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Abstract - In this study, we investigate trajectories involving multiple impulses within the framework of a generic spacecraft dynamics. Revisiting the age-old query of “How many impulses?”, we present novel manipulations heavily leveraging on the properties of the state transition matrix. Surprisingly, we are able to rediscover classical results leading to the introduction of a primer vector, albeit not making use of Pontryagin Maximum Principle as in the original developments by Lawden. Furthermore, our mathematical framework exhibits great flexibility and enables the introduction of what we term a “surrogate primer vector” extending a well known concept widely used in mission design. This enhancement allows to derive new simple optimality conditions that provide insights into the possibility to add and/or move multiple impulsive manoeuvres and improve the overall mass budget. This proves especially valuable in scenarios where a baseline trajectory arc is, for example, limited to a single impulse—an instance where traditional primer vector developments become singular and hinder conclusive outcomes. In demonstrating the practical application of the surrogate primer vector, we examine a specific case involving the four-body dynamics of a spacecraft within an Earth-Moon-Sun system. The system is characterized by the high-precision and differentiable VSOP2013 and ELP2000 ephemerides models. The focal point of our investigation is a reference trajectory representing a return from Mars, utilizing the weak stability boundary (WSB) of the Sun-Earth-Moon system. The trajectory incorporates two consecutive lunar flybys to insert the spacecraft into a lunar distant retrograde orbit (DRO). Conventionally, this trajectory necessitates a single maneuver at the DRO injection point. Prior to implementing the surrogate primer vector, a local optimization of the tra-

jectory is performed. Upon application of the surrogate primer vector, we successfully identify potential maneuver injection points, strategically reducing the overall mission cost. The introduction of these additional maneuvers, followed by local optimization, validates that the revised trajectory indeed incurs a lower cost compared to the original configuration.

I. INTRODUCTION

In this work, we will revisit the concept of the primer vector. The concept, originally developed back in the 60s by D. Lawden in a seminal work [1] has been more recently described in details by J. Prussing [2] who provides a more accessible work. The original derivation uses Pontryagin optimal control theory [3] to define an expression for the primer vector and derive necessary conditions for optimality in the context of multiple impulsive trajectories. In this work we propose an alternative elementary analysis of the optimality of multiple-impulse trajectory, showing how the concept of a primer vector emerges naturally without resorting to the introduction of co-states or any other concept from variational calculus (e.g. Pontryagin’s theory).

Our derivation is based on the direct analysis of the sensitivities of a multiple impulsive trajectory and provides a novel view on the fundamental concept introduced by Lawden. Using this analytical tool we are able to derive classical expressions for the primer vector for fixed time impulsive arcs [4], as well as to extend the analysis to arcs with a single impulse, deriving new expressions for the primer vector.

II. CONTEXT

We consider the problem of designing optimal multi-impulsive trajectories under a generic dynamics [5]. While there can be multiple nonlinear constraints imposed on the spacecraft, a general statement of trajectory optimization with impulsive maneuvers can be described as follows:

Given a dynamical system $\mathbf{f}(\mathbf{t}, \mathbf{x})$ and a spacecraft trajectory $\mathbf{x}(\mathbf{t})$ that obeys this dynamics

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{t}, \mathbf{x}) \quad (1)$$

We want to find a set of N impulsive maneuvers $\Delta\mathbf{V}_i$ at epochs τ_i such that boundary conditions are met

$$\mathbf{x}(\mathbf{t}_s) = \mathbf{x}_s \quad (2)$$

$$\mathbf{x}(\mathbf{t}_f) = \mathbf{x}_f \quad (3)$$

that also minimizes the total Δv

$$\Delta v = \sum_i |\Delta\mathbf{V}_i| \quad (4)$$

This can be formulated in a standard nonlinear programming (NLP) problem and optimized with various algorithms such as SNOPT, WORHP, SLSQP etc. However what this formulation fails to capture is the possibility of adding new maneuvers. While it is possible to add zero velocity maneuvers a posteriori and reoptimize with these new maneuvers, the optimization result will heavily depend on the location of these new maneuvers, choosing the location poorly can result in sub-optimal configurations. Exhaustively running a search for the location of these new maneuvers can prove to be prohibitively expensive. These are the questions the primer aims to answer: Can adding a new maneuver reduce the total Δv ? And if so: where should these new maneuvers be located?

A. Notation

Before going further, we need to establish notations and conventions we are going to use throughout the paper. With reference to Fig. 1 we consider a generic reference multiple-impulse trajectory we indicate with $\mathbf{x}_{\text{ref}}(t), t \in [\tau_s, \tau_f]$. The boundary states are indicated with the subscript s for start and f for final. The N impulses are given at intermediate points at epochs τ_i . Quantities immediately before the impulse are indicated with a minus (i.e. \mathbf{x}^-) and those immediately after with a plus (i.e. \mathbf{x}^+). The 6×6 time dependent matrix which maps the perturbation $\delta\mathbf{x}_1$ of the state applied at some epoch t_1 to the effect $\delta\mathbf{x}_2$ on the state at a later time t_2 is the State Transition Matrix (STM) denoted with $M(t_1, t_2)$ or simply M when possible:

$$\delta\mathbf{x}_2 = M(t_1, t_2)\delta\mathbf{x}_1$$

The STM satisfies the following equations of motion:

$$\dot{M} = \frac{\partial \mathbf{f}}{\partial \mathbf{x}} \Big|_{\mathbf{x}_{\text{ref}}} M$$

In the case of our generic multiple impulse reference trajectory (Fig. 1), we will denote with M_i the STM from \mathbf{x}_j^+ to \mathbf{x}_f . This can also be computed as the product of all intermediate STMs as follows:

$$M_i = \prod_{j=i}^N \tilde{M}_j$$

where \tilde{M}_j is the state transition matrix from \mathbf{x}_j^+ to \mathbf{x}_{j+1}^- .

