Quaternion Representation of Rotation from the Aspect of Group Theory

Yuki Ishihara(1), Masaki Nakamiya(2)

(1)*Teikyo University*
Utsunomiya, Japan
Email: 23fm10002m6@stu.teikyo-u.ac.jp

(2) *Teikyo University*
Utsunomiya, Japan
Email: nakamiya@ase.teikyo-u.ac.jp

***Abstract* – There are several methods to express three-dimensional rotations of spacecraft and aircraft. In particular, quaternions are used for the representation of rotations because of having no gimbal lock and a low calculation cost. The quaternion representation of rotations has a different form than the other representations of rotations. For instance, a three-dimensional rotation of a vector is expressed as $\boldsymbol{Q(\theta)n_qQ^{-1}(\theta)}$, where $\boldsymbol{Q(\theta)}$ is a rotation quaternion and $\boldsymbol{n_q}$ is a vector quaternion, respectively. However, the derivations of these representations are only superficially provided in most books. Therefore, in this study, we discuss the mathematical structures of the quaternion representation of rotations from the viewpoint of Group theory. As a result, it was found that a map of rotation for the vector quaternion in the Lie algebra can be obtained by an adjoint representation for the Lie group. This mapping is caused by Inner automorphism, leading to the expression $\boldsymbol{Q(\theta)n_qQ^{-1}(\theta)}$.**

I. INTRODUCTION

There are several methods to represent three-dimensional rotations of spacecraft and aircraft, such as Euler angles, rotation matrices, and quaternions. Among them, the rotation expression using Euler angles is commonly used to express three-dimensional rotations because of its advantages such as fewer parameters, and is easier to understand intuitively. Euler angle representation, however, has the disadvantage of the gimbal lock occurring when two of three axes are aligned. The gimbal lock, however, can be avoided by adding yet another axis. For this reason, rotation representation using quaternions, which uses four parameters, is also often used. Here, the equation expressing three-dimensional rotations using quaternions is as follows:

$$n_q = n_{q_1} i + n_{q_2} j + n_{q_3} k, \tag{1.1}$$

$$Q(\theta) = \cos\frac{\theta}{2} + (w_1 i + w_2 j + w_3 k)\sin\frac{\theta}{2}, \tag{1.2}$$

$$n'_q = Q(\theta) n_q Q^{-1}(\theta). \tag{1.3}$$

Equation (1.1) is a point n_q in three-dimensional space expressed using a vector quaternion, (1.2) is the rotation quaternion $Q(\theta)$ related to the rotation of the angle θ with the unit vector, $\vec{w} = (w_1, w_2, w_3)$ as a rotation axis, and (1.3) represents the point n'_q after rotating n_q by an angle θ by the rotation quaternion $Q(\theta)$. The quaternion representation of rotation has different forms from the other representations of rotations. However, the derivation of these representations is only superficially provided in most books. Hence, the reason for the equation is unclear why $\cos\frac{\theta}{2}$ and $\sin\frac{\theta}{2}$ are used instead of $\cos\theta$ and $\sin\theta$ in (1.2), and the reason why a vector quaternion n_q is sandwiched between rotation quaternion $Q(\theta)$ and $Q^{-1}(\theta)$ in (1.3) [1].
This study, therefore, considers the reasons why equations by quaternions expressing three-dimensional rotation are used and how to derive them.

II. $\frac{\theta}{2}$ ROTATION

Regarding (1.2) shown in the introduction, let us consider the reason for using $\cos\frac{\theta}{2}$ and $\sin\frac{\theta}{2}$ are used instead of $\cos\theta$ and $\sin\theta$. Equation (1.2) is a rotation quaternion related to the rotation of the angle θ around $\vec{w} = (w_1, w_2, w_3)$ and is also an element of the Lie group $U(1)$ in group theory. In addition, a general three-dimensional rotation matrix that represents a similar rotation uses θ, and is also an element of the Lie group $SO(3)$. Therefore, consider the reason why $\frac{\theta}{2}$ is used in (1.2) by comparing the two rotation-related elements from the perspective of the group theory of $SO(3)$ and $U(1)$. Here, we consider the Lie algebra $so(3)$ corresponding to the Lie group $SO(3)$ and the Lie algebra $u(1)$ corresponding to Lie group $U(1)$.
First, using the alternation matrix R_3 of the Lie algebra $so(3)$ the rotation axis unit vector $\vec{w}$ is expressed as follows:

