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Design of Quasi-Satellite Orbits: Analytical Alternatives

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Abstract

An analytical solution to the quasi-satellite orbit problem is provided based on the secular terms of the restricted three body problem in the Hill problem approximation. The validity of this solution restricts to the case of co-orbital motion in the 1:1 resonance with either small or moderate oscillations of the orbit about the primary. The evaluation of the secular terms alone is computationally undemanding, and provides an instant way of exploring quasi-satellite orbit evolution. Furthermore, the secular frequencies of the orbital motion can be used as design parameters in the search for initial conditions of periodic orbits. Application of the analytical solution to the computation of distant retrograde orbits is illustrated with several examples, and its performance is compared with the Lindstedt series solution already existing in the literature.

Keywords: Distant Retrograde Orbits; Quasi-Satellite Orbits; Hill Problem; Perturbation Methods; Lindstedt Series; Special Functions.

Introduction

Quasi-satellite orbits [1], also called distant retrograde orbits [2], have been pointed out as useful solutions for different solar system missions. In particular, because these kinds of orbits may enjoy stability for very long times, distant retrograde orbits are appealing candidates for quarantine orbits [3]. But quasi-satellite orbits have been also pointed out as prospective orbits for surveying the Martian moons, a case in which direct orbiting is not viewed as an option due to their low mass [4,5]. Analytical solutions to the quasi-satellite orbit problem have not being found yet, not even in the simplifications provided by the restricted three body problem model. Still, rough solutions based on averaging the higher frequencies of the motion have revealed useful in the qualitative explanation of the long-term dynamics [6,7]. On the other hand, it has been recently shown that analytical approximations of the dynamics that can be useful in the preliminary mission design of distant retrograde orbits (DRO) can be computed by perturbation techniques, and a high order Lindstedt series solution to the DRO problem has been brought to the scene [8,9].

As an alternative to the Lindstedt series approach, it is shown that standard perturbation theory can be used to provide an analytical solution to the DRO problem by finding the secular terms of the restricted three body problem Hamiltonian in the Hill problem approximation. The validity of the perturbation solution restricts to the case of co-orbital motion in the 1:1 resonance with either small or moderate oscillations of the quasi-satellite orbit about the primary of lesser mass, hereafter called librations.

The evaluation of the secular terms of the new analytical solution is computationally undemanding and provides an instant way of exploring DRO evolution. In addition, when required, the computation of ephemeris is straightforward, because osculating elements are easily obtained by adding periodic corrections to the secular solution. These corrections comprise both long- and short-period effects, and are obtained analytically as a result of the perturbation procedure. Furthermore, the secular frequencies of the motion can be used as design parameters in the search for initial conditions of periodic orbits. Indeed, commensurability between the (secular) periods of the orbital and librational motions results in approximately periodic orbits of the Hill problem whose periodicity is easily improved by the standard computation of differential corrections to the initial conditions. Application of the analytical solution to the computation of DRO is illustrated with several examples, and its performance is compared with the capabilities of the Lindstedt series solution already existing in the literature.

Body of the Paper

Long-Period Hamiltonian

The 1:1 co-orbital motion in the planar Hill problem approximation is traditionally approached in epicyclic canonical variables (ϕ,q,Φ,Q), whose relation with Cartesian variables (x,y,X,Y) is

$$x = a\xi + b\sin\phi \quad (1)$$
$$y = a\eta + a\cos\phi \quad (2)$$
$$X = -2B\eta - B\cos\phi \quad (3)$$
$$Y = -B\xi - B\sin\phi \quad (4)$$

in which

$$B = \omega b, \qquad a = 2b, \qquad b = \sqrt{2\Phi/\omega}, \quad (5)$$

and

$$\xi = \xi(Q,\Phi) \equiv \frac{Q}{2kB}, \qquad \eta = \eta(q,\Phi) \equiv \frac{kq}{b}, \quad (6)$$

where ω is the rotation rate of the Hill problem system, and the parameter k scales the transformation, and is conveniently chosen as $k = \sqrt{3/4}$.

In the epicyclic variables, quasi-satellite orbits are viewed like ellipses whose semi-major axis a, in the y axis direction, and semi-minor axis b are in the ratio 2:1, and whose guiding center

$$x_{\text{center}} = a\xi = Q/(k\omega), \qquad y_{\text{center}} = a\eta = 2kq, \quad (7)$$

oscillates slowly about the origin.

