Analytical Modeling of Gravitational Potential of Irregularly Shaped Celestial Bodies: Application to (101955) Bennu

M. L. Mota(1,2) S. Aljbaae(2) F. B. A. Prado(2,3)

(1) Federal Institute of São Paulo, IFSP, Hortolândia, SP, Brasil. (E-mail: prof.mlmota@ifsp.edu.br)
(2) National Institute for Space Research, São José dos Campos, SP, Brazil.
(3) Academy of Engineering, RUDN University, Miklukho-Maklaya street 6, Moscow, Russia, 117198.

Abstract

This study aims to investigate the location and stability of equilibrium points in the potential field of an irregularly shaped celestial body in the solar system. The potential series expansion method is used, which involves decomposing the asteroid into homogeneous tetrahedral elements with a constant rotation speed. The results are compared with those obtained using the classical polyhedron method, and we found good agreement outside the body, even in close proximity to its surface. Our model can be applied to any polyhedral-shaped body as it expresses the potential analytically, which makes it easier to manipulate algebraically for determining acceleration and velocity components. Moreover, it significantly reduces computational costs in terms of CPU time requirements.

1 Introduction

Modeling the gravitational potential of a spacecraft close to a small, irregularly shaped celestial body is one of the most challenging concepts in orbital property studies. This is due to the difficulty in creating a mathematical model that can accurately reproduce the distribution of mass. Numerous works have been developed to model the gravitational field around non-spherical bodies, such as those by MacMillan [1930], Kellogg [1953], Waldvogel [1976], Werner [1994], Balmino [1994], Werner and Scheeres [1996], Dechambre and Scheeres [2002], Hu and Scheeres [2004], Mota [2017], Venditti [2013], Chanut et al. [2015], Aljbaae et al. [2017], and Aljbaae et al. [2021].

This study uses the potential series expansion method (PSEM), as presented in Mota [2017], Mota and Rocco [2019], Mota et al. [2023], to explicitly display the approximate gravitational potential function close to the asteroid (101955) Bennu as an example of an irregularly shaped body. This approach allows for a comprehensive study through symbolic manipulation of the gravitational potential function. In section 2, we present the physical properties of the polyhedral model of the asteroid (101955) Bennu, considering a constant density of 1.25 g.cm^{-3} [Chesley et al., 2014]. In section 3, we used the PSEM to calculate the potential on a grid of 1,002,000 points close to the target and compared our results with the classical polyhedron method as presented in Tsoulis and Petrovic [2001] and the gravitation model as presented in Aljbaae et al. [2021]. The latter authors represented the gravitational field of the central body using a cloud of point masses system distributed inside a polyhedral shape, dividing each tetrahedron into 20 parts. We also calculated the coordinates of the equilibrium points and examined their stability. Finally, in section 4, we present our conclusion, summarizing the results obtained in this study.

2 Physical properties of the asteroid (101955) Bennu

A non-convex polyhedral shape model of the asteroid Bennu, with 12288 triangular faces, is available in the Planetary Data System (PDS[1], [Nolan et al., 2013]).

Initially, to perform a dynamical study, it is necessary to refine the initial data of the coordinates of the vertices of the model by performing a translation and a rotation so that the center of mass and the main axes of inertia of the asteroid, respectively, coincide with the origin and with the axes of the system of coordinates fixed to the asteroid. The total dimensions of this asteroid's shape in the main directions, in km, are $(-0.2800, 0.2787) \times (-0.2617, 0.2671) \times (-0.2446, 0.2540)$. We used the method of Lien and Kajiya [1984] to calculate the integrals. Table 1 presents the main physical properties of the target, considering a homogeneous structure with uniform density.

[1] https://pds.nasa.gov/

Table 1: The physical properties of polyhedral models of the asteroid (101955) Bennu using the polyhedral model with 12288 triangular faces and 6146 vertices.