We also make heavy use of the following notation in manipulations involving STMs:

$$M = \left[\begin{array}{c|c} M^{rr} & M^{rv} \\ \hline M^{vr} & M^{vv} \end{array} \right] = [M^{xr} \mid M^{xv}] = \left[\begin{array}{c} M^{rx} \\ \hline M^{vx} \end{array} \right]$$

so that we may write, for example:

$$\delta\mathbf{r}_f = M^{rr}\delta\mathbf{r}_s + M^{rv}\delta\mathbf{v}_s \quad (5a)$$

$$\delta\mathbf{v}_f = M^{vr}\delta\mathbf{r}_s + M^{vv}\delta\mathbf{v}_s \quad (5b)$$

as well as the more compact form:

$$\delta\mathbf{x}_f = M(\tau_s, \tau_f)\delta\mathbf{x}_s$$

B. Sensitivities in a multi-impulsive trajectory

A generic multiple-impulse trajectory is fully defined by the initial state, the dynamics and the various $\Delta\mathbf{V}_i$ applied at τ_i . The differential of the final state with respect to variations of the free parameters is:

$$\delta\mathbf{x}_f = \sum_{i=1}^N M_i^{xv}\delta\Delta\mathbf{V}_i + \sum_{i=1}^N M_i(\mathbf{f}_i^- - \mathbf{f}_i^+)\delta\tau_i \quad (6)$$

with

$$M_i = M(\tau_i, \tau_f) \quad (7)$$

as it can be readily seen applying the definition of the STM and singling out the various contributions of the resulting state changes. Whenever the dynamics has the form $\mathbf{f} = [\mathbf{v}, \mathbf{g}(\mathbf{t}, \mathbf{r})]$, then the previous formula simplifies to:

$$\delta\mathbf{x}_f = \sum_{i=1}^N M_i^{xv}\delta\Delta\mathbf{V}_i - \sum_{i=1}^N M_i^{xr}\Delta\mathbf{V}_i\delta\tau_i \quad (8)$$

This form is essentially widely applicable unless there is a force that depends on velocity (e.g. a drag term, a rotating frame, etc.). When designing a multiple impulsive trajectory, it is often the case that the sum of all velocity increment magnitudes $\Delta v = \sum_i |\Delta\mathbf{V}_i|$ is considered as an objective function to minimize. Let us now assume that we are aiming to lower the total Δv by adding

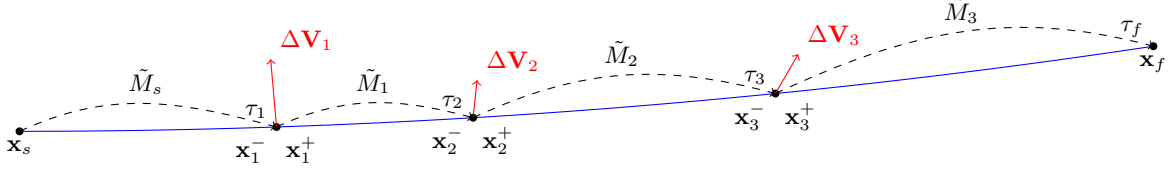


Fig. 1: A multiple impulsive trajectory with $N = 3$ impulses. State \mathbf{x}_i is considered just before the impulse is applied.

a number of infinitesimally small velocity increments to the already existing impulses, as well as some additional ones at some other epochs (i.e. in correspondence to zero norm impulses).

The total variation of the objective function can then be expressed in the following general form:

$$\delta\Delta v = \sum_{|\Delta\mathbf{V}_i| \neq 0} \frac{\Delta\mathbf{V}_i}{|\Delta\mathbf{V}_i|} \cdot \delta\Delta\mathbf{V}_i + \sum_{|\Delta\mathbf{V}_i| = 0} |\delta\Delta\mathbf{V}_i| \quad (9)$$

Note how the impulses added in correspondence of the finite $\Delta\mathbf{V}_i$ have a much different effect on the total Δv variation. This, as we shall see, has profound consequences in the optimality conditions we will derive.

III. ADDING/CHANGING IMPULSES

To simplify the initial analysis we will consider the possibility to add 3 correction maneuvers $\delta\Delta\mathbf{V}_i$ at the epochs τ_i to an already existing multiple impulsive arc. Later we will see that this, in fact, cover all possible cases. The three correction maneuvers (velocity increments) are applied at epochs where the impulses $\Delta\mathbf{V}_1$, $\Delta\mathbf{V}_2$, $\Delta\mathbf{V}_3$ are present. We allow the possibility that some of those $\Delta\mathbf{V}_i$ are actually zero, in which case the correction manoeuvre corresponds to the addition of one impulse. Without loss of generality, we impose the order that $\Delta\mathbf{V}_1$ occurs at the **beginning** of the trajectory, $\Delta\mathbf{V}_3$ occurs at the **end** of the segment and that $\Delta\mathbf{V}_2$ lies in between. The total Δv , before applying the correction manoeuvres, is given by:

$$\Delta v = |\Delta\mathbf{V}_1| + |\Delta\mathbf{V}_2| + |\Delta\mathbf{V}_3| \quad (10)$$

after:

$$\Delta v = |\Delta\mathbf{V}_1 + \delta\Delta\mathbf{V}_1| + |\Delta\mathbf{V}_2 + \delta\Delta\mathbf{V}_2| + |\Delta\mathbf{V}_3 + \delta\Delta\mathbf{V}_3| \quad (11)$$

Is it possible to find the $\delta\Delta\mathbf{V}$ as to reduce the total Δv while still reaching the very same final conditions? This is the fundamental question we will be answering definitively in the rest of our developments, and that ties closely to Edelbaum's question "How many impulses?" [6].