After removing the short-period terms by a canonical transformation from osculating to mean variables, the Hill problem Hamiltonian in mean variables only depends on long-period effects and is conveniently rearranged in the form of a truncated Taylor series

$$\mathcal{H} = \sum_{m\geq 0} \frac{\epsilon^m}{m!} H_m(-, q, \Phi, Q), \tag{8}$$

where $\epsilon \equiv 1$ is a formal small parameter used to show the strength of each perturbation term. When constraining to the terms that have been used in [9] for the construction of the high order Lindstedt series solution, the summands H_m are

$$\begin{aligned}
H_0 &= \mathcal{F} - \omega\Phi\left[3\xi^2 + \frac{4}{3}(\tilde{K} - \tilde{E})\delta\eta^2\right], \\
H_1 &= \omega\Phi\left[\frac{1}{9}(14\tilde{E} - 11\tilde{K})\delta\eta^4 + \frac{4}{3}(\tilde{K} - 4\tilde{E})\delta\xi^2\right], \\
H_2 &= \omega\Phi\left\{\frac{4}{3}\left[\frac{8}{3}(\tilde{E} - \tilde{K})(3\tilde{K} + 2\tilde{E}) + \frac{7}{4}\right]\delta^2\eta^2\right. \\
&\quad \left. + \frac{4}{81}(71\tilde{E} - 50\tilde{K})\delta\eta^6 + \frac{8}{3}\left[2(2\tilde{K} - 5\tilde{E}) - \tilde{K} - 26\tilde{E}\right]\delta\xi^2\eta^2\right\},
\end{aligned}$$

where

$$\mathcal{F} = \mathcal{F}(\Phi) \equiv \omega\Phi\left[1 - 2\tilde{K}\delta + \left(\frac{1}{2} - 2\tilde{K}^2\right)\delta^2 + 5\left(\frac{4}{15} - \tilde{K}^3 + \tilde{K} - \tilde{E}\right)\delta^3\right],$$

$\tilde{K} = K(k^2)/\pi$, $\tilde{E} = E(k^2)/\pi$, with K and E denoting the complete elliptic integrals of first and second kind, respectively, of modulus k, and the auxiliary non-dimensional variable

$$\delta = \delta(\Phi) \equiv \frac{\mu\omega}{B^3}, \tag{9}$$

where μ is the gravitational parameter of the Hill problem model, is used for convenience.

Because ϕ is a cyclic coordinate in Eq. (8), its conjugate momentum Φ is an integral of the averaged motion, and hence the size and shape of the ellipse remain constant, cf. Eq. (5). In consequence, the flow decouples into the fast motion determined by the phase ϕ along this constant ellipse, and the slow motion of the guiding center of the reference ellipse, determined by (q, Q) as follows from Eq. (7).

The term H_0 of the long-period Hamiltonian (8) clearly shows that the guiding center of the ellipse evolves with perturbed harmonic motion. Indeed,

$$H_0 = \mathcal{F} - \frac{1}{2}\left(Q^2 + \Omega^2 q^2\right) \tag{10}$$

where $\mathcal{F}(\Phi)$ comprises all the terms that are free from q and Q in H_0, $\Omega \equiv \Omega(\Phi)$ is the libration frequency

$$\Omega = \omega\sqrt{\delta(\tilde{K} - \tilde{E})}, \tag{11}$$

and $\delta \equiv \delta(\Phi)$ is computed fromEq. (9).

Solutions to the Hamiltonian flow in Eq. (8) are trivial for the zeroth order truncation given by Eq. (10), and can be computed by the Lindsted series technique for the higher orders of the long-term Hamiltonian (8), cf. [8,9]. Alternatively, a new canonical transformation can be applied such that it removes the remaining periodic effect from the Hamiltonian (8) in mean elements, in this way reducing it to its secular terms. This last approach is the contribution of the current paper.

