8)) formula with stepsize control is as following:

$$\begin{cases} \boldsymbol{X}_{n+1} = \boldsymbol{X}_n + \sum_{i=0}^{10} c_i k_i + O(h^8) \\ \hat{\boldsymbol{X}}_{n+1} = \boldsymbol{X}_n + \sum_{i=0}^{12} \hat{c}_i k_i + O(h^9) \end{cases} \qquad (8)$$

$$\begin{cases} k_0 = h \cdot f_0(t_n, \boldsymbol{X}_n) \\ k_i = h \cdot f_i\left(t_n + \alpha_i h, \boldsymbol{X}_n + h\sum_{j=0}^{i-1}\beta_{ij} f_j\right), \ i = 1, 2, \cdots, 12. \end{cases} \qquad (9)$$

The coefficients α_i, β_{ij}, c_i and $\hat{c}_i$ can be referenced in Ref. 17).

In the following, we respectively choose two different orbits, one of which orbits around the Earth-Moon collinear libration points L_1 and the other orbits around the Earth-Moon triangular libration points L_4, to compare the results with different solution techniques. The results are given as below.

Firstly, RK4 and RKF78 are used to solve the Lagrangian satellite motion equation for 30 days respectively, and the results obtained from the two methods are displayed in Figs. 2 and 3. Then the simulation time extended to 60 days and the results are shown in Figs. 4 and 5. The errors introduced by different solution techniques are shown in Figs. 6 and 7.

As shown in Figs. 2 and 3, for 30 days, two Lagrangian satellites' trajectories seem to be same and stable with two different solution techniques under the same initial conditions. For 60 days, as shown in Figs. 4 and 5, both of the motion trajectories introduced by RK4 and RKF78 are diverging and unstable. We can see from Fig. 6 that the difference between two methods is about 1000 meters in 30 days. For 60 days, the calculation errors between two methods are about the order of 1×10^8 m as shown in Fig. 7. However, the errors introduced by RKF78 are smaller than those introduced by RK4, which can be confirmed from the Figs. 8 and 9. Though the almost same results of Eq. (5) are got in 20 days by using RK4 and RKF78, the results show big differences in the days after 20 and the errors are bigger and bigger.

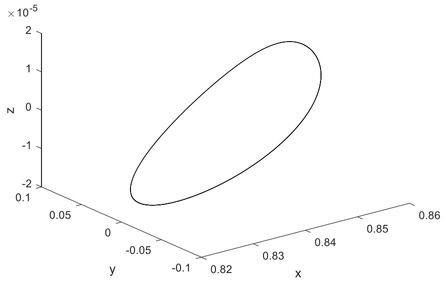

Fig. 2. Orbit around L_1 using RK4 for 30 days.

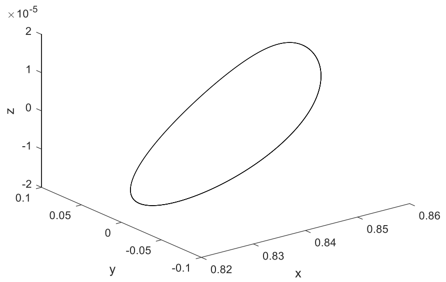

Fig. 3. Orbit around L_1 using RKF78 for 30 days.

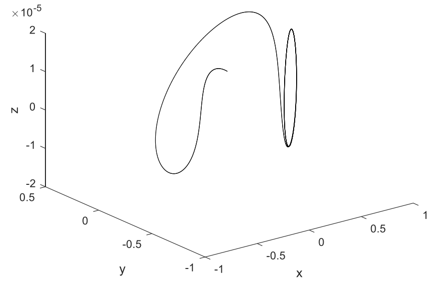

Fig. 4. Orbit around L_1 using RK4 for 60 days.

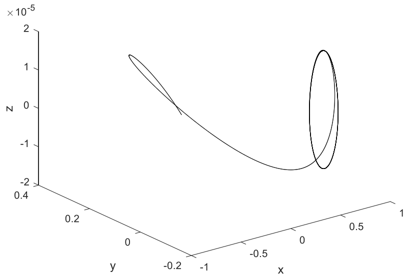

Fig. 5. Orbit around L_1 using RKF78 for 60 days.

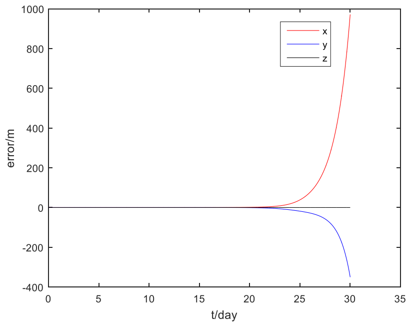

Fig. 6. Errors of the prediction orbit between RK4 and RKF78 for 30 days.