$$R_3 = w_1 \cdot I + w_2 \cdot J + w_3 \cdot K, \tag{2.1}$$

where I, J and K are the bases of the Lie algebra $so(3)$. Calculating the commonly seen three-dimensional rotation matrix, they can be expressed as follows:

$$I = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix},\ J = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix},$$
$$K = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \qquad (2.2)$$

The commutator product between the bases of this Lie algebra $so(3)$ is calculated as [2]

$$[I, J] = K\,,\ [J, K] = I\,,\ [K, I] = J. \qquad (2.3)$$

Besides, when (2.1) is expressed as a matrix, it becomes

$$\begin{aligned} R_3 &= w_1 \cdot I + w_2 \cdot J + w_3 \cdot K \\ &= w_1 \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} + w_2 \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} \\ &\quad + w_3 \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\ &= w_1 \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} + w_2 \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} \\ &\quad + w_3 \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & -w_3 & w_2 \\ w_3 & 0 & -w_1 \\ -w_2 & w_1 & 0 \end{pmatrix}. \end{aligned} \qquad (2.4)$$

R_3 is expressed as $(R_3)^\vee \equiv (w_1, w_2, w_3)$ (from now on, the vector notation of Lie algebra will be expressed like this), the square of R_3 is calculated as by the vector triple product,

$$\begin{aligned} {R_3}^2 &= (R_3)^\vee \{ (R_3)^\vee\}^T \{ (R_3)^\vee\}^T - \{ (R_3)^\vee\}^T (R_3)^\vee E \\ &= (R_3)^\vee \{ (R_3)^\vee\}^T - E, \end{aligned} \qquad (2.5)$$

where the unit matrix E is written as $E = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.
Furthermore, the third power, fourth power, and fifth power of R_3 are each calculated as follows:

$${R_3}^3 = {R_3}^2 R_3 = ((R_3)^\vee \{ (R_3)^\vee\}^T - E) R_3 = -R_3, \qquad (2.6)$$
$${R_3}^4 = {R_3}^3 R_3 = -R_3 R_3 = -{R_3}^2, \qquad (2.7)$$
$$\begin{aligned} {R_3}^5 &= {R_3}^4 R_3 \\ &= -{R_3}^2 R_3 \\ &= -((R_3)^\vee \{ (R_3)^\vee\}^T - E) R_3 \\ &= R_3. \end{aligned} \qquad (2.8)$$

R_3 to the sixth power or higher can be similarly expressed by R_3 and ${R_3}^2$. Here, the elements of the Lie group $SO(3)$, which represent rotation of the angle θ around the axis $(R_3)^\vee = (w_1, w_2, w_3)$, can be obtained by exponential mapping of (2.1), which is a Lie algebra [3]. Therefore, the elements of the Lie group $SO(3)$ are calculated from (2.4) to (2.8) as

$$\begin{aligned} \exp(\theta \cdot R_3) &= E + (\theta R_3) + \frac{1}{2!}(\theta R_3)^2 + \frac{1}{3!}(\theta R_3)^3 \\ &\quad + \frac{1}{4!}(\theta R_3)^4 + \frac{1}{5!}(\theta R_3)^5 + \cdots \\ &= E + \left(\theta - \tfrac{1}{3!}(\theta)^3 + \tfrac{1}{5!}(\theta)^5 - \cdots\right) R_3 \\ &\quad + \left(\tfrac{1}{2!}(\theta)^2 - \tfrac{1}{4!}(\theta)^4 + \cdots\right) {R_3}^2. \end{aligned} \qquad (2.9)$$

In addition,

$$\begin{aligned} cos\theta &= 1 - \tfrac{1}{2!} \cdot \theta^2 + \tfrac{1}{4!} \cdot \theta^4 - \cdots, \\ sin\theta &= \theta - \tfrac{1}{3!} \cdot \theta^3 + \tfrac{1}{5!} \cdot \theta^5 - \cdots. \end{aligned} \qquad (2.10)$$