Secular Hamiltonian

The transformation from mean to secular variables is better achieved when using harmonic oscillator-type variables, given by

$$\begin{aligned} \Phi &= \Psi, && (12)\\ \phi &= \psi + k^2\frac{\Theta}{\Phi}\sin\theta\cos\theta, && (13)\\ Q &= \sqrt{2\Theta\Omega}\cos\theta, && (14)\\ q &= \sqrt{2\Theta/\Omega}\sin\theta, && (15)\end{aligned}$$

After applying the canonical transformation in Eqs. (12)–(15) to the long-period Hamiltonian (8), viz.

$$\mathcal{H} = \sum_{m\geq 0}\frac{\epsilon^m}{m!}H_m(-,\theta,\Phi,\Theta), \tag{16}$$

a new canonical transformation $(\psi,\theta,\Psi,\Theta) \rightarrow (\psi',\theta',\Psi'\ \Theta';\ \epsilon)$ is applied to remove the long-period angle θ. Because $\Psi = \Phi$ is already an integral of the mean elements Hamiltonian, this variable is not affected by the transformation. Thus, $\Psi' = \Psi = \Phi$, and, in consequence, Ω remains unaltered.

The transformation to prime variables is obtained by perturbation theory. After applying this transformation to Eq. (16), the secular Hamiltonian

$$\mathcal{H} = \omega\Psi\sum_{m=0}^{3}h_m\left(\frac{\Omega}{\omega}\right)^{2m} - \Omega\Theta'\sum_{i=0}^{2}\left[\sum_{j=0}^{2-i}h_{i,j}\left(\frac{\Omega}{\omega}\right)^{2j}\right]\left(\frac{\Theta'/\Psi}{\Omega/\omega}\right)^{i} \tag{17}$$

is obtained, which discloses the secular motion as the combination of two (perturbed) harmonic oscillations of frequencies ω, for the motion along the reference ellipse, and Ω, for the motion of its guiding center.

The exact numeric coefficients h_m and $h_{i,k}$ in the secular Hamiltonian (17) are functions of the complete elliptic integrals of the first and second kind. Their exact, as well as their approximate values, are

$$
\begin{aligned}
h_0 &= 1 \\
h_1 &= -\frac{2\tilde{K}}{\tilde{K}-\tilde{E}} = -4.56184 \\
h_2 &= -\frac{4\tilde{K}^2-1}{2(\tilde{K}-\tilde{E})^2} = -4.8846 \\
h_3 &= \frac{1-5k^2(\tilde{K}^3-\tilde{K}+\tilde{E})}{k^2(\tilde{K}-\tilde{E})^3} = 44.7893 \\
h_{0,0} &= 1 \\
h_{0,1} &= \frac{64(\tilde{K}-\tilde{E})(4\tilde{K}+5\tilde{E})-63}{144(\tilde{K}-\tilde{E})^2} = 2.07095 \\
h_{0,2} &= -\frac{2(4\tilde{E}-\tilde{K})^2}{81(\tilde{K}-\tilde{E})^2} = -0.199537 \\
h_{1,0} &= \frac{3(11\tilde{K}-14\tilde{E})}{128(\tilde{K}-\tilde{E})} = 0.167748 \\
h_{1,1} &= \frac{214\tilde{K}\tilde{E}-23\tilde{K}^2-200\tilde{E}^2}{96(\tilde{K}-\tilde{E})^2} = 1.8482 \\
h_{2,0} &= \frac{1829\tilde{K}^2-3652\tilde{K}\tilde{E}+1364\tilde{E}^2}{49152(\tilde{K}-\tilde{E})^2} = 0.0220455
\end{aligned}
$$

Both Ψ and Θ' are constant in Eq. (17), whereas ψ' and θ' grow linearly with time in the prime variables. That is

$$
\begin{aligned}
\psi' &= \psi'_0 + n_\psi t, &(18)\\
\theta' &= \theta'_0 + n_\theta t, &(19)
\end{aligned}
$$

where the constant secular frequencies n_ψ and n_θ are computed from Eq. (17) using Hamilton equations

$$
n_\psi = \partial\mathcal{H}/\partial\Psi, \qquad n_\theta = \partial\mathcal{H}/\partial\Theta'. \tag{20}
$$

Orbit design parameters

Remarkably, the constant actions Ψ and Θ' can be used like orbit design parameters [10,11]. These two parameters give control to the mission designer on the size of the orbit, on the one hand, and on how close the orbiter can approach to the origin, on the other. Indeed, using Eq. (14) and (15), the guiding center of the reference ellipse Eq. (7) turns into

$$
\begin{aligned}
x_{\text{center}} &= \frac{1}{k}\frac{\Omega}{\omega}M\cos(\theta'_0+n_\theta t), &(21)\\
y_{\text{center}} &= 2kM\sin(\theta'_0+n_\theta t), &(22)
\end{aligned}
$$

where

$$M = \sqrt{2\Theta'/\Omega}, \tag{23}$$

is constant because both Θ' and Ψ, and hence $\Omega \equiv \Omega(\Psi)$ are constant.