	12288 faces
Density (g.cm^{-3})	1.25
Effective diameters (km)	0.49010142
Areas estimation (km^2)	0.78734094
Polyhedral volumes (km^3)	0.06234399
Masses estimation ($\times 10^{10}$ kg)	7.79299999
Dynamical polar flattening: $J_2(-C_{20})$	0.05797096
Dynamical equatorial flattening C_{22}	0.00313350
Moments of inertia I_{xx}/M (km^2)	0.02308779
Moments of inertia I_{yy}/M (km^2)	0.02384619
Moments of inertia I_{zz}/M (km^2)	0.02697463
	$a = 0.2633$
Equivalent ellipsoid (km)	$b = 0.2560$
	$a = 0.2234$

3 Potential model and equilibrium points

In this work, only the asteroid's gravitational field is considered in a body-fixed frame of reference. The expansion series potential method was used, associated with the decomposition of the asteroid into tetrahedral elements, generating 1:

$$\begin{aligned} U &= \sum_{k=1}^{n} U^{(k)} \\ &= \sigma \sum_{k=1}^{n} \sum_{i=0}^{m} \iiint_{Q_k} P_i(u) \frac{\rho'^i \rho^i}{\rho^{(2i+1)}} dV + \epsilon \end{aligned} \tag{1}$$

where G is the gravitational constant, σ is the density of the asteroid, Q_k is a tetrahedral element, and ρ is the distance from a point outside the body to its center of mass, ρ' is the distance of a point belonging to the body to its center of mass, as shown in Fig. 1, $P_i(u)$ is the Legendre polynomials and ϵ is the truncation error. For more details on this expression and the mathematical development, we refer the reader to Mota [2017].

To demonstrate the efficiency of our method, we calculated the gravitational potential close to our target. We performed a series of tests comparing the potential (U_{PSEM}) calculated by our PSEM method (considering several orders: 5, 6, 7, ...) or the gravitation model presented in Aljbaae et al. [2021] (U_{TC20}), which divides the asteroid into 20 layers with the same density, with the classical polyhedron method (U_T) as calculated in Tsoulis and Petrovic [2001]. We calculated the relative errors between U_{PSEM} (orders 5, 6, 7, ...) or U_{TC20} and U_T as follows:

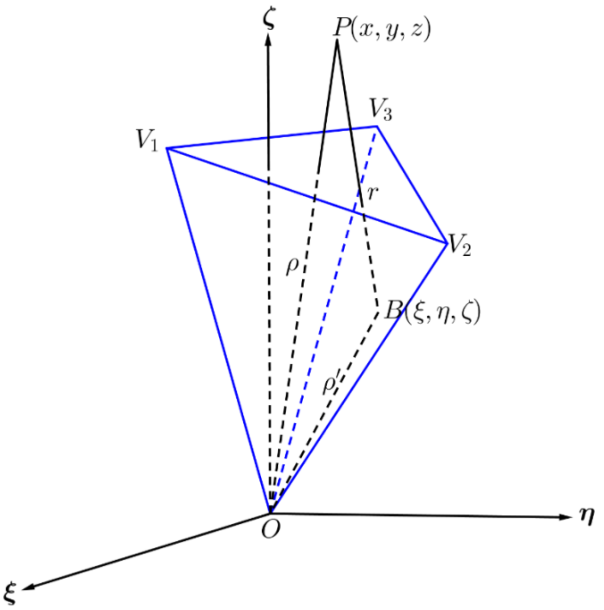

Figure 1: Tetrahedral element Q_k of vertices V_1, V_2, V_3 and O, indicating the distances ρ, ρ' and r .

$$RE = \frac{U - U_T}{U_T} \tag{2}$$

where RE is the relative error and U is either U_{PSEM} or U_{TC20}. Our results are presented in Fig. 2, where we can observe a good agreement with these models outside the body (right side of the red line). We can see that our model provides better results than the approach presented in Aljbaae et al. [2021] if the potential is developed with an order higher than 9. In Table 2, we present the CPU time needed to compute the potential of a grid of 1,002,000 points outside the asteroid using a CPU Pentium 3.10 GHz. It is worth mentioning that our method considerably reduces the computation processing time with respect to the classical polyhedron method while keeping the accuracy at a very acceptable level.

Table 2: Execution time for calculating the gravitational potential on a 1,002,000 point close to Bennu using a Pentium 3.60GHz CPU.

Tsoulis and Petrovic [2001]	Aljbaae et al. [2021]	This work
42m2.000s	0m30.180s	0m3.190s

We applied our method to obtain the zero-velocity surfaces and the equilibrium points of the target and assessed their stability. In Fig. 3, we present the projection of the zero-velocity surface onto the xy-plane. The position and the relative error of each equilibrium point's vector position with respect to the clas-