Equation (2.9) becomes as follows:

$$(right\ hand\ side) = E + R_3 sin\theta + {R_3}^2 (1 - cos\theta). \qquad (2.11)$$

For simplicity, $\cos\theta$ is written as $c\theta$, and $\sin\theta$ is written as $s\theta$. The element of the Lie group $SO(3)$ is calculated from (2.4), (2.5), and (2.11) as

$$\begin{aligned} &\exp(\theta \cdot R_3) \\ &= \begin{pmatrix} c\theta + w_1^2(1 - c\theta) & w_1 w_2 (1 - c\theta) - w_3 s\theta & w_2 s\theta + w_3 w_1 (1 - c\theta) \\ w_3 s\theta + w_1 w_2 (1 - c\theta) & c\theta + w_2^2 (1 - c\theta) & -w_1 s\theta + w_2 w_3 (1 - c\theta) \\ -w_2 s\theta + w_3 w_1 (1 - cc\theta) & w_1 s\theta + w_2 w_3 (1 - c\theta) & c\theta + w_3^2 (1 - c\theta) \end{pmatrix}, \end{aligned} \qquad (2.12)$$

where (2.12) is a commonly seen rotation matrix. For example, the rotation of the angle θ around the z-axis $(R_{3z})^\vee = (0,0,1)$ in (2.12) can be written as

$$exp(\theta \cdot R_{3z}) = \begin{pmatrix} c\theta & -s\theta & 0 \\ s\theta & c\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}. \qquad (2.13)$$

Thus, it is evident that θ is used in the rotation representation by the Lie group $SO(3)$, which is attached to the Lie algebra using the basis of (2.2).
Next, let us consider the rotation representation using quaternions. Exponential mapping of the Lie algebra $so(3)$ gives the Lie group $SO(3)$. From this relationship, it is expected that the Lie group $U(1)$, which is an exponential mapping of the Lie algebra $u(1)$, is also related to rotation. Therefore, consider the Lie group $U(1)$ from the Lie algebra $u(1)$ corresponding to the Lie algebra $so(3)$ mentioned above. The rotation axis vector $\vec{w}$ is written by the vector quaternion of the Lie algebra $u(1)$ in a form similar to (2.1) as

$$r_q = w_1 \cdot i + w_2 \cdot j + w_3 \cdot k. \qquad (2.14)$$

However, the commutator products using these bases i, j and k are calculated as

$$[i,j]=2k\,,\ [j,k]=2i\,,\ [k,i]=2j. \quad (2.15)$$

This (2.15) does not correspond to (2.3). Thus, assuming that the bases of $u(1)$ are $\frac{i}{2},\frac{j}{2}$ and $\frac{k}{2}$ the commutator products are calculated as

$$\left[\frac{i}{2},\frac{j}{2}\right]=\frac{k}{2}\,,\ \left[\frac{j}{2},\frac{k}{2}\right]=\frac{i}{2}\,,\ \left[\frac{k}{2},\frac{i}{2}\right]=\frac{j}{2}. \quad (2.16)$$

This (2.16) corresponds to (2.3). Therefore, using these bases and the vector quaternion of the Lie algebra $u(1)$, the rotation axis vector $\vec{w}=(w_1,w_2,w_3)$ can be expressed as

$$r_q=w_1\cdot\frac{i}{2}+w_2\cdot\frac{j}{2}+w_3\cdot\frac{k}{2}. \quad (2.17)$$

Furthermore, the square, cube, fourth, and fifth power of r_q are calculated as follows:

$$r_q{}^2=\left(w_1\frac{i}{2}+w_2\frac{j}{2}+w_3\frac{k}{2}\right)^2=-\frac{1}{4}, \quad (2.18)$$
$$r_q{}^3=r_q{}^2r_q=-\frac{1}{8}(w_1i+w_2j+w_3k), \quad (2.19)$$
$$r_q{}^4=r_q{}^3r_q=r_q{}^2r_q{}^2=\frac{1}{16}, \quad (2.20)$$
$$r_q{}^5=r_q{}^4r_q=r_q{}^2r_q{}^2r_q=\frac{1}{32}(w_1i+w_2j+w_3k). \quad (2.21)$$