The maximum elongation of y_{center} in the y axis direction happens each time θ' takes the value $\theta' = \theta_0 + n_\theta\, t = (m - 1/2)\,\pi$, with m integer, and this fact can be used to fix the minimum distance

$$y_{\min} = a - 2kM, \tag{24}$$

or, from Eq. (7), $y_{\min} = a\,(1 - \eta_{\max})$, that the orbiter will approach the origin along the y axis direction.

Other useful design parameter comes from the ratio $R = R(\Psi,\Theta') \equiv n_\psi/n_\theta$ between the orbital and libration secular frequencies. When R is a rational number, the secular solution will correspond to a secular periodic orbit. In general, this ratio will not be rational for arbitrary values of a and $y_{\min}$. Still, fixing, for instance, $y_{\min}$, and varying a will result in the desired commensurability from a simple root finding procedure in Ψ. Indeed, in view of $a = a(\Psi)$, as follows from Eq. (5), and $\Theta' = \Theta'(\Psi; y_{\min})$, as follows from Eqs. (23) and (24), it happens that the ratio depends only on Ψ in addition to design parameter $y_{\min}$, viz. $R = R(\Psi; y_{\min})$.

The periodicity obtained in this way is not constrained to the secular elements orbit and applies also to the complete analytical solution. Certainly, because the long-period corrections only depend on the secular elements, which have been made periodic by design, the periodicity will be preserved also in mean elements with the same period. In the same way, for the short-period corrections are function of the mean elements, which are periodic by construction, the periodicity in the osculating orbit is likewise achieved. However, the true orbit corresponding to the initial conditions of the periodic orbit in osculating elements will be only approximately periodic due to the truncation order of the perturbation solution. Nevertheless, if desired, straightforward differential corrections can be applied to the initial conditions in order to obtain a true periodic orbit with the desired characteristics.

Performance Evaluation

The performance of the analytical solution has been compared with the higher order Lindstedt series solution computed in [11], finding that, as expected, it produces solutions of comparable accuracy. However, the procedure is expedited now because the number of operations involved in the evaluation of the solution is abridged. Indeed, the evaluation of secular terms from Eqs. (18) and (19) is computationally inexpensive while the long- period corrections are notably shorter than the Lindstedt series in [11] (the short-period corrections are the same in both cases).

On the other hand, the computation of a periodic orbit is straightforward in the secular variables, while it involves the evaluation of two different Lindsted series in [11]. The computation of periodic DROs using the current analytical solution is illustrated in the following examples.

Large amplitude libration

A periodic DRO with initial design parameters a = 10 and $y_{\min}$ = 2.5, in Hill's problem units ($\mu = \omega = 1$), that is, with large amplitude librations, is computed first. Starting from these values, the approximate values: M = 4.33, from Eq. (24), $\Psi = \Phi$ = 12.5, from the definition of a in Eq. (5), Ω = 0.049, from Eqs. (9) and (11), and, finally Θ' = 0.46, from Eq. (23), are computed. Then, the secular frequencies $n_\theta = -0.0638$, and n_ψ = 1.0072 are computed from Eq. (20). These values of the secular frequencies are not commensurable, yielding a ratio libration period orbit period R_1 = 15.784. However, successive iterations of the secant method

$$\Psi = \Psi_n + \frac{\Psi_{n+1} - \Psi_n}{R_{n+1} - R_n}(R - R_n) \tag{25}$$

for the closest integer value R = 16 to the ratio R_1 obtained in the initial design, result into the Ψ value that yields the required commensurability of the secular frequencies $n_\psi = 16 \times n_\theta$. Straightforward evaluation of Eqs. (18) and (19) will produce the desired secular, periodic DRO that, by direct application of Eqs. (12)–(15), first, and Eqs. (1)–(4), then, is displayed in (secular) Cartesian coordinates in Fig. 1