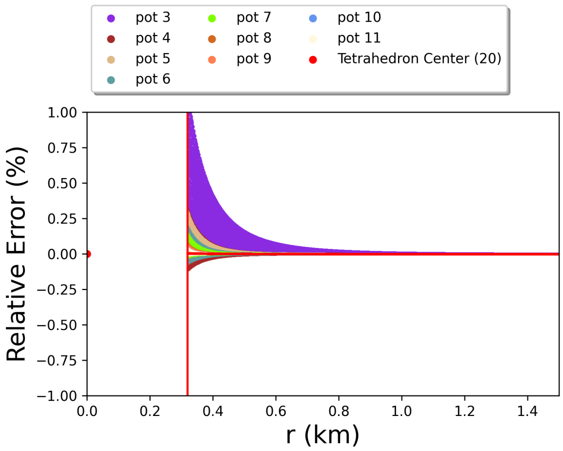

Figure 2: Relative error of the gravitational potential U_{PSEM} or U_{TC20} with respect to the classical polyhedron method (U_T)

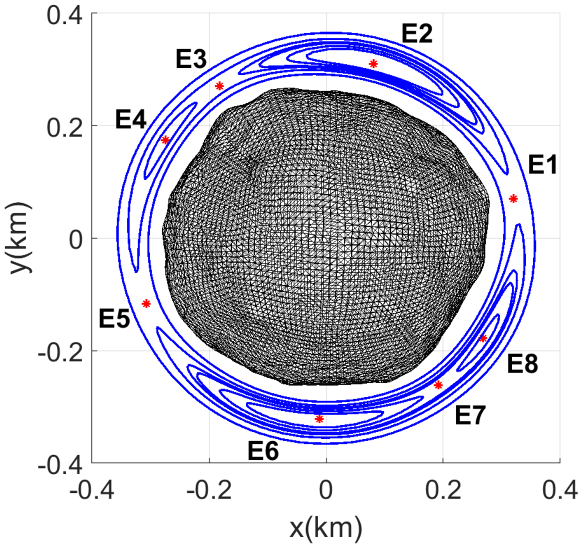

Figure 3: Zero-velocity curves and equilibrium points of (101955) Bennu

sical polyhedron method [Tsoulis and Petrovic, 2001] are given in table 3.

Examining Table 4, according to the topological classification established by and Jiang et al. [2014] and Wang et al. [2014], we found that points E_1, E_3, E_5 and E_7 have two real eigenvalues and four pure imaginary eigenvalues, belonging to Case 2, saddle-center-center. This reveals two families of periodic orbits and a family of quasi-periodic orbits in the vicinity of each of these points. On the other hand, points E_2, E_4 and E_8 present two pure imaginary eigenvalues and four complex eigenvalues, belonging to Case 5, sink-source-center, indicating the existence of only a family of periodic orbits in the neighborhood of each of these points. Finally, the linearly stable equilibrium point E6 presents three pairs of pure imaginary eigenvalues, indicating three families of periodic orbits, Case 1.

Table 3: Position of the Equilibrium points (in km) outside (101955) Bennu, using the potential series expansion method and their relative errors (RE) with respect to the classical polyhedron method.

		Pot 10	$\|RE\|(\%)$	Pot 11	$\|RE\|(\%)$
	x	0.320084		0.320299	
E_1	y	0.071455	0.13214	0.070627	0.12291
	z	-0.00061		-0.00064	
	x	0.080495		0.081055	
E_2	y	0.310212	0.00653	0.310079	0.00222
	z	0.001453		0.001456	
	x	-0.181241		-0.182605	
E_3	y	0.271724	0.09276	0.270832	0.08712
	z	-0.000054		-0.000066	
	x	-0.272636		-0.273951	
E_4	y	0.175824	0.04120	0.173731	0.03518
	z	-0.001907		-0.001939	
	x	-0.307399		-0.306813	
E_5	y	-0.114678	0.04371	-0.116349	0.03169
	z	-0.002318		-0.002425	
	x	-0.010622		-0.011002	
E_6	y	-0.321361	0.03645	-0.321341	0.03395
	z	-0.002497		-0.002405	
	x	0.193843		0.192142	
E_7	y	-0.260166	0.03380	-0.261382	0.03069
	z	-0.001056		-0.001163	
	x	0.269660		0.268367	
E_8	y	-0.176038	0.02938	-0.178066	0.02727
	z	-0.0009202		-0.0009307	

4 Conclusion

This study applied the PSEM to model the gravitational potential of the asteroid (101955) Bennu, an irregularly shaped body. The physical properties of the asteroid were analyzed, and the PSEM was used to calculate the potential on a grid close to the asteroid. The results were compared with the classical polyhedron method. The coordinates of the equilibrium points were also calculated and examined for stability. The PSEM proved to be an efficient method, reducing computation processing time while maintaining accuracy. The results suggest that the PSEM can be a valuable tool for modeling the gravitational potential of other irregularly shaped bodies.