When (2.17) is exponentially mapped, the rotation quaternion of the Lie group $U(1)$ is obtained from (2.10) and (2.17) to (2.21) as follows:

$$\begin{aligned}
&\exp(\theta\cdot r_q)\\
&=1+(\theta\cdot r_q)+\frac{1}{2!}(\theta\cdot r_q)^2+\frac{1}{3!}(\theta\cdot r_q)^3\\
&\quad+\frac{1}{4!}(\theta\cdot r_q)^4+\frac{1}{5!}(\theta\cdot r_q)^5+\cdots\\
&=1+(w_1i+w_2j+w_3k)\cdot\frac{\theta}{2}-\frac{1}{2!}\cdot\left(\frac{\theta}{2}\right)^2\\
&\quad-(w_1i+w_2j+w_3k)\cdot\frac{1}{3!}\cdot\left(\frac{\theta}{2}\right)^3+\frac{1}{4!}\cdot\left(\frac{\theta}{2}\right)^4\\
&\quad+(w_1i+w_2j+w_3k)\cdot\frac{1}{5!}\cdot\left(\frac{\theta}{2}\right)^5-\cdots\\
&=\left(1-\frac{1}{2!}\cdot\left(\frac{\theta}{2}\right)^2+\frac{1}{4!}\cdot\left(\frac{\theta}{2}\right)^4-\cdots\right)\\
&\quad+(w_1i+w_2j+w_3k)\left(\frac{\theta}{2}-\frac{1}{3!}\cdot\left(\frac{\theta}{2}\right)^3+\frac{1}{5!}\cdot\left(\frac{\theta}{2}\right)^5-\cdots\right)\\
&=cos\frac{\theta}{2}+(w_1i+w_2j+w_3k)sin\frac{\theta}{2}. \quad (2.22)
\end{aligned}$$

Here, (2.22) corresponds to (1.2) expressed in terms of $Q(\theta)$ and can be rewritten as

$$Q(\theta)\equiv e^{\theta r_q}. \quad (2.23)$$

Consequently, the unit quaternion $U(1)$ related to rotation obtained from the Lie algebra $u(1)$ which corresponds to the Lie algebra $so(3)$ of the rotation matrix, has terms of $\cos\frac{\theta}{2}$ and $\sin\frac{\theta}{2}$, so that $\frac{\theta}{2}$ is used in (1.2), which expresses rotation using the quaternions shown in the introduction.

III. FORM SANDWICHED BY QUATERNIONS IN THREE-DIMENSIONAL ROTATION REPRESENTATION

A. Representation method based on two-dimensional rotation representation

In two dimensions, a vector after rotation can be represented by the product of a complex number representing rotation ($e^{\theta r_c}=cos\theta+psin\theta$ (p: imaginary unit)) and a complex number representing vector ($e^{\theta n_c}=x+py$ (x,y: real number)). In this chapter, we see whether a three-dimensional rotation, like a two-dimensional rotation, can be expressed as a simple multiplication of the quaternion associated with the rotation and the quaternion representing the vector.
The rotated vector (x,y,z) in Euclidean space is expressed using a pure imaginary quaternion as

$$n_q=x\frac{i}{2}+y\frac{j}{2}+z\frac{k}{2}. \quad (3.1)$$

Here, let us calculate it as a simple multiplication with the rotation quaternion $e^{\theta r_q}$ related to the rotation of the angle θ around the unit vector $\vec{w}=(w_1,w_2,w_3)$. From (2.22) and (3.1), it can be calculated as follows

$$\begin{aligned}
&e^{\theta r_q}n_q\\
&=\left(\cos\frac{\theta}{2}+(w_1i+w_2j+w_3k)\sin\frac{\theta}{2}\right)\left(x\frac{i}{2}+y\frac{j}{2}+z\frac{k}{2}\right)\\
&=\frac{i}{2}\left(x\cos\frac{\theta}{2}-w_3y\sin\frac{\theta}{2}+w_2z\sin\frac{\theta}{2}\right)\\
&\quad+\frac{j}{2}\left(w_3x\sin\frac{\theta}{2}+y\cos\frac{\theta}{2}-w_1z\sin\frac{\theta}{2}\right)\\
&\quad+\frac{k}{2}\left(-w_2xsin\frac{\theta}{2}+w_1ysin\frac{\theta}{2}+zcos\frac{\theta}{2}\right)\\
&\quad-\frac{1}{2}\left(w_1xsin\frac{\theta}{2}+w_2ysin\frac{\theta}{2}+w_3zsin\frac{\theta}{2}\right). \quad (3.2)
\end{aligned}$$