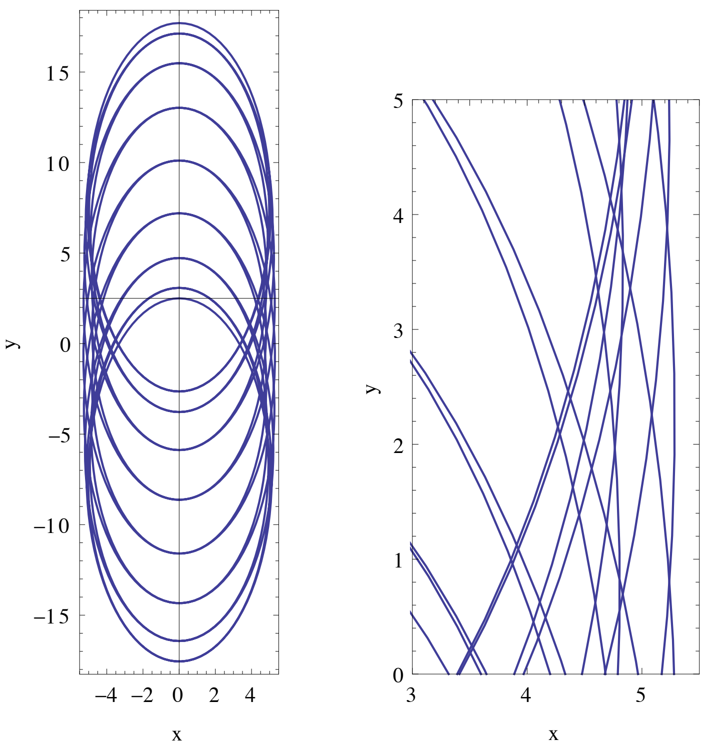

Fig. 1: Left: 16 revolutions secular periodic orbit. The x axis is centered at $y = y_{\min}$ = 2.5 to highlight that the secular orbit agrees with the orbit design stipulation. Right: Detail showing that ascending and descending trajectories get very close to each other.

The orbit also remains periodic in mean elements, which are obtained after applying the direct long-period corrections, because these corrections only depend on secular terms, which have been made periodic by design. The differences between the mean and secular orbits are mainly due to the differences in the trajectories of the centers of the corresponding reference ellipses, as illustrated in Fig. 2.

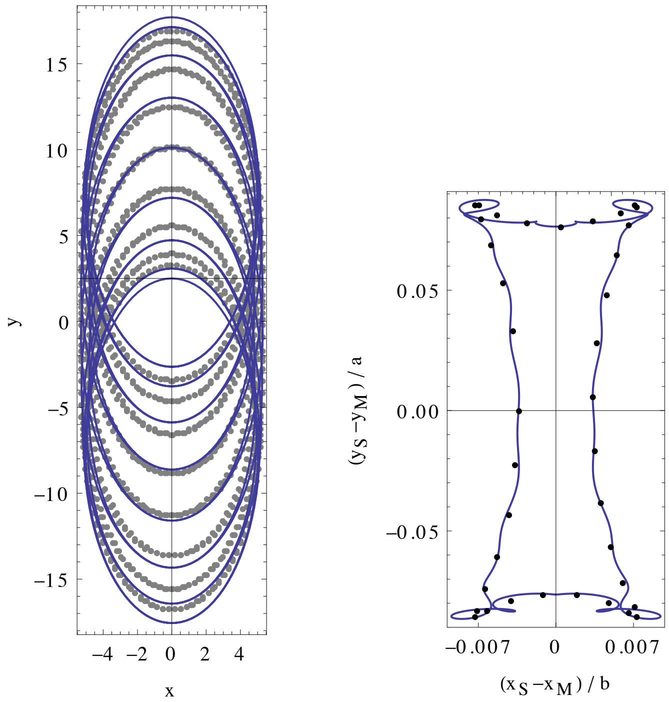

Fig. 2: Left: Sample periodic orbit after 16 revolutions in mean elements (dots) superimposed to its secular partner (full line). Right: differences between the secular and mean orbits in Cartesian coordinates (full line) with corresponding differences between their guiding centers superimposed (dots).

The analytical solution also remains periodic in osculating elements, since the transformation from mean to osculating elements only depends on mean elements, which have been already shown to be periodic. However, one must note, that due to the truncation order of the perturbation theory, when propagating the osculating initial conditions predicted by the analytical solution directly in the original, non averaged Hill problem model, the orbit obtained is only almost periodic after one libration period.