Since we need to reach the same end state, applying Eq.(6), the boundary conditions differentials can be written as:

$$\delta\mathbf{r}_f = M_1^{rv} \delta\Delta\mathbf{V}_1 + M_2^{rv} \delta\Delta\mathbf{V}_2 = 0 \quad (12a)$$

$$\delta\mathbf{v}_f = M_1^{vv} \delta\Delta\mathbf{V}_1 + M_2^{vv} \delta\Delta\mathbf{V}_2 + \delta\Delta\mathbf{V}_3 = 0 \quad (12b)$$

Where

$$M_1 = M(\tau_1, \tau_3)$$

$$M_2 = M(\tau_2, \tau_3)$$

Solving for $\delta\Delta\mathbf{V}_1$ in terms of $\delta\Delta\mathbf{V}_2$, we obtain

$$\delta\Delta\mathbf{V}_1 = \left[(M_1^{rv})^{-1} M_2^{rv} \right] \delta\Delta\mathbf{V}_2 = A_{12} \delta\Delta\mathbf{V}_2 \quad (13)$$

According to Prussing [2] the set of times for which M_1^{rv} is not invertible is discrete and so we can neglect these cases. Now $\delta\Delta\mathbf{V}_3$ can be solved in terms of $\delta\Delta\mathbf{V}_2$

$$\delta\Delta\mathbf{V}_3 = [-M_1^{vv} A_{12} - M_2^{vv}] \delta\Delta\mathbf{V}_2 \quad (14)$$

For ease of notation, we will rewrite those equations in the following way:

$$\delta\Delta\mathbf{V}_1 = A_{12} \delta\Delta\mathbf{V}_2 \quad (15a)$$

$$\delta\Delta\mathbf{V}_3 = A_{32} \delta\Delta\mathbf{V}_2 \quad (15b)$$

with

$$A_{12} = (M_1^{rv})^{-1} M_2^{rv} \quad (16a)$$

$$A_{32} = -M_1^{vv} A_{12} - M_2^{vv} \quad (16b)$$

These linear relations, efficiently computed from the state transition matrix (available via the variational equations or even analytically for simpler dynamics), introduce a relation between the three correction manoeuvres, so that we are only free to decide the middle one (or any other) as the other two will follow necessarily as to satisfy the requested boundary conditions. In order to search for the existence of a $\delta\Delta\mathbf{V}_2$ that lowers the total objective, and with reference to Eq.(9), there are 4 qualitatively different cases which is interesting to analyze here (there are more in the most general case, but we here restrict ourselves to these), as seen in table 1:

Tab. 1: Case taxonomy

Case	# $\Delta\mathbf{V}$		Comment
	= 0	$\neq 0$	
1	3	0	Trivial case
2	0	3	Local optimality
3	1	2	Primer vector
4	2	1	Surrogate primer vector

A. Case 1: 3 zero $\Delta\mathbf{V}$'s

The first case of 3 zero $\Delta\mathbf{V}$ is trivial as adding any new $\Delta\mathbf{V}$ will automatically increase the total Δv

B. Case 2: 3 nonzero $\Delta\mathbf{V}$'s

The most straightforward case is 3 nonzero $\Delta\mathbf{V}$'s – Eq.(9) reduces to:

$$\delta\Delta v = \frac{\Delta\mathbf{V}_1}{|\Delta\mathbf{V}_1|} \cdot \delta\Delta\mathbf{V}_1 + \frac{\Delta\mathbf{V}_2}{|\Delta\mathbf{V}_2|} \cdot \delta\Delta\mathbf{V}_2 + \frac{\Delta\mathbf{V}_3}{|\Delta\mathbf{V}_3|} \cdot \delta\Delta\mathbf{V}_3 \quad (17)$$

This can be written in terms of $\delta\Delta\mathbf{V}_2$ only

$$\delta\Delta v = \left(\frac{\Delta\mathbf{V}_1}{|\Delta\mathbf{V}_1|} \cdot A_{12} + \frac{\Delta\mathbf{V}_2}{|\Delta\mathbf{V}_2|} + \frac{\Delta\mathbf{V}_3}{|\Delta\mathbf{V}_3|} \cdot A_{32} \right) \cdot \delta\Delta\mathbf{V}_2 \quad (18)$$

The total Δv can be reduced if the vector \mathbf{b} is nonzero, where \mathbf{b} is

$$\mathbf{b} = - \left(\frac{\Delta\mathbf{V}_1}{|\Delta\mathbf{V}_1|} \cdot A_{12} + \frac{\Delta\mathbf{V}_2}{|\Delta\mathbf{V}_2|} + \frac{\Delta\mathbf{V}_3}{|\Delta\mathbf{V}_3|} \cdot A_{32} \right) \quad (19)$$