Next, let us verify the vector rotation using a general rotation matrix. Furthermore, when the point in the Euclidean space corresponding to (3.1) is expressed as a vector, it can be expressed as follows:

$$(n_q)^{\vee}\equiv\begin{pmatrix}x\\y\\z\end{pmatrix} \quad (x,y,z{:}\,real\ number). \quad (3.3)$$

Thereby, the vector after rotation of $(n_q)^{\vee}$ by the three-dimensional rotation orthogonal matrix $e^{\theta R_3}$ representing the rotation of the angle θ around the above-mentioned axis $\vec{w}=(w_1,w_2,w_3)$ is given by (2.12) and (3.1) and can be calculated as follows:

$$e^{\theta R_3}\left(n_q\right)^{\vee}$$
$$= \begin{pmatrix} c\theta + w_1{}^2(1-c\theta) & w_1w_2(1-c\theta) - w_3s\theta & w_2s\theta + w_3w_1(1-c\theta) \\ w_3s\theta + w_1w_2(1-c\theta) & c\theta + w_2{}^2(1-c\theta) & -w_1sin\theta + w_2w_3(1-c\theta) \\ -w_2s\theta + w_3w_1(1-c\theta) & w_1s\theta + w_2w_3(1-c\theta) & c\theta + w_3{}^2(1-c\theta) \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$
$$= \begin{pmatrix} x\left(c\theta + w_1{}^2(1-c\theta)\right) + y(w_1w_2(1-c\theta) - w_3s\theta) + z\left(w_2s\theta + w_3w_1(1-c\theta)\right) \\ x\left(w_3s\theta + w_1w_2(1-c\theta)\right) + y\left(c\theta + w_2{}^2(1-c\theta)\right) + z\left(-w_1s\theta + w_2w_3(1-c\theta)\right) \\ x\left(-w_2s\theta + w_3w_1(1-c\theta)\right) + y\left(w_1s\theta + w_2w_3(1-c\theta)\right) + z\left(c\theta + w_3{}^2(1-c\theta)\right) \end{pmatrix}, \tag{3.4}$$

where $\cos\theta$ is written as $c\theta$ and $\sin\theta$ is written as $s\theta$. Equations (3.2) and (3.4) are not isomorphic.

$$e^{\theta r_q} n_q \ncong e^{\theta R_3}\left(n_q\right)^{\vee}. \tag{3.5}$$

This shows that in three-dimensional rotation, rotation cannot be expressed by a simple multiplication of the quaternion representing the rotation and the quaternion representing the vector.

B. Representation method based on the representation of group theory

In this section, we will discuss why rotation can be expressed in the form of (1.3) in three-dimensional rotation expressions using quaternions, which cannot be expressed by simple multiplications. In the first place, an element n_q of Lie algebra $u(1)$ is different from an element $Q(\theta)$ of the Lie group $U(1)$ in both structure and definition. For that reason, the result of these simple multiplications are not necessarily an element of the Lie algebra $u(1)$ ((3.2) contains terms other than the basis of the Lie algebra $u(1)$ so it should not become an element of $u(1)$). In a three-dimensional rotation using quaternions, the quaternion representing the rotation is a Lie group, and the quaternion representing the vector mapped by the automorphism is a Lie algebra. In general, to generate an automorphism of a vector space from a group, a homomorphism using a group representation is used. Thereby, we will discuss this based on the group representation. In general, the adjoint representations of groups can be applied to the group representation that generates an origin-invariant automorphism. For that reason, a rotation by a quaternion is also a rotation that does not change the origin. Therefore, the adjoint representation of the group is used to generate an automorphism, which acts on a quaternion which is an element of the Lie algebra $u(1)$ representing the vector to be rotated, using a quaternion which is an element of the Lie group $U(1)$ representing the rotation. From here, use Fig.1 below to see how the automorphism of the Lie algebra based on the adjoint representation of the Lie group represents a rotation by focusing on the relationships between the groups.