This lack of periodicity of the true orbit is clearly noted in Fig. 3, where the initial and final points are highlighted with black dots. The periodicity error is about six hundredths the size of the orbit, for the coordinates, and one hundredth the size of the hodograph for the conjugate momenta.

The periodicity of the numerical orbit is easily improved using a differential corrections algorithm, as, for instance, the one in [12]. The improved periodic orbit remains very close to the analytical osculating prediction, as shown in Fig. 4.

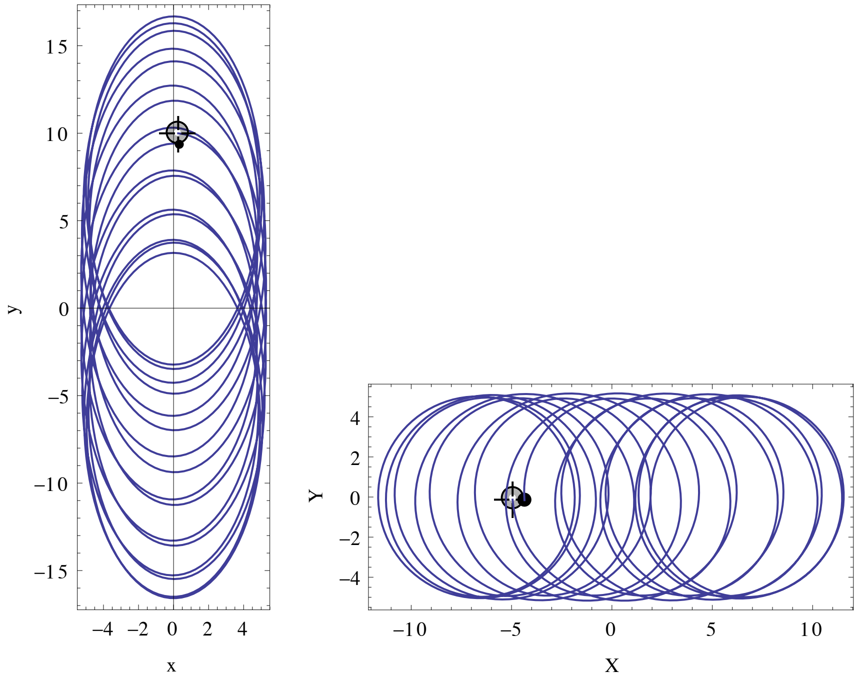

Fig. 3: Non-periodic, true orbit of the Hill problem numerically propagated from the initial conditions and libration period predicted by the analytical sample solution periodic after 16 revolutions.

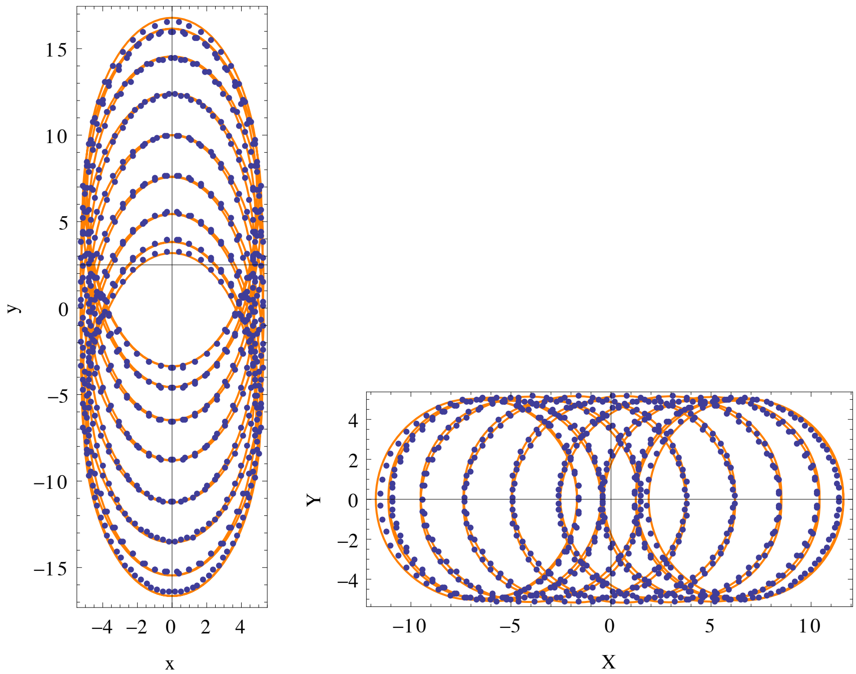

Fig. 4: Improved, periodic orbit of the Hill problem (dots) with the osculating analytical sample periodic solution after 16 revolutions superimposed (full, bright line).