In the case that $|\mathbf{b}| > 0$ we can choose

$$\delta\Delta\mathbf{V}_2 = |\delta\Delta\mathbf{V}_2| \frac{\mathbf{b}}{|\mathbf{b}|} \quad (20)$$

And $\delta\Delta\mathbf{V}_1, \delta\Delta\mathbf{V}_3$ from equations 15. These directions will guarantee to locally reduce the total Δv . If the vector \mathbf{b} is 0 then we know, at least locally, the total Δv cannot be reduced

C. Case 3: 2 nonzero $\Delta\mathbf{V}$'s, 1 zero $\Delta\mathbf{V}$

Assuming $\delta\Delta\mathbf{V}_2 = \mathbf{0}$, this is the case that will result in finding the classical primer vector – Eq.(9) reduces to:

$$\delta\Delta v = \frac{\Delta\mathbf{V}_1}{|\Delta\mathbf{V}_1|} \cdot \delta\Delta\mathbf{V}_1 + |\delta\Delta\mathbf{V}_2| + \frac{\Delta\mathbf{V}_3}{|\Delta\mathbf{V}_3|} \cdot \delta\Delta\mathbf{V}_3 \quad (21)$$

Writing everything in terms of $\delta\Delta\mathbf{V}_2$

$$\delta\Delta v = \left[\frac{\Delta\mathbf{V}_1}{|\Delta\mathbf{V}_1|} \cdot A_{12} + \frac{\Delta\mathbf{V}_3}{|\Delta\mathbf{V}_3|} \cdot A_{32} \right] \cdot \delta\Delta\mathbf{V}_2 + |\delta\Delta\mathbf{V}_2| \quad (22)$$

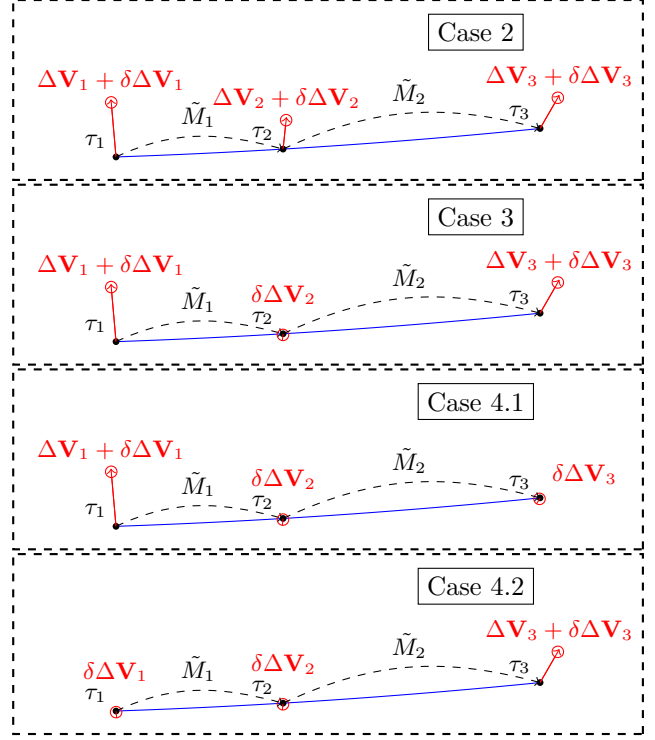


Fig. 2: Non trivial cases considered. From Case 2 we find local optimality conditions. From Case 3 we are able to derive an expression for the primer vector. From Case 4 we find a new necessary condition for optimality allowing, also for this case where a primer vector is not available, to conclude on the possibility to add impulses.

Letting $\delta\Delta\mathbf{V}_2 = |\delta\Delta\mathbf{V}_2| \hat{\mathbf{u}}$ where $\hat{\mathbf{u}}$ is a unit vector, allows us to write

$$\delta\Delta v = \left(\left[\frac{\Delta\mathbf{V}_1}{|\Delta\mathbf{V}_1|} \cdot A_{12} + \frac{\Delta\mathbf{V}_3}{|\Delta\mathbf{V}_3|} \cdot A_{32} \right] \cdot \hat{\mathbf{u}} + 1 \right) |\delta\Delta\mathbf{V}_2| \quad (23)$$

Let us introduce the vector \mathbf{p} defined as:

$$\mathbf{p} = - \left(\frac{\Delta\mathbf{V}_1}{|\Delta\mathbf{V}_1|} \cdot A_{12} + \frac{\Delta\mathbf{V}_3}{|\Delta\mathbf{V}_3|} \cdot A_{32} \right) \quad (24)$$

We get:

$$\delta\Delta v = (-\mathbf{p} \cdot \hat{\mathbf{u}} + 1) |\delta\Delta\mathbf{V}_2| \quad (25)$$

The best direction to lower the total Δv is $\hat{\mathbf{u}} = \mathbf{p}/|\mathbf{p}|$, in which case the total reduction of Δv is given by

$$\delta\Delta v = (-|\mathbf{p}| + 1) |\delta\Delta\mathbf{V}_2| \quad (26)$$

This means if $|\mathbf{p}| > 1$ there exist a direction that can lower the total Δv . Note that Eq. (24), accounting for

Eq.(16), is the same as Eq.(2.50) in [2]. We have thus found, in this case, an expression for the *primer vector*, remarkably circumventing entirely Pontryagin's theory and without introducing co-states. The corresponding change in the $\Delta\mathbf{V}$'s are given by

$$\delta\Delta\mathbf{V}_2 = |\delta\Delta\mathbf{V}_2| \frac{\mathbf{p}}{|\mathbf{p}|} \quad (27a)$$

And $\delta\Delta\mathbf{V}_1, \delta\Delta\mathbf{V}_3$ from equations 15. If everywhere on the trajectory $|\mathbf{p}| \leq 1$ then, locally, it is not possible to add a new maneuver and reduce the total Δv .