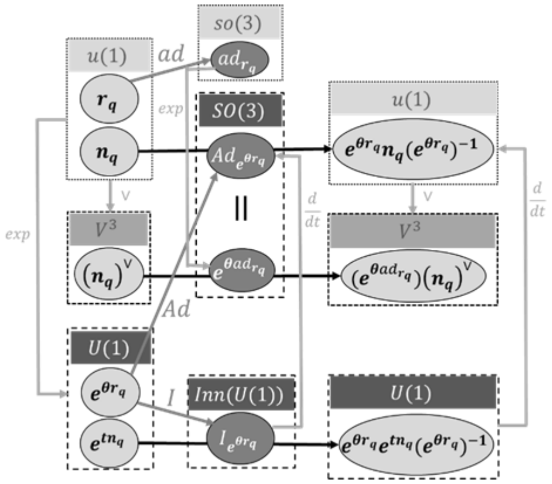

Fig. 1. *Diagram of the Relationship between Adjoint representations, Automorphisms, and Rotation groups.*

First, look at the elements in the route of $u(1) \to so(3) \to SO(3)$. The element r_q of the pure imaginary quaternion related to the rotation of the Lie algebra $u(1)$ defined by (2.17) in the upper left of Fig. 1 is mapped by the adjoint representation ad of the Lie algebra [3]. This becomes an element of the Lie algebra $so(3)$ at the center of the diagram and is written as ad_{r_q}. Here, the bases of $u(1)$ can be calculated as

$$ad_{\frac{i}{2}} = I, ad_{\frac{j}{2}} = J, ad_{\frac{k}{2}} = K \ . \tag{3.6}$$

Furthermore, since both $u(1)$ and $so(3)$ are vector spaces, they can be written in the form of a linear combination of bases. Thereby, we know that ad_{r_q} is an element of the Lie algebra $so(3)$ (diagram top center). Incidentally, the unit vector written as R_3 in (2.1) corresponds to this ad_{r_q} and can be rewritten as

$$R_3 \equiv ad_{r_q} \ . \tag{3.7}$$

When this $ad_{r_q} \in so(3)$ is exponentially mapped, it can be written as $e^{\theta ad_{r_q}}$, and this is the element of the Lie group $SO(3)$. In addition, this $e^{\theta ad_{r_q}}$ corresponds to the rotation matrix $e^{\theta R_3}$ in (2.12).

Next, consider the elements generated by the route of $u(1) \to U(1) \to SO(3)$. When the element r_q of the Lie algebra $u(1)$ in the upper left of the diagram is exponentially mapped, it becomes the element $e^{\theta r_q}$ of the Lie group $U(1)$ in the lower left of the diagram. When $e^{\theta r_q}$ is mapped by the group adjoint representation Ad, it becomes an element of the Lie group $SO(3)$ (center of the diagram), and we write it as $Ad_{e^{\theta r_q}}$ [1]~[3]. This $Ad_{e^{\theta r_q}} \in SO(3)$ is equal to the rotation matrix $e^{\theta ad_{r_q}} \in SO(3)$ generated by the route of $u(1) \to so(3) \to SO(3)$ that we illustrated previously.

$$e^{\theta ad_{r_q}} = Ad_{e^{\theta r_q}} \in SO(3)\,. \qquad (3.8)$$

It, therefore, can be seen that the mapping $Ad_{e^{\theta r_q}}$ generated from the adjoint expression Ad represents a rotation.