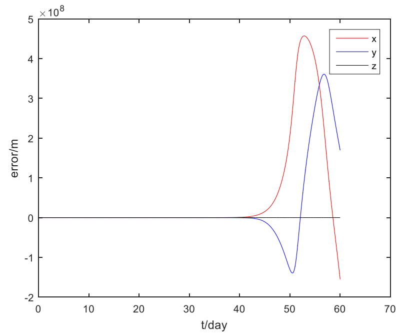

Fig. 7. Errors of the prediction orbit between RK4 and RKF78 for 60 days.

The comparisons with the results calculated by RK4 and RKF78 in 30 days and satellite ephemeris are shown in Figs. 8 and 9. The same comparisons in 60 days are shown in Figs. 10 and 11. Figures 8 and 9 illustrate that the errors between the results calculated by RK4 and RKF78 in 30 days and the satellite ephemeris are almost identical. The error is about 10^6m and the accuracy is not good. We also can draw a conclusion from Figs. 10 and 11 that the RK4 method produces large error after 40 days, but RKF78 produces the big errors after 50 days.

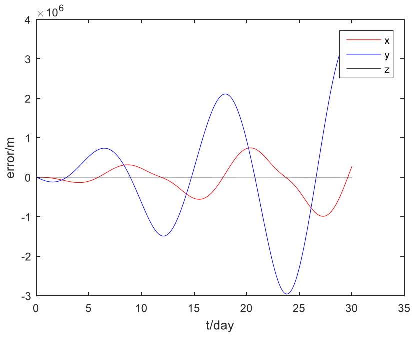

Fig. 8. Errors of RK4 and true ephemeris for 30 days.

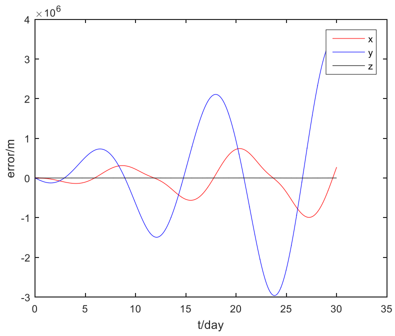

Fig. 9. Errors of RKF78 and true ephemeris for 30 days.

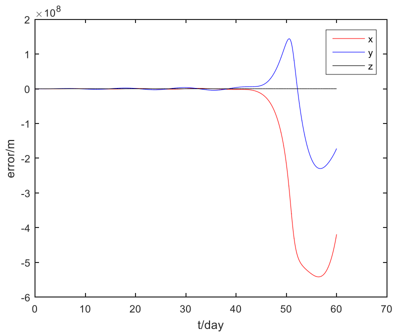

Fig. 10. Errors of RK4 and true ephemeris for 60 days.

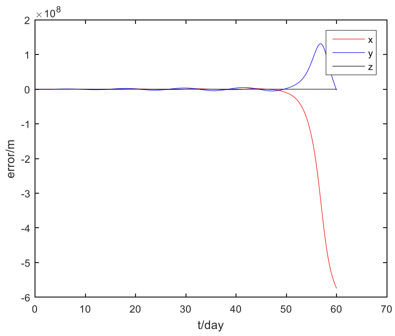

Fig. 11. Errors of RKF78 and true ephemeris for 60 days.

The satellite's trajectories around L_4 for 30 days, which are obtained respectively from RK4 and RKF78, are displayed in Figs. 12 and 13. Then the results for 60 days are shown in Figs. 14 and 15. Figures 12-15 illustrate that RKF45 and RKF78 can introduce the convergence trajectory within 60 days for orbit around L_4. The errors introduced by two methods are shown in Figs. 16 and 17. We can see that the calculation results of two methods are similar.

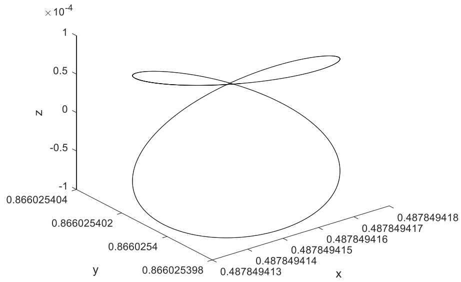

Fig. 12. Orbit around L_4 using RK4 for 30 days.