D. Case 4: 1 nonzero $\Delta\mathbf{V}$, 2 zero $\Delta\mathbf{V}$'s

This is a novel application of the primer vector. Here we have only 1 non-zero $\Delta\mathbf{V}$. We can still solve for $\delta\Delta\mathbf{V}_1$ and $\delta\Delta\mathbf{V}_3$ in terms of $\delta\Delta\mathbf{V}_2$ with the formula 15 and 16.

1. $\Delta\mathbf{V}_1 \neq 0, \Delta\mathbf{V}_3 = 0$
Eq.(9) reduces to:

$$\delta\Delta v = \frac{\Delta\mathbf{V}_1}{|\Delta\mathbf{V}_1|} \cdot \delta\Delta\mathbf{V}_1 + |\delta\Delta\mathbf{V}_3| + |\delta\Delta\mathbf{V}_2| \quad (28)$$

Writing everything in terms of $\delta\Delta\mathbf{V}_2$

$$\delta\Delta v = \frac{\Delta\mathbf{V}_1}{|\Delta\mathbf{V}_1|} \cdot A_{12} \cdot \delta\Delta\mathbf{V}_2 + |A_{32} \cdot \delta\Delta\mathbf{V}_2| + |\delta\Delta\mathbf{V}_2| \quad (29)$$

Similarly to what done previously we have:

$$\delta\Delta v = \left(\left[\frac{\Delta\mathbf{V}_1}{|\Delta\mathbf{V}_1|} \cdot A_{12} \right] \cdot \hat{\mathbf{u}} + |A_{32} \cdot \hat{\mathbf{u}}| + 1 \right) |\delta\Delta\mathbf{V}_2| \quad (30)$$

where

$$\mathbf{b} = - \left(\frac{\Delta\mathbf{V}_1}{|\Delta\mathbf{V}_1|} \cdot A_{12} \right) \quad (31)$$

and $\delta\Delta\mathbf{V}_2 = |\delta\Delta\mathbf{V}_2| \hat{\mathbf{u}}$.

2. Case $\Delta\mathbf{V}_1 = 0, \Delta\mathbf{V}_3 \neq 0$

This case result in the same developments as in the previous case where the indexes are swapped $1 \leftrightarrow 3$.

In all cases, an expression in the following form is found:

$$\delta\Delta v = (-\mathbf{b} \cdot \hat{\mathbf{u}} + |B\hat{\mathbf{u}}| + 1) |\delta\Delta\mathbf{V}_2| \quad (32)$$

We may then obtain a reduction in our total Δv if and only if $\min_{\hat{\mathbf{u}}} (-\mathbf{b} \cdot \hat{\mathbf{u}} + |B\hat{\mathbf{u}}| + 1) < 0$. Trivially, if $|\mathbf{b}| < 1$ then we can exclude such a possibility efficiently avoiding to make any computation. Besides, since:

$$\delta\Delta v \geq (-|\mathbf{b}| + \sigma_{\min} + 1) |\delta\Delta\mathbf{V}_2| \quad (33)$$

where σ_{\min} is the smallest singular value of the matrix B we have that if

$$|\mathbf{b}| < 1 + \sigma_{\min} \quad (34)$$

we can also exclude to be able to reduce the total ΔV . In other cases, we must find numerically the minimum. There are 2 good initial guesses to the minimization problem.

- $\hat{\mathbf{u}} = \mathbf{b}/|\mathbf{b}|$
- $\hat{\mathbf{u}} = \pm \mathbf{s}_{\min}$

where \mathbf{s}_{\min} is the singular vector that corresponds to the smallest singular value of the matrix B . The sign in front of \mathbf{s}_{\min} is chosen so that

$$\hat{\mathbf{u}} \cdot \mathbf{b} \geq 0 \quad (35)$$

With these 2 initial guesses, numerical analysis indicates the global minima of the function is always obtain using a few iterations of a gradient based solver. Once this optimization is done and $\hat{\mathbf{u}}_{\text{optimal}}$ is obtained, we need to check if the **primer condition** is satisfied

$$\mathbf{b} \cdot \hat{\mathbf{u}}_{\text{optimal}} - |B\hat{\mathbf{u}}_{\text{optimal}}| > 1 \quad (36)$$

In case the condition is satisfied, the directions that produces a lower total Δv are given by

$$\delta\Delta\mathbf{V}_2 = |\delta\Delta\mathbf{V}_2| \hat{\mathbf{u}}_{\text{optimal}} \quad (37)$$

And $\delta\Delta\mathbf{V}_1, \delta\Delta\mathbf{V}_3$ from equations 15. To be consistent with the classical primer case, it may be useful to define a **surrogate primer vector** \mathbf{p} as:

$$\mathbf{p} = \left(\mathbf{b} \cdot \hat{\mathbf{u}}_{\text{optimal}} - |B\hat{\mathbf{u}}_{\text{optimal}}| \right) \hat{\mathbf{u}}_{\text{optimal}} \quad (38)$$

E. Extra commentary

Please note that when one of the $\Delta\mathbf{V} = 0$, its location is arbitrary along the trajectory, so the above computation should be done for each possible location (Or at least densely enough). For case 3), there is one zero $\Delta\mathbf{V}$ so the primer vector is parameterized by the location of the new maneuver along the orbit. In case 4) the novel application, there are 2 zero maneuvers. This means the analysis must be performed for all possible pairs of time $t_1 < t_2$ along the trajectory.