In addition, from the definition of the adjoint representation of a group, the mapping $Ad_{e^{\theta r_q}}$ is a mapping onto itself on the Lie algebra, and it acts on the element n_q of the Lie algebra $u(1)$ in the upper left as follows:

$$Ad_{e^{\theta r_q}}(n_q) \equiv e^{\theta r_q} n_q (e^{\theta r_q})^{-1}\,. \qquad (3.9)$$

Thus, the reason that n_q in (1.3) is sandwiched between $e^{\theta r_q}$ and $(e^{\theta r_q})^{-1}$ is due to the mapping $Ad_{e^{\theta r_q}}$ generated by the adjoint representation.

Here, we check why the action (3.9) using the adjoint representation of the group is defined in such a way that n_q is sandwiched between $e^{\theta r_q}$ and $(e^{\theta r_q})^{-1}$. First, the automorphism that acts from the Lie group $U(1)$ at the bottom left of the diagram to the Lie group $U(1)$ at the bottom right is inner automorphism in group theory. Let the inner automorphism be $I_{e^{\theta r_q}} \in Inn(U(1))$, is generated by the mapping I that generates an inner automorphism from the element n_q of the Lie group $U(1)$ at the bottom left of the diagram. Then, the element e^{tn_q} of the same lower left Lie group $U(1)$ is defined as follows [3]:

$$I_{e^{\theta r_q}}(e^{tn_q}) \equiv e^{\theta r_q} e^{tn_q} (e^{\theta r_q})^{-1}\,. \qquad (3.10)$$

As a result, the Lie group $U(1)$ on the lower right has a sandwiching form, and when it is differentiated concerning the identity element, it becomes an element of the Lie algebra on the upper right, which can be written as

$$\frac{d}{dt}\left(e^{\theta r_q} e^{tn_q} (e^{\theta r_q})^{-1}\right)\Big|_{t=0} = e^{\theta r_q} n_q (e^{\theta r_q})^{-1}\,. \quad (3.11)$$

It, therefore, can be seen that the form in which n_q defined in (3.9) is sandwiched between $e^{\theta r_q}$ and $(e^{\theta r_q})^{-1}$ is due to an inner automorphism on the Lie group. Actually, when calculated by substituting (3.1) and (2.22) into the right-hand side of (3.11) as

$$\begin{aligned}
&e^{\theta r_q} n_q (e^{\theta r_q})^{-1} \\
&= \left(\cos\frac{\theta}{2} + (w_1 i + w_2 j + w_3 k)\sin\frac{\theta}{2}\right) \\
&\left(x\frac{i}{2} + y\frac{j}{2} + z\frac{k}{2}\right)\left(\cos\frac{\theta}{2} + (w_1 i + w_2 j + w_3 k)\sin\frac{\theta}{2}\right)^{-1} \\
&= \frac{i}{2}\{x\left(\cos\theta + w_1{}^2(1-\cos\theta)\right) \\
&\quad + y(w_1 w_2(1-\cos\theta) - w_3\sin\theta) \\
&\quad + z(w_2\sin\theta + w_3 w_1(1-\cos\theta))\} \\
&\quad + \frac{j}{2}\{x\left(w_3\sin\theta + w_2 w_1(1-\cos\theta)\right) \\
&\quad + y\left(\cos\theta + w_2{}^2(1-\cos\theta)\right) \\
&\quad + z(w_2 w_3(1-\cos\theta) - w_1\sin\theta)\} \\
&\quad + \frac{k}{2}\{x(w_3 w_1(1-cos\theta) - w_2 sin\theta) + y\left(w_1 sin\theta + \right. \\
&\left. w_2 w_3(1-cos\theta)\right) + z\left(cos\theta + w_3{}^2(1-cos)\right)\}, \quad (3.12)
\end{aligned}$$

$e^{\theta ad_{r_q}}(n_q)^{\vee}$ and (3.12) are isomorphic.

$$e^{\theta ad_{r_q}}(n_q)^{\vee} \cong e^{\theta r_q} n_q (e^{\theta r_q})^{-1}. \qquad (3.13)$$

Therefore, the equation for expressing three-dimensional rotations using quaternions cannot be expressed by a simple multiplication since (1.3) has the form where n_q is sandwiched between quaternions $e^{\theta r_q}$ and $(e^{\theta r_q})^{-1}$.