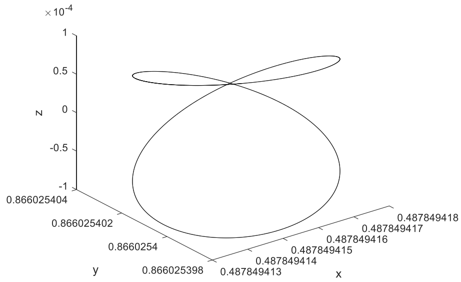

Fig. 13. Orbit around L_4 using RKF78 for 30 days.

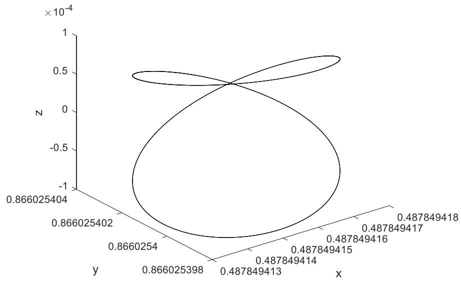

Fig. 14. Orbit around L_4 using RK4 for 60 days.

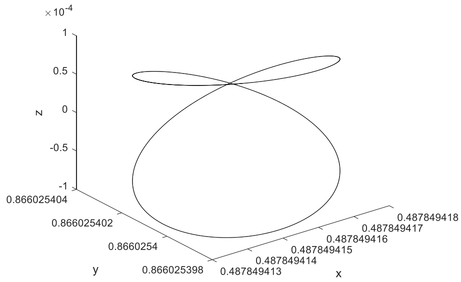

Fig. 15. Orbit around L_4 using RKF78 for 60 days.

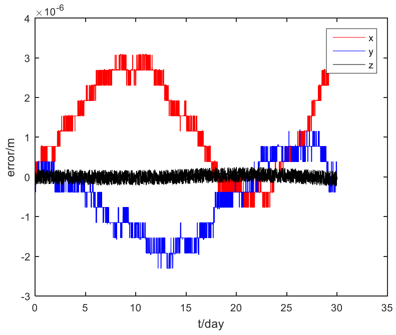

Fig. 16. Errors of RK4 and RKF78 for 30 days.

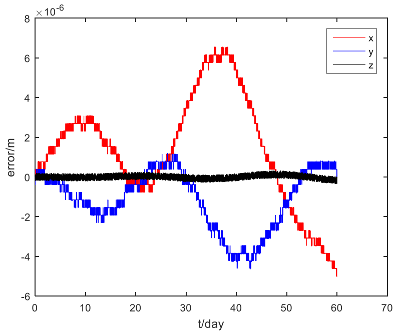

Fig. 17. Errors of RK4 and RKF78 for 60 days.

4. Analysis of the AOD error caused by different solution techniques

Without loss of generality, two Lagrangian satellites are considered. The state variable is defined as the following equation, in which the subscripts represent the satellite number.

$$\boldsymbol{x} = \begin{bmatrix} x_i & y_i & z_i & \dot{x}_i & \dot{y}_i & \dot{z}_i & x_j & \cdots & \dot{z}_j \end{bmatrix}^T, \quad (10)$$

The equations of motion can be written as

$$\dot{\boldsymbol{x}} = F(\boldsymbol{x}, t), \ \boldsymbol{x}(t_0) = \boldsymbol{x}_0 \quad (11)$$

Here the crosslink range is used as observation. The crosslink range can be got by using the satellite-links. The observations ρ is a nonlinear function of $\boldsymbol{x}$, which is represented by $h(\boldsymbol{x},t)$. The measured crosslink range can be written as

$$\rho = h(\boldsymbol{x},t) + v \quad (12)$$

Take satellite i and satellite j for example, the distance theoretically should be calculated as

$$\rho_{i,j} = h(\boldsymbol{x},t) = \sqrt{(x_i - x_j)^2 + (y_i - y_j)^2 + (z_i - z_j)^2} \quad (13)$$

But the measurement noise must be taken into account for the real instance, so the zero-mean Gaussian white noise is added to $\rho_{i,j}$, $v \in N(0,R)$.

For 180 days' simulation, the AOD errors of lagrangian satellite L_1 using two methods are shown in Figs. 18 and 19. 10 m initial state error is added in the simulation. The step of AOD is chosen as 1 hour. As we can see from Figs. 18 and 19, the AOD errors are almost smaller than 100m during most of the time when the measurement noise is about 10m. All of the AOD error curves show a common feature, that is, the errors in x-axis and y-axis show the good convergence. Although z-axis has a visible trend of convergence before 120 days, it diverges gradually. There is not obvious difference between the AOD error curves of L_1 satellite with two methods.