F. Independence of the primer vector

In the above analysis, we considered the case of only 3 maneuvers. However, there are trajectories with arbitrary many maneuvers. How does the primer vector behave in such a case? Concentrating on the classical

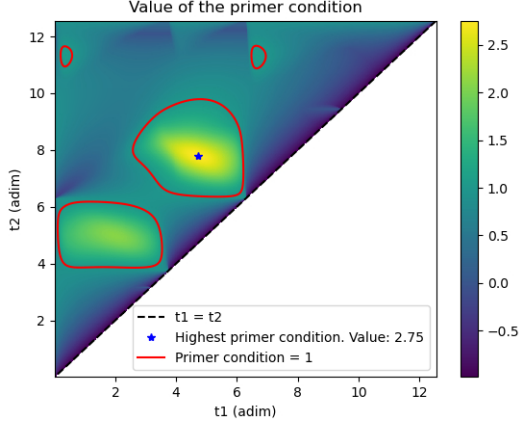


Fig. 3: Surrogate primer vector map for the simple transfer

primer vector case: at any given time along the trajectory, the primer vector can be calculated by choosing any two nonzero maneuvers. In general, the primer vector will be dependent of the choice of the two maneuvers. However, we have shown that if the trajectory is locally optimal with respect to all $\Delta \mathbf{V}_i$ then the primer vector is independent of the choice of the two maneuvers chosen to calculate it. In fact, if the trajectory is locally optimal, the primer vector, at any time, has a simple form

$$\mathbf{p}(t) = \boldsymbol{\lambda}_r \cdot M(t, t_f)^{rv} + \boldsymbol{\lambda}_v \cdot M(t, t_f)^{vv} \quad (39)$$

where $\boldsymbol{\lambda}_r$ and $\boldsymbol{\lambda}_v$ are constant vectors, their values are

$$\boldsymbol{\lambda}_v = \frac{\Delta \mathbf{V}_f}{|\Delta \mathbf{V}_f|} \quad (40a)$$

$$\boldsymbol{\lambda}_r = \left(\frac{\Delta \mathbf{V}_s}{|\Delta \mathbf{V}_s|} - \boldsymbol{\lambda}_v \cdot M(t_s, t_f)^{vv} \right) \left[M(t_s, t_f)^{rv} \right]^{-1} \quad (40b)$$

$\Delta \mathbf{V}_s$ and $\Delta \mathbf{V}_f$ are the first and last nonzero maneuver respectively. Interestingly, only optimality with respect to $\Delta \mathbf{V}_i$ and not the epoch τ_i is needed. The proof of independence, while not complicated, is a bit tedious and so is omitted.

IV. EXAMPLES

A. A simple transfer

In this chapter, we will cover a simple application of the novel primer condition in a Keplerian dynamics transfer. Although the case itself is trivial, it serves as a toy model that can be well understood to illustrate the concept. The system in question will be scaled so that the center

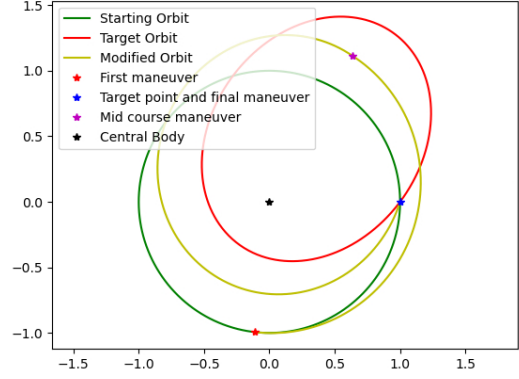


Fig. 4: Example of the change in the trajectory following surrogate primer vector for the simple transfer.

body has $\mu = GM = 1$. The transfer start with the state

$$r_0 = [1, 0, 0] \quad (41)$$

$$v_0 = [0, 1, 0] \quad (42)$$

The spacecraft is propagated for 4π units of time (and reaches the same state) and then the maneuver

$$\Delta \mathbf{V}_3 = [0.6, -0.2, 0] \quad (43)$$

is applied. The transfer goes from a circular orbit of semi-major of 1 to an eccentric orbit of the same semi-major axis of 1. This transfer cost $\Delta v_{\text{total}} = \sqrt{2/5} \approx 0.632$. Applying the surrogate primer vector calculation for $0 \leq t_1 \leq t_2 \leq 4\pi$, we obtain the graph in 3. The maximum of the value of

$$\mathbf{b} \cdot \hat{\mathbf{u}}_{\text{optimal}} - |B\hat{\mathbf{u}}_{\text{optimal}}| \quad (44)$$

is reached at $t_1 = 4.708$ and $t_2 = 7.783$ with value 2.754. The corresponding $\delta \Delta \mathbf{V}'_s$ are

$$\delta \Delta \mathbf{V}_1 / |\delta \Delta \mathbf{V}_2| = [0.941, 0.036, 0]$$

$$\delta \Delta \mathbf{V}_2 / |\delta \Delta \mathbf{V}_2| = [0.997, -0.078, 0]$$

$$\delta \Delta \mathbf{V}_3 / |\delta \Delta \mathbf{V}_2| = [-3.878, 0.05834, 0]$$