1:1 resonance

In spite of low order resonances are excluded from the phase space encompassed by the perturbation solution, which implicitly assumes that $\Omega << \omega$, one can find the periodic analogs of 1:1 resonant orbits directly from secular predictions of orbits with negligible libration amplitude.

For instance, the choice of the design parameter $a = 10$ is made as in the previous example, but now $y_{\min} = a$. This new selection of the design parameters produces the values: $M = 0$, $\Psi = 12.5$, $\Omega \approx 0.049$, and $\Theta' = 0$, after which the secular frequencies $n_\theta = -0.0493$, and $n_\psi = 1.0056$ are computed from Eq. (20). The non commensurability of n_θ, and n_ψ yields a ratio libration period orbit period $R = 20.3916$. But a slightly shorter semi-major axis of the reference ellipse is computed by successive iterations of the secant method making $R = 20$ (the closest integer to the value obtained from the initial design) in Eq. (25).

Then, straightforward evaluation of Eqs. (18) and (19) with the new value of a will produce the desired secular, periodic DRO. This orbit is depicted in Fig. 5, where it is shown that the amplitude of the librational motion after the prefixed 20 orbital periods, ore the equivalent one libration period, is very small.

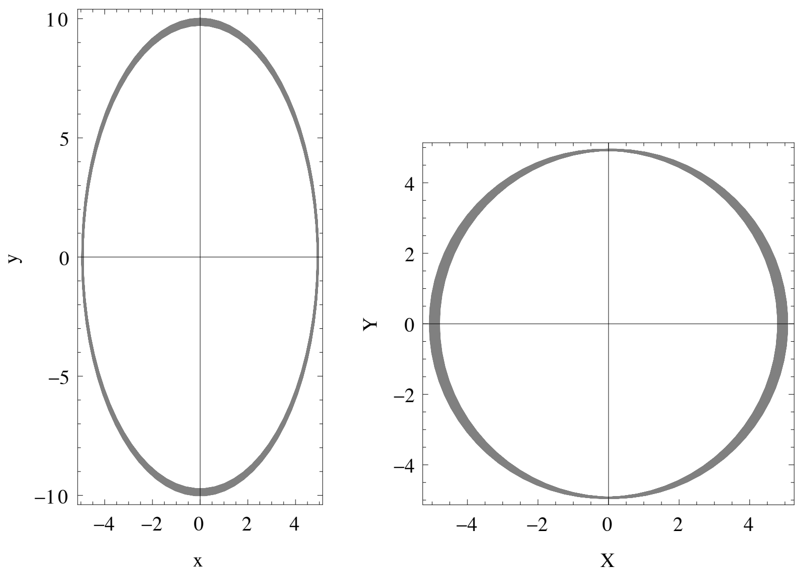

Fig. 5: Periodic analytical solution with small amplitude libration after 20 revolutions.

Due to the small amplitude of the libration, and to the strong stability character of 1:1 DROs, when differential corrections are applied to the approximate initial conditions of this orbit in Cartesian coordinates they will naturally converge to find a true periodic orbit after one, contrary to 20, orbital period. In fact, the differential corrections algorithm fails when trying to converge to the 1:20 resonant periodic orbit.

Conclusion

The dynamics of a large class of distant retrograde orbits of the Hill problem is efficiently characterized by an analytical perturbation solution that provides the secular terms of the Hamiltonian flow. This analytical solution captures the bulk of the dynamics of common quasi-satellite orbits even in the case of orbits with large oscillations about the origin. Besides, the size of the orbit as well as the size of its oscillations are easily turned into mission design parameters in the secular variables space. Furthermore, periodic orbits are easily obtained by adding to the design parameters a commensurability condition. Since the analytical solution is only approximate, the obtained periodic orbits are only almost periodic in the original space. Still, these approximate orbits are amenable of improvement by usual differential corrections procedures to get true periodic distant retrograde orbits of the Hill problem.

The Hill problem has been chosen as a demonstration problem for its simplicity and generality, but the same methods can be applied to more realistic dynamical models.

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