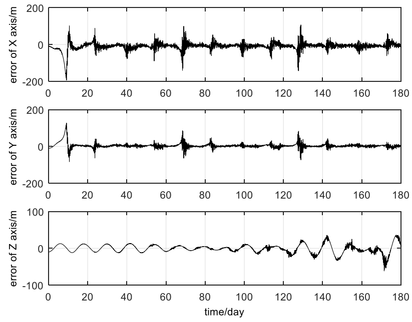

Fig. 18. The AOD error of L_1 using RK4.

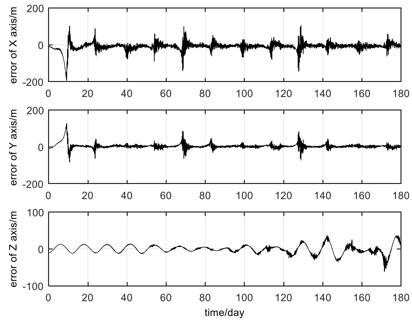

Fig. 19. The AOD error of L_1 using RKF78.

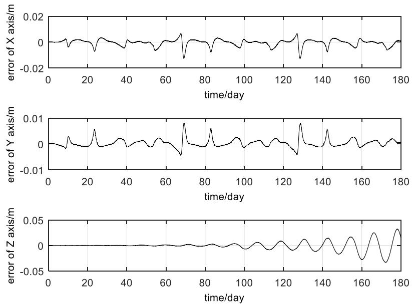

Fig. 20. The difference of the AOD errors of L_1 using RK4 and RKF78.

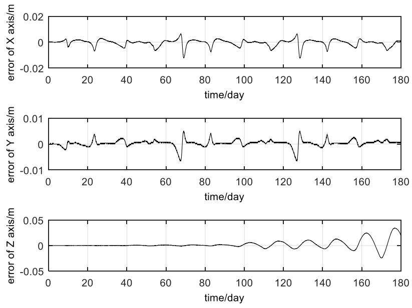

Fig. 21. The difference of the AOD errors of L_2 using RK4 and RKF78.

The difference between the AOD errors of L_1 and L_2 caused by RK4 and RKF78 are shown in Figs.20 and 21.From figure 20 and 21, we can get that the difference of the AOD errors between RK4 and RKF78 are smaller than 0.05m in 180 days.

There are few AOD errors caused by different solution techniques in a relatively short time. However, the prediction errors produced by solving dynamical equation of Lagrangian navigation satellite by RK4 and RKF78 are over 8×10^7m at 60 days. It means that calculation errors have been corrected after Kalman filter, and AOD accuracy of two methods are almost same. Besides, the arithmetic speed of RK4 is nearly 4 times faster than that of RKF78. It seems better to use RK4 when dealing with AOD problem in a suitable time.

When the step of AOD is increased to 10 hours, the AOD errors of L_1 between RK4 and RKF78 are shown in Fig. 22. As we can see, the AOD errors are much bigger than that shown in Fig. 20. That is because the truncation errors of two different methods are different. Since the truncation error of RK4 is much bigger, the Kalman filter cannot correct the calculation errors as good as AOD with 1hour step. Therefore, if the AOD step is bigger than 10 hours, RKF78 is a better choice.

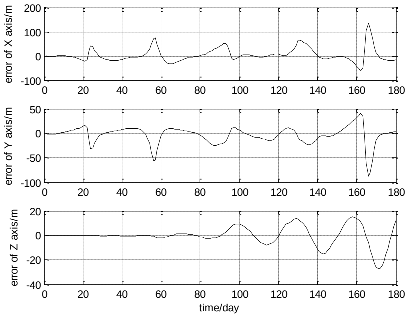

Fig. 22. The difference of the AOD errors of L1 using RK4 and RKF78 with 10 hours AOD step.

5. Conclusion

In order to investigate the effect of the solution technique on AOD accuracy of Lagrangian navigation satellite, we choose RK4 method and RKF78 methods to solve the motion equation of CR3BP. The simulations indicate that different solution technique accuracy will cause much different prediction error for orbits around collinear libration point. In 30 days, the difference is not obvious. After 30 days the difference increases quickly with time. However the difference caused by different solution technique accuracy will not cause obvious difference on AOD accuracy when the AOD step is small. That is because the EKF will correct the truncation error very AOD step. Under this condition, RK4 is a better choice because the arithmetic speed of RK4 is nearly 4 times faster than that of RKF78. However if the AOD step is bigger than 10 hours, the AOD accuracy of RKF78 is better. Therefore the choice of solution technique will based on the condition of the actual AOD.

Acknowledgments

This study was supported by the basic scientific research fund of national defense (No. 2016110C019), the natural science foundation of jiangsu province (No. BK20160811) and shanghai deep space detection technology key laboratory open funding(No.DS2016-01).

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