Incidentally, ad_{r_q} generated from r_q, which is an element of Lie algebra $u(1)$, by the adjoint representation ad of the Lie algebra becomes an element of the Lie algebra $so(3)$. When this is exponentially mapped, it becomes an element of the rotation group $SO(3)$. In addition, $Ad_{e^{\theta r_q}}$ is generated by the adjoint representation Ad from $e^{\theta r_q}$, which is obtained by exponentially mapping the element r_q of Lie algebra $u(1)$ to the Lie group $U(1)$ and becomes an element of the rotation group $SO(3)$. Hence, it is equal to $e^{\theta ad_{r_q}}$ obtained by the exponential mapping of ad_{r_q}. Thus, since $e^{\theta ad_{r_q}}$ is a rotation matrix, it can be said that $Ad_{e^{\theta r_q}}$, which is an element of the same group, also represents rotation. Furthermore, the reason that $Ad_{e^{\theta r_q}}$ is defined as a form in which an element n_q of the Lie algebra is sandwiched between quaternions $e^{\theta r_q}$ and $(e^{\theta r_q})^{-1}$ is due to an inner automorphism $I_{e^{\theta r_q}}$ on the Lie group. It is defined that the elements after mapping of $I_{e^{\theta r_q}}$ are mapped between quaternions. Furthermore, by differentiating this concerning the identity element, it becomes equal to the element after mapping $Ad_{e^{\theta r_q}}$ that acts on the Lie algebra generated by the adjoint representation Ad. These actions lead to the rotation representation (1.3) using quaternions in the Lie algebra.

IV. SUMMARY

This study examined the reason why the angle $\frac{\theta}{2}$ is used instead of the angle θ in the formula expressing three-dimensional rotation by quaternions, the reason why the formula is sandwiched between quaternions, and the method for deriving the three-dimensional rotation by quaternions. Firstly, we gave the reason why $\frac{\theta}{2}$ is used instead of θ when expressing rotation using a rotation quaternion. Equation (1.2) where $\frac{\theta}{2}$ is used is an element of the Lie group $U(1)$, and the rotation matrix where θ is used is an element of the Lie group $SO(3)$. Therefore, if we compare the commutator product of the bases of the Lie algebra $so(3)$ corresponding to $SO(3)$ and the Lie algebra $u(1)$ corresponding to $U(1)$, the bases of $u(1)$ need to be $\frac{i}{2}, \frac{j}{2}, \frac{k}{2}$ instead of i, j, k. When mapping the Lie algebra $u(1)$ expressed by these bases to the Lie group $U(1)$ by exponential mapping, $\frac{\theta}{2}$ appears in cos and sin. Accordingly, we found that $\frac{\theta}{2}$ is used to express rotation in the rotation quaternion.

Next, the reason why (1.3) is sandwiched between quaternions was investigated. In three-dimensional rotation representation using quaternions, the quaternion representing the rotation is a Lie group, while the quaternion representing the vector to be rotated is a Lie algebra. Based on this, we considered the group representation and used the adjoint representation of the group. From the relationship between the groups, we can see that the element of $SO(3)$ which is an exponential mapping generated from the element of the Lie algebra $u(1)$ by the adjoint representation of the algebra, is equal to the element of $SO(3)$ created by exponentially mapping the elements of Lie algebra $u(1)$ to the element of the Lie group $U(1)$ and using the Lie group adjoint representation. It, therefore, can be said that the mapping generated from the element of the Lie group $U(1)$ by adjoint representation can represent three-dimensional rotation. The reason why this mapping is defined in such a way that the elements of the Lie algebra are sandwiched between the quaternions $e^{\theta r_q}$ and $(e^{\theta r_q})^{-1}$ is that the inner automorphism on the Lie algebra has a form in which the element of the Lie group $U(1)$ is sandwiched between the quaternions $e^{\theta r_q}$ and $(e^{\theta r_q})^{-1}$, and it is differentiated concerning the identity element. Thus, the reason why (1.3) is sandwiched between the quaternions is due to the mapping generated by the adjoint representation, and this can be said to be attributed to the action of inner automorphism on the Lie group.

Based on the results of this study, in the future we would like to apply the group theory approaches to understanding of rotational representations in other dimensions and reveal their mathematical structures.

V. REFERENCES

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