An example of the trajectory that is obtained following the direction of the surrogate primer vector is shown in 4. The trajectory shown has a $\Delta v_{\text{total}} = 0.487$

B. The MSR-ERO case

We now turn our attention to a more operational case. In the context of the MSR-ERO mission [7], one of the proposed trajectories for the Earth Return Orbiter (ERO), coming back from Mars makes use of a lunar fly to insert into the weak stability boundary [8] of the Earth-Moon-Sun system and then a second flyby of the Moon

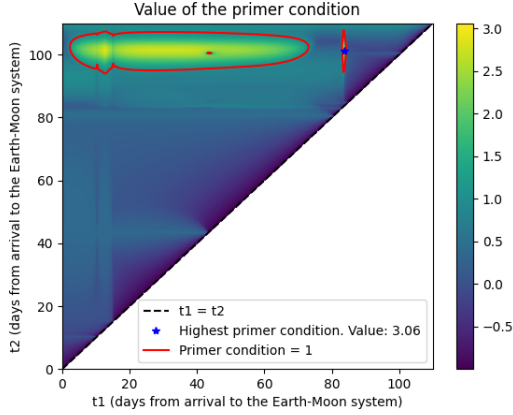


Fig. 5: Surrogate primer vector map for the MSR-ERO case

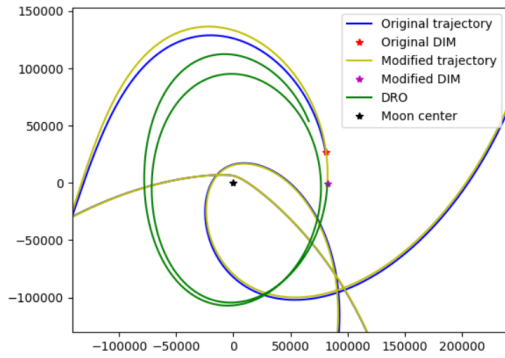


Fig. 6: End part of the trajectories inserting into the DRO. The trajectories are displayed in the Earth-Moon rotating system, centered on the Moon. For the modified trajectory the other maneuvers are outside this plot.

to insert in a Distant Retrograde Orbit (DRO) [9] of the Earth-Moon system. This trajectory has nominally only 1 maneuver, a burn at the insertion into the DRO. The total Δv for this trajectory is 72.89 m/s. We reproduced the trajectory under a dynamics accounting only for the Earth, the Sun and the Moon gravity using the VSOP and ELP analytical ephemerides in a Taylor integration scheme [10].

The surrogate primer vector was calculated and is shown in figure 5. Several locations exist that can lower the total Δv . Numerical optimization of the trajectory using these locations shows the Δv can be reduced by ≈ 3 m/s. While this is a modest reduction in Δv , this case shows the possibility and general applicability of our method to generic multiple impulse trajectories. The resulting trajectories are shown in 6. [tb]

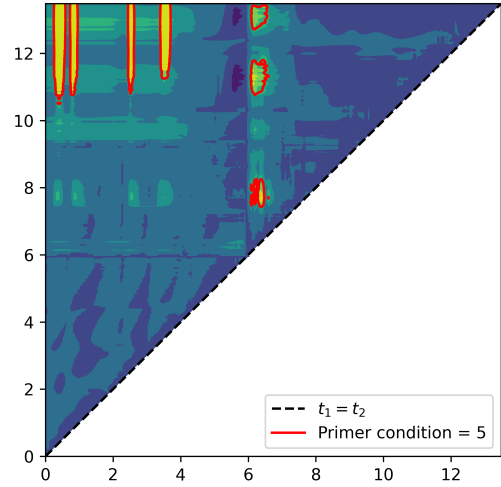


Fig. 7: Surrogate primer vector map for the Halo-Vertical Lyapunov case. Areas where the primer condition is higher than 5 are highlighted. The sensible nature of these transfers allow for larger values of the primer.

C. A Halo-Vertical Lyapunov heteroclinic transfer

As a third example, we consider the circular restricted three-body problem in the Earth-Moon system. We start with an almost heteroclinic connection between a Halo orbit around L_1 and a vertical Lyapunov orbit around L_2 : this trajectory connects the two periodic orbits but requires a small ΔV to be applied along the trajectory. The transfer requires a 29.31 m/s correction maneuver after 25.83 days and reaches the vertical Lyapunov orbit along the corresponding stable manifold after 58.63 days in total. This trajectory was taken among those presented in the work of Tagliaferri et al. [11], where a systematic search procedure is outlined for these cases. However, the small mid-course position discontinuity (characteristic of the approach in [11]), was canceled with a few correction steps. The state transition matrix is computed via the variational equations. Due to the chaotic nature of the transfer, the STM has a rather large numerical instability associated that needs to be taken care of by resetting the corresponding initial conditions frequently. This, in connection to the high precision Taylor scheme employed [10] allows to contain this intrinsic instability to a tolerable amount. Formally, this case was not explicitly covered above, as it displays an impulse in the middle of the trajectory arc, however, the same formalism can be easily adapted to this situation. The results are shown in Figure 7, in which areas where the primer

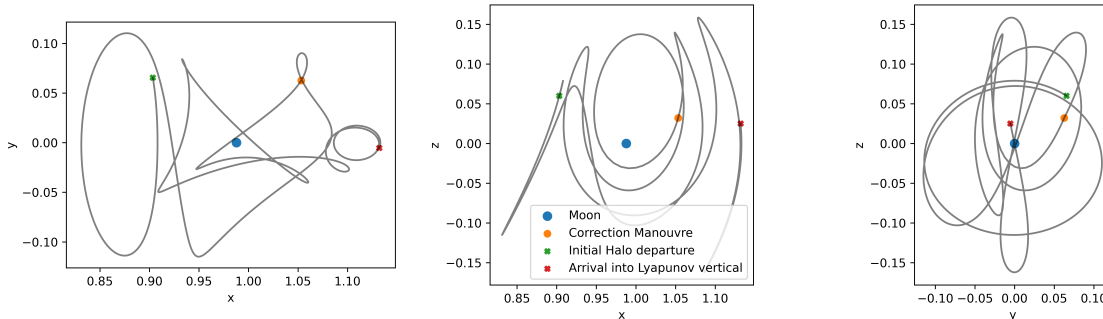


Fig. 8: The reference heteroclinic connection between a Halo and a vertical Lyapunov orbit. A single impulsive correction manoeuvre is present.

condition rises above values of 5 are highlighted. From this quick analysis, it is apparent that adding two impulses: a first impulse, before the mid-course correction, and a second one towards the vertical Lyapunov orbit arrival, results in the biggest ΔV gains.

V. CONCLUSION

In the context of multiple impulse trajectories, the classical primer vector, rooted in Pontryagin’s optimal control theory, traditionally facilitates the addition or removal of impulses. In our work, we demonstrate an alternative approach, leveraging only first-order information surrounding a reference trajectory, as provided by the state-transition matrix, and without introducing adjoint variables (co-states). Our derivation aims for simplicity and clarity while achieving the same conditions. Subsequently, we utilize this new formalism to derive novel optimality conditions, covering previously intractable cases. Specifically, we construct a surrogate primer vector that allows us to assess the benefits of adding two new impulses to a single $\Delta \mathbf{V}$ trajectory. Although the construction involves a simple numerical optimization, it proves highly efficient and unveils the intricate structure of multiple impulse trajectories. We then successfully apply the new surrogate primer vector to three cases: a simple transfer in Keplerian dynamics, a more complex scenario involving the MSR-ERO mission in an accurate ephemeris model, and an almost heteroclinic transfer between a Halo orbit and a vertical Lyapunov orbit in the Earth-Moon system

Due to its broad derivation applicable to any multi-impulsive trajectory, our findings seamlessly extend to diverse applications. Notably, our developments offer insights into optimizing station-keeping maneuvers around Lagrange points and refining quasi-heteroclinic connections between Halo orbits. This generality underscores the versatility and potential impact of our approach

across various spaceflight mechanics scenarios.

VI. CODE AVAILABILITY

All of the figures and quantitative results in this paper can be reproduced running the python based code distributed freely under a permissive GPLv3 licence here: <https://gitlab.com/EuropeanSpaceAgency/mit>.

REFERENCES

- [1] Lawden, D. F. “Optimal trajectories for space navigation.” (No Title), 1963.
- [2] Prussing, J. E. “Primer vector theory and applications.” *Spacecraft trajectory optimization*, Vol. 29, p. 16, 2010.
- [3] Pontryagin, L. S. *Mathematical theory of optimal processes*. Routledge, 2018.
- [4] Lion, P. M. and Handelsman, M. “Primer vector on fixed-time impulsive trajectories.” *Aiaa Journal*, Vol. 6, No. 1, pp. 127–132, 1968.
- [5] Ocampo, C. and Munoz, J.-P. “Variational equations for a generalized spacecraft trajectory model.” *Journal of guidance, control, and dynamics*, Vol. 33, No. 5, pp. 1615–1622, 2010.
- [6] Edelbaum, T. “How many impulses/ques.” “3rd and 4th Aerospace Sciences Meeting,” p. 7. 1967.
- [7] You, H., Sarli, B., Gough, K., Hagedorn, A., Hudgins, P., Yew, C., Neuman, M., Lin, J., Green, T., Bowman, E., et al. “Mars Sample Return (MSR) Capture, Containment, and Return System (CCRS) Overview.” “Taiwan International Assembly of Space Science, Technology, and Industry (TASTI) 2023,” 2023.

- [8] Garcia, F. and Gomez, G. “A note on weak stability boundaries.” *Celestial Mechanics and Dynamical Astronomy*, Vol. 97, pp. 87–100, 2007.
- [9] Capdevila, L. R. and Howell, K. C. “A transfer network linking Earth, Moon, and the triangular libration point regions in the Earth-Moon system.” *Advances in Space Research*, Vol. 62, No. 7, pp. 1826–1852, 2018.
- [10] Biscani, F. and Izzo, D. “Revisiting high-order Taylor methods for astrodynamics and celestial mechanics.” *Monthly Notices of the Royal Astronomical Society*, Vol. 504, No. 2, pp. 2614–2628, 2021.
- [11] Tagliaferri, F., Blazquez, E., Acciarini, G., and Izzo, D. “Global Optimization for Trajectory Design via Invariant Manifolds in the Earth-Moon Circular Restricted Three-Body Problem.”, 2024.