In conclusion, this study demonstrated the effective-

Table 4: Eigenvalues of the equilibrium points around the asteroid (101955) Bennu.

$\times 10^{-4}$	E_1	E_2	E_3	E_4
λ_1	2.168	4.436i	1.999	4.569i
λ_2	-2.168	-4.436i	-1.999	-4.569i
λ_3	4.082i	0.713 + 2.676i	3.975i	0.755 + 2.575i
λ_4	-4.082i	0.713 - 2.676i	-3.975i	0.755 - 2.575i
λ_5	4.585i	-0.713 + 2.676i	4.602i	-0.755 + 2.575i
λ_6	-4.585i	-0.713 - 2.676i	-4.602i	-0.755 - 2.575i
	E_5	E_6	E_7	E_8
λ_1	1.937	2.012i	1.830	4.416i
λ_2	-1.937	-2.012i	-1.830	-4.416i
λ_3	3.889i	3.033i	4.018i	0.695 + 2.688i
λ_4	-3.889i	-3.033i	-4.018i	0.695 - 2.688i
λ_5	4.650i	4.443i	4.494i	-0.695 + 2.688i
λ_6	-4.650i	-4.443i	-4.494i	-0.695 - 2.688i

ness of the PSEM for modeling the gravitational potential of the asteroid (101955) Bennu. The results showed that the PSEM is a useful tool for reducing computation processing time while maintaining accuracy in the modeling of irregularly shaped bodies. This study adds to the growing body of knowledge on the use of the PSEM in various fields and highlights its potential for future research in gravitational potential modeling.

Acknowledgements

The authors wish to express their appreciation for the support provided by: grant 309089/2021-2 from the National Council for Scientific and Technological Development (CNPq) and the Coordination for the Improvement of Higher Education Personnel (CAPES). This publication has been supported by the RUDN University Scientific Projects Grant System, project No 202235-2-000.

Data Availability

All the data and the code used in this work will be available from the first author upon reasonable request.

References

S. Aljbaae, T. G. G. Chanut, V. Carruba, J. Souchay, A. F. B. A. Prado, and A. Amarante. The dynamical environment of asteroid 21 Lutetia according to different internal models. , 464:3552–3560, Jan. 2017. doi: 10.1093/mnras/stw2619. URL `http://adsabs.harvard.edu/abs/2017MNRAS.464.3552A`.

S. Aljbaae, D. M. Sanchez, A. F. B. A. Prado, J. Souchay, M. O. Terra, R. B. Negri, and L. O. Marchi. First approximation for spacecraft motion relative to (99942) Apophis. *Romanian Astronomical Journal*, 31(3):241–264, Nov. 2021. ISSN 2285-3758. URL `https://ui.adsabs.harvard.edu/abs/2021RoAJ...31..241A`.

G. Balmino. Gravitational Potential Harmonics from the Shape of an Homogeneous Body. *Celestial Mechanics and Dynamical Astronomy*, 60(3):331–364, Nov. 1994. doi: 10.1007/BF00691901. URL `https://ui.adsabs.harvard.edu/abs/1994CeMDA..60..331B`.

T. G. G. Chanut, S. Aljbaae, and V. Carruba. Mascon gravitation model using a shaped polyhedral source. , 450:3742–3749, July 2015. doi: 10.1093/mnras/stv845. URL `http://adsabs.harvard.edu/abs/2015MNRAS.450.3742C`.

S. R. Chesley, D. Farnocchia, M. C. Nolan, D. Vokrouhlický, P. W. Chodas, A. Milani, F. Spoto, B. Rozitis, L. A. M. Benner, W. F. Bottke, M. W. Busch, J. P. Emery, E. S. Howell, D. S. Lauretta, J.-L. Margot, and P. A. Taylor. Orbit and bulk density of the OSIRIS-REx target Asteroid (101955) Bennu. , 235: 5–22, June 2014. doi: 10.1016/j.icarus.2014.02.020.

D. Dechambre and D. J. Scheeres. Transformation of spherical harmonic coefficients to ellipsoidal harmonic coefficients. , 387:1114–1122, June 2002. doi: 10.1051/0004-6361:20020466. URL `https://ui.adsabs.harvard.edu/abs/2002A&A...387.1114D`.

W. Hu and D. J. Scheeres. Numerical determination of stability regions for orbital motion in uniformly rotating second degree and order gravity fields. , 52 (8):685–692, July 2004. doi: 10.1016/j.pss.2004.01.003. URL `https://ui.adsabs.harvard.edu/abs/2004P&SS...52..685H`.

Y. Jiang, H. Baoyin, X. Wang, and H. Li. Stability and Motion around Equilibrium Points in the Rotating Plane-Symmetric Potential Field. *arXiv e-prints*, art. arXiv:1403.1967, Mar. 2014. doi: 10.48550/arXiv.1403.1967. URL `https://ui.adsabs.harvard.edu/abs/2014arXiv1403.1967J`.

O. Kellogg. *Foundations of Potential Theory*. Die Grundlehren der mathematischen Wissenschaften in Einzeldarstellungen. Dover Publications, 1953. ISBN 9780486601441. URL `https://books.google.com.br/books?id=TxlfQi46CvEC`.

S.-l. Lien and J. T. Kajiya. A symbolic method for calculating the integral properties of arbitrary nonconvex polyhedra. *IEEE Computer Graphics and Applications*, 4(10):35–42, Oct. 1984. ISSN 1558-1756. doi: 10.1109/MCG.1984.6429334.

W. MacMillan. *The Theory of the Potential*. (MacMillan: Theoretical Mechanics). McGraw-Hill, 1930. URL https://books.google.com.br/books?id=-PEgAAAAMAAJ.

M. L. Mota. *Modelo do campo gravitacional de um corpo com distribuição de massa irregular utilizando o método da expansão do potencial em série e determinaçao de seus coeficientes dos harmônicos esféricos*. PhD thesis, INPE, São José dos Campos, 2017.

M. L. Mota and E. M. Rocco. Equilibrium points stability analysis for the asteroid 21 Lutetia. In *Journal of Physics Conference Series*, volume 1365 of *Journal of Physics Conference Series*, page 012007, Oct. 2019. doi: 10.1088/1742-6596/1365/1/012007. URL https://ui.adsabs.harvard.edu/abs/2019JPhCS1365a2007L.

M. L. Mota, S. Aljbaae, and A. F. B. A. Prado. The potential series expansion method: application to the asteroid (87) Sylvia. *European Physical Journal Special Topics*, 232(18-19):2961–2966, Dec. 2023. doi: 10.1140/epjs/s11734-023-01026-w.

M. C. Nolan, C. Magri, E. S. Howell, L. A. M. Benner, J. D. Giorgini, C. W. Hergenrother, R. S. Hudson, D. S. Lauretta, J. L. Margot, S. J. Ostro, and D. J. Scheeres. Asteroid (101955) Bennu Shape Model V1.0. NASA Planetary Data System, id. EAR-A-I0037-5-BENNUSHAPE-V1.0, Sept. 2013.

D. Tsoulis and S. Petrovic. On the singularities of the gravity field of a homogeneous polyhedral body. *Geophysics*, 66(2):535, Jan 2001. doi: 10.1190/1.1444944. URL https://ui.adsabs.harvard.edu/abs/2001Geop...66..535T.

F. C. F. Venditti. *Manobras orbitais ao redor de corpos irregulares*. PhD thesis, INPE, São José dos Campos, 2013.

J. Waldvogel. The Newtonian potential of a homogeneous cube. *Zeitschrift Angewandte Mathematik und Physik*, 27(6):867–871, Nov. 1976. doi: 10.1007/BF01595137. URL https://ui.adsabs.harvard.edu/abs/1976ZaMP...27..867W.

X. Wang, Y. Jiang, and S. Gong. Analysis of the potential field and equilibrium points of irregular-shaped minor celestial bodies. , 353(1):105–121, Sept. 2014. doi: 10.1007/s10509-014-2022-8. URL https://ui.adsabs.harvard.edu/abs/2014Ap&SS.353..105W.

R. A. Werner. The Gravitational Potential of a Homogeneous Polyhedron or Don't Cut Corners. *Celestial Mechanics and Dynamical Astronomy*, 59(3):253–278, July 1994. doi: 10.1007/BF00692875. URL https://ui.adsabs.harvard.edu/abs/1994CeMDA..59..253W.

R. A. Werner and D. J. Scheeres. Exterior gravitation of a polyhedron derived and compared with harmonic and mascon gravitation representations of asteroid 4769 Castalia. *Celestial Mechanics and Dynamical Astronomy*, 65(3):313–344, Sept. 1996. doi: 10.1007/BF00053511. URL https://ui.adsabs.harvard.edu/abs/1996CeMDA..65